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ÚSTAV MATEMATIKY

MATHEMATICAL MODELS OF RELIABILITY IN TECHNICAL APPLICATIONS

MATEMATICKÉ MODELY SPOLEHLIVOSTI V TECHNICKÉ PRAXI

MASTER'S THESIS

DIPLOMOVÁ PRÁCE

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ABSTRACT
This master's thesis is describing and applying parametric and nonparametric reliability models for censored data. It shows the implementation of reliability in the Six Sigma methodology. The methods are used in survival/reliability of real technical data.

KEYWORDS
Kaplan-Meier, Six Sigma, reliability, survival, Markov chain Monte Carlo, MCMC, Integrated Nested Laplace Approximation, INLA, Minitab, R

ABSTRAKT
Tato práce popisuje a aplikuje parametrické a neparametrické modely spolehlivosti na cenzorovaná data. Ukazuje implementaci spolehlivosti v metodologii Six Sigma. Metody jsou využity pro přežití/spolehlivost reálných technických dat.

KLÍČOVÁ SLOVA
Kaplan-Meier, Six Sigma, spolehlivost, přežití, Markov chain Monte Carlo, MCMC, Integrated Nested Laplace Approximation, INLA, Minitab, R

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DECLARATION

I declare that I have written my master's thesis on the theme of "Mathematical Models of Reliability in Technical Applications" independently, under the guidance of the master's thesis supervisor and using the technical literature and other sources of information which are all cited in the thesis and detailed in the list of literature at the end of the thesis.

As the author of the master's thesis I furthermore declare that, as regards the creation of this master's thesis, I have not infringed any copyright. In particular, I have not unlawfully encroached on anyone's personal and/or ownership rights and I am fully aware of the consequences in the case of breaking Regulation § 11 and the following of the Copyright Act No 121/2000 Sb. of the Czech Republic, and of the rights related to intellectual property right and changes in some Acts (Intellectual Property Act) and formulated in later regulations, inclusive of the possible consequences resulting from the provisions of Criminal Act No 40/2009 Sb. of the Czech Republic, Section 2, Head VI, Part 4.

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Brno 26.5.2017 . . . .

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CONTENTS

INTRODUCTION

This work will show will make the reader acquainted with the concept of reliability models. In the first part we will describe, how reliability data can be accessed by parametric and non-parametric methods. In the second part the gained knowledge will be applied on real world data, acquired from an industrial partner. The problematic of durability in statistics and production is important, because it helps to predict, improve and analyze data in order make reliable long lasting products.

The most used fields of reliability/survival models are factory production and medical research. In production we describe the time to failure of a component, whereas in medical research the focus is given on modeling the time to death of a ill patient or the time of healing from a certain disease. However counting the remaining time of a patent is a risky undertaking and take not in account the faith and determination to heal and live on. This work will analyze in the section 5 data from a real industrial process.

0.1 Survival Analysis

In survival analysis, we try to understand the reliability and durability of components via different methods. In this thesis the main attention is dedicated to the parametric and non-parametric methods. Through this methods we are trying to reveal the data's reliability characteristics. A typical hazard function of a component is the bathtub function, see figure 1.2. On the x-axes is the time and on the y-axes is the failure rate. After a higher failure rate at the beginning, called infant mortality or burn in phase, follows a steady-state operating time. On the end of the components lifetime comes finally the wear-out phase. The burn-in phase follows the Weibull distribution, the steady-state phase has commonly an exponential distribution, which is a special case of the Weibull distribution. The wear-out phase follows often the lognormal distribution. Examples of other bathtub functions are shown in figure 1.

0.2 Basic definitions

The following definitions are summarized in table 1.

Common technical definition of reliability

The probability that a system or a component will perform its intended task, under given operational conditions, for a specified time period.

Survival time (lifetime in medical research)

Time to occurrence of some *event of interest* for individuals in some population. The event may or may not be "death", and is often referred to as "failure" [7]. There are two states: "functioning" and "failed". Both can be clearly decided. We will consider only one way of transition. From "functioning" to "failed". The event of interest is random and can be described with statistic instruments.

Definition 1 (Component)**.** An element on which we are observing the time to failure.

Definition 2 (Time to failure)**.** A random variable X, which can have values $x \in \langle 0, \infty)$. This is a continuous random variable and represents the time between the beginning of the life (usage) of the component and the failure (death, event of interest).

Definition 3 (Distribution function $F(x)$)**.** The distribution function of the random variable X is $F(x) = P(X < x)$, for all $x \in (-\infty, \infty)$ It shows the probability that the time to failure is smaller than x. $F(x) = 0$ for all $x \in (-\infty, 0)$ [6].

Definition 4 (Survival function $S(x)$)**.** The survival function is also called reliability function: $S(x) = P(X \geq x)$

Definition 5 (Hazard function)**.** The hazard function $h(x)$ explains the probability of survival from time x into the next moment.

$$h(x) = \lim_{\Delta \to 0} \frac{P\left(x < X \leq x + \Delta x | X > x\right)}{\Delta x}$$

Theorem 1 (Change of variable formula)**.** *Let X be a continuous random variable with density $f_X(x)$. Consider the random variable $Y = g(X)$. Then the density of Y is*

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left|\frac{dg^{-1}(y)}{d(y)}\right|$$

Likelihood

Let's state that $\mathbf{x}$ is a random observation from a distribution with parameter $\boldsymbol{\theta}$. In the discrete case $P(\boldsymbol{x}|\boldsymbol{\theta})$ or $P_{\boldsymbol{\theta}}\mathbf{x}$ represents the probability of realizations $\mathbf{x}$ for a specific value $\boldsymbol{\theta}$ (from the distribution with specified parameter theta). We want to find the parameter $\boldsymbol{\theta}$ that fits the best the data. Thus we use the likelihood function 1 of the parameter $\boldsymbol{\theta}$ with fixed (conditioned on) observations $L(\boldsymbol{\theta}|\mathbf{x}) = P_{\boldsymbol{\theta}}(\mathbf{x})$ and maximize it. Similarly in the continuous case. We assume the terms $x_1, ..., x_n$ admitting a joint density of the random vector $\boldsymbol{x}$ are i.i.d. random variables. The

Functional characteristics	$f(t)$	$F(t)$	$S(t)$	$h(t)$
$f(t)$	$=$	$\frac{\mathrm{d}F(t)}{\mathrm{d}t}$	$-\frac{\mathrm{d}S(t)}{\mathrm{d}t}$	$h(t)\exp\left[-\int_0^t h(\tau)d\tau\right]$
$F(t)$	$\int_0^t f(\tau)d\tau$	$=$	$1-S(t)$	$1-\exp\left[-\int_0^t h(\tau)d\tau\right]$
$S(t)$	$1-\int_0^t f(\tau)d\tau$	$1-F(t)$	$=$	$\exp\left[-\int_0^t h(\tau)d\tau\right]$
$h(t)$	$\frac{f(t)}{1-\int_0^t f(\tau)d\tau}$	$\frac{\frac{\mathrm{d}F(t)}{\mathrm{d}t}}{1-F(t)}$	$\frac{-\frac{\mathrm{d}S(t)}{\mathrm{d}t}}{S(t)}$	$=$

Table 1: Overview of relationships [6]

joint density can be factorized using $\forall i \in \{1, ..., n\} : f_{x_i|\cdot}(x_i|\cdot) = f_{x_i}(x_i)$ with the chain rule, see [11].We define the likelihood function as stated in definition 6, using lemma 1.

If we want denote in the text that we are not considering the normalizing constant c (only the core is used) in the way $f(x) = c \cdot \tilde{f}$, we write $\tilde{f}$ and the likelihood function as $\pi(\boldsymbol{\theta}|\boldsymbol{x})$. We use also $\pi(\mathbf{x}|\boldsymbol{\theta}) = \tilde{f}_{\boldsymbol{\theta}}(\mathbf{x})$.

$$L(\boldsymbol{\theta}|\mathbf{x}) = f_{\boldsymbol{\theta}}(\mathbf{x}) \tag{1}$$

Definition 6 (Likelihood function)**.** The likelihood function of parameter value θ and realizations $\mathbf{x}$ is

$$L(\boldsymbol{\theta}|\mathbf{x}) = \prod_{i=1}^{n} f(x_i, \theta) \tag{2}$$

Lemma 1. Continuous random variables $X_1, ..., X_n$ admitting a joint density are independent from each other if and only if $f_{X_1,...,X_n}(x_1, ..., x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$

Anderson-Darling statistics

The Anderson-Darling statistics describes, how well data follow a proposed distribution. The smaller the statistic is, the better the distribution fits the data. For multiple censored times (censored at different times) the p-value can not be calculated. It is the squared distance between the plot points and the nonparametric step function. In addition, it is weighted more in the tails. The statistic is calculated in Minitab after equation 3. When we want to determine which distribution fits the

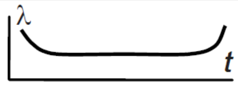
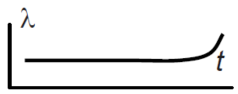
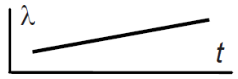
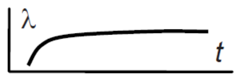
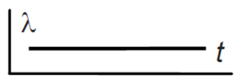
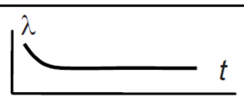
	Failure behaviour		General characteristics	Typical examples
wearout failures	A		• abnormal curve	• old steam engine (late 18th to early 19th century)
	B		• simple devices • complex machines with bad design (one single dominating type of failure)	• car water pump • shoelace • 1974 Vega engine
	C		• structures • wearout element	• car bodies • airplane and automobile tires
random failures	D		• complex machines with high-stress trials after start of operation	• high pressure relief valves
	E		• well designed complex machines	• gyro compass • multiple sealing high pressure centrifugal pump
	F		• electronic components • complex components after corrective maintenance	• computer “mother boards” • programmable controls

Figure 1: Different bathtub functions [1]

data the best, we choose the one with the lowest Anderson-Darling statistic. If the statistics don’t differ very much, as in our case in chapter 5, we look on additional criteria such as probability plots.

$$AD^* = n \sum_{i=1}^{n+1} (A_i + B_i + C_i) \tag{3}$$

where

$$n = \text{number of plotted points}$$
$$A_i = -Z_i - \ln(1 - Z_i) + Z_{i-1} + \ln(1 - Z_{i-1})$$
$$B_i = 2\ln(1 - Z_i)F_n(Z_{i-1})2\ln(1 - Z_{i-1})F_n(Z_{i-1})$$
$$C_i = \ln(Z_i)F_n(Z_{i-1})^2 - \ln(1 - Z_i)F_n(Z_{i-1})^2 - \ln(Z_{i-1})F_n(Z_{i-1})^2 + \ln(1 - Z_{i-1})F_n(Z_{i-1})^2$$
$$Z_i = \text{fitted estimate of the cumulative distribution function in the i}^{th}\text{ data point}$$
$$F_n(Z_i) = \text{i}^{th}\text{ data point}$$
$$Z_0 = F_n(Z_0) = \ln(Z_0) = 0$$
$$Z_{n+1} = 1 - (1\text{E-}12)$$

0.2.1 Censoring

The studies usually end earlier before by all subjects can occur the event of interest. A situation when the time from the beginning to the event of interest is not known, is called *censoring*. A situation when incomplete information is available occur often in practice. From censored data is obtained partial information.

We distinguish between left-censoring, right-censoring and interval censoring. We can speak about **left-censoring** by e.g. patients with detected cancer. In figure 2, situation A. We don't know when the disease exactly started. An example for **right-censoring** could be, when we are modeling the life-time of cars. Some cars have got an accident and can't serve to their usual life expectancy as in figure 2, situation C. **Interval censoring** occur if investigate the population of birds and in the winter times we lose track of them, because they fly South. Figure 2, situation B. In the technical praxis this situation equals to failure findings by regular car service checks.

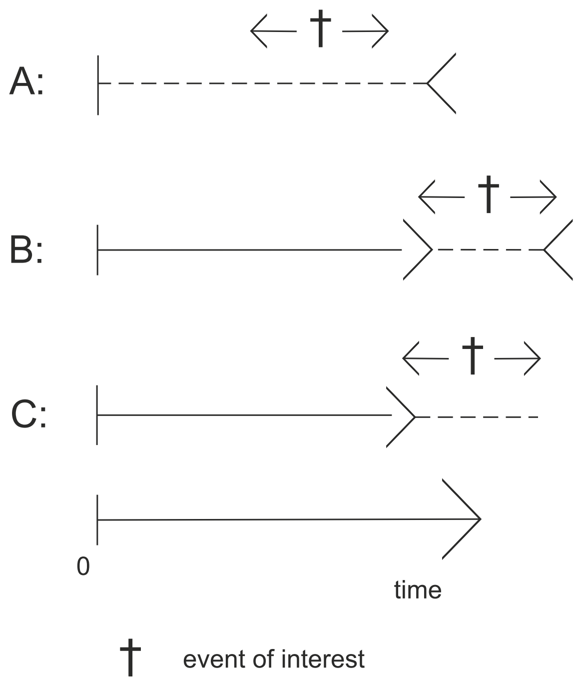


Figure 2: Censoring illustration

1 DISTRIBUTIONS

1.1 Exponential distribution

$$f(x) = \lambda e^{-\lambda x} \tag{1.1}$$

The pdf is described by formula 1.1. Sometimes instead of the rate parameter λ in $Exp(\lambda)$ is used $Exp(1/\lambda)$, where $1/\lambda$ is the mean time between events. For simulation purposes, it is possible to transform samples from a uniform distribution into an exponential by the transformation: $X = -(1/\lambda)ln(1-U)$, where $u \sim \text{Unif}(0,1)$.

The exponential distribution has a significant contribution to reliability analysis and forms the basis of reliability methods , which are introduced and formalized by it and them formed into more advanced methods. However the use of the exponential distribution reveals often inappropriate. Most real world data doesn't have a constant failure rate as e.g. the human mortality rate (see [1].) Despite this fact, the exponential distribution is still widely used in today's approaches and standards, [24].

1.2 Weibull distribution

The exponential distribution is a special case of the Weibull distribution for $\beta = \eta = 1$. There are more possibilities, how to write the pdf. The most used form for the Weibull distribution is:

$$f(x) = \frac{\beta}{\lambda}\left(\frac{x-\gamma^{\beta-1}}{\lambda}\right)\exp^{-\left(\frac{x-\gamma}{\lambda}\right)^{\beta}} \tag{1.2}$$

where $x \geq 0$ is the time, the shape parameter (1/slope) $\beta > 0$, the scale parameter $\lambda > 0$ and the location parameter $-\infty < \gamma < \infty$. The location parameter is frequently not used. For analysis of small data sets we use the one-parameter Weibull distribution, where the parameter β has to be estimated. The estimation of β is recommended to be a good and justifiable estimate before use. The one-parameter Weibull distribution allows analysis of small data sets [26]. The bathtub function in figure 1.2 illustrates a mixed failure rate of the Weibull distribution with $\beta < 1$ in the infant mortality time, $\beta = 1$ in the random failures time and $\beta > 1$ in the wearout failures time. By the Weibull distribution the cumulative distribution function is known, equation 1.3. Another characteristics are the survival function $(1 - F(t))$ and hazard function, see equation 1.4. On figure 1.3 the probability

[1] `https://www.science-of-aging.com/timelines/gompertz-aging-human-mortality.php`

density function with varying scale parameter and on figure 1.4 with varying shape parameter. Since the problematic in section 5 is dealing with quantities. We are for better understanding of the later part denoting the x-axes with x instead of t.

$$F(x;\beta,\lambda) = 1 - e^{-(x/\lambda)^{\beta}} \tag{1.3}$$

$$h(x;\beta,\lambda) = \frac{\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1} \tag{1.4}$$

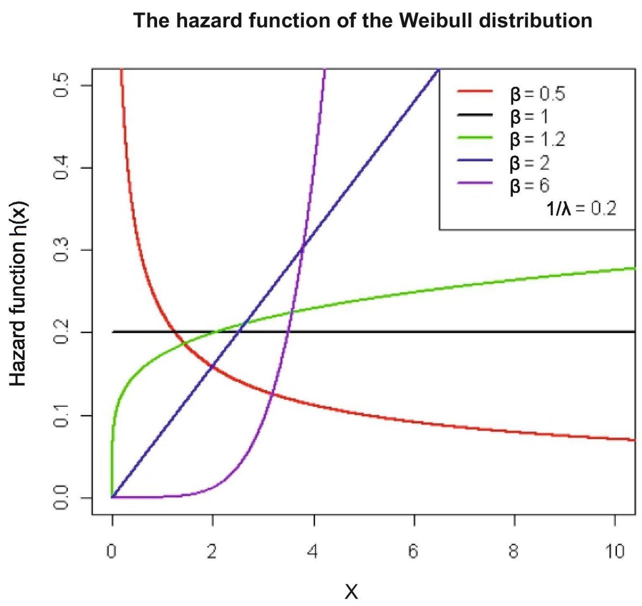


Figure 1.1: The risk function of the Weibull distribution

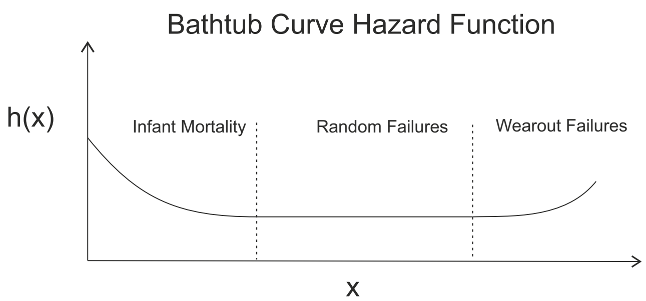


Figure 1.2: Usual bathtub function.

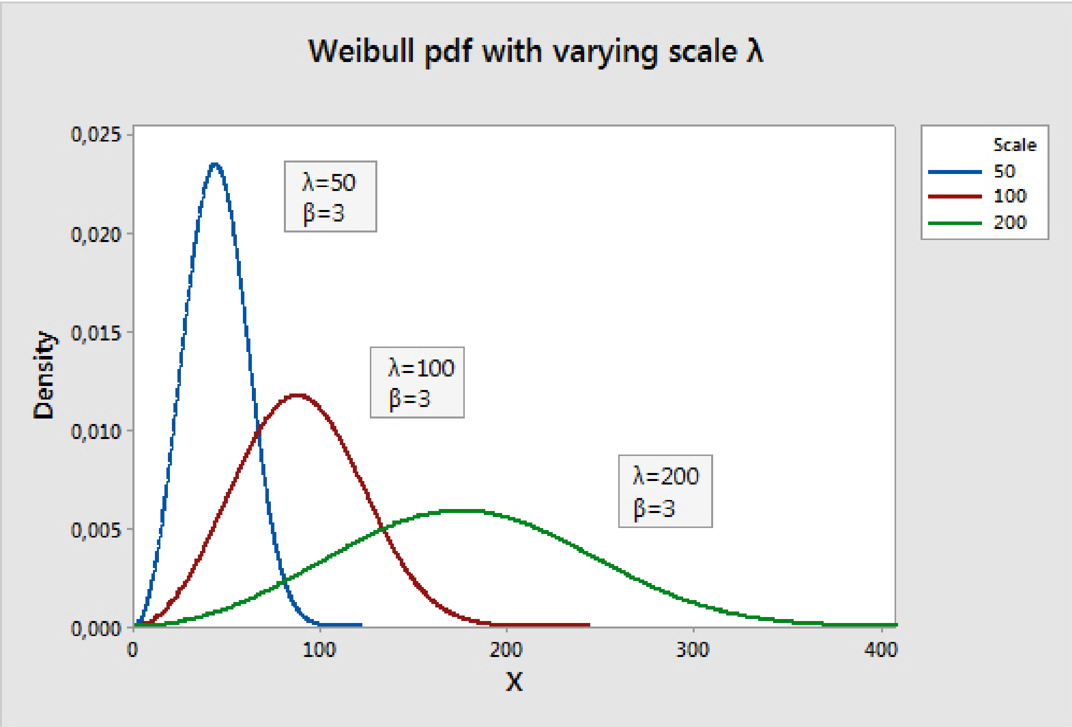


Figure 1.3: The Weibull density with varying scale parameter

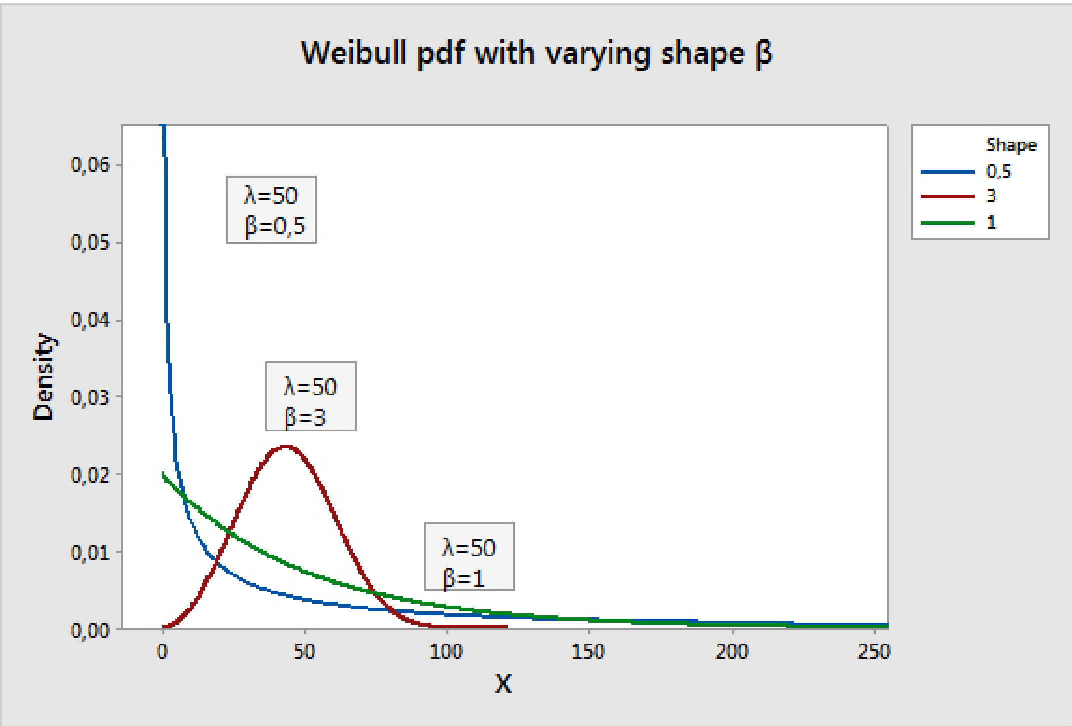


Figure 1.4: The Weibull density with varying shape parameter

2 SIX SIGMA

Six sigma is a management system for improving processes and controlling quality. The objective of the Six Sigma methodology is to have 3.4 defects on 10^6 products. This comes from an effort that a process (with a Gaussian error, figure 2.3) should stay within 6 standard deviations from both sides, 6σ on the left and 6σ on the right hand side. From the short-term view, an empirically based 1.5 σ shift is introduced. We can see it as a short term process bias. With this shift, the process will have 3.4 on 10^6 products outside the limits as on figure 2.2. The effort is not only to minimize the defects, but also the costs related to them. Six Sigma provides increases of profit and reduction of costs, as described here [17]. The methodology is based on understanding the needs and expectations of the customer, collection of good data which provides good information for statistical analysis. This provides a helpful tool in manufacturing, business, logistic and other fields. A clear commitment is given to make decisions based on verifiable data and statistical methods. The goals could be summarized as:

- maximization of profit
- increase of productivity and decrease of variation in the process
- effectively use of sources
- monitoring and controlling processes
- minimization of defects and prevention of their generation

Six Sigma is composed of 3 main areas, as illustrated on the graphic 2.1.

2.1 DMAIC

Improvement of existing products is done in 5 steps - **D**efine, **M**easure, **A**nalyze, **I**mprove, **C**ontrol. This set is abbreviated **DMAIC**. The process flow visualized in the schema 2.4. In the define phase, we try at first to set up all possible causes that could have impact on the process. We ask ourselves clauses, which start with a "w". What, who, why, how much and till when? For this purposes might also be useful a *Ishikawa diagram* or *flow chart*, called also a cause-and-effect diagram. Let's see such a diagram on figure 2.5. The main terms are placed at the tips of the branches. Related terms are attached to them in order to form a hierarchical structure. In reliability a similar concept is made by *FTA* (Fault Tree Analysis), explained in subsection FTA 2.2. Software as CAFTA 5.3 is used. In the measure phase we obtain data. Usually is done so by selecting $1-3$ dependent variables and $10-15$ independent variables (can be passively known about $30-50$ that are not measured) which shrink in the analyze phase to $2-3$. The analyze phase focuses also of finding the functional dependencies $Y = f(x)$ and identifies the causes of the

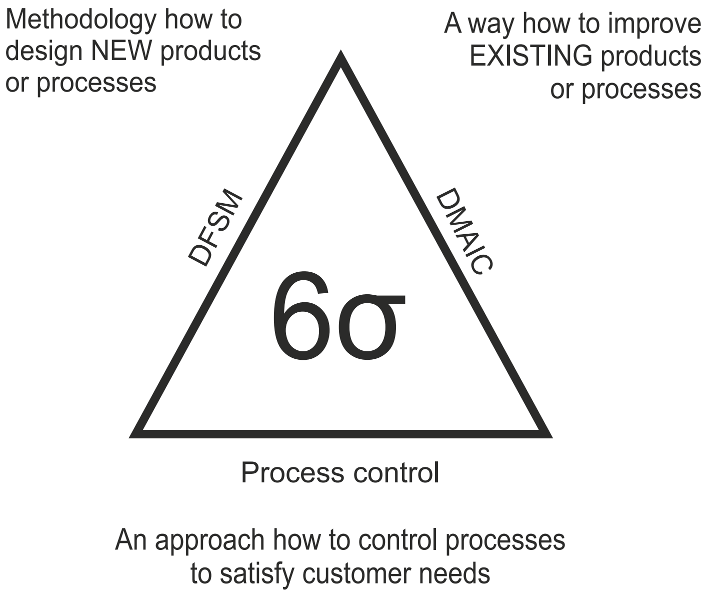


Figure 2.1: 3 main areas of Six Sigma

process [14]. In the phase analysis, we are accessing specific durability data with parametric and non-parametric methods.

2.2 Key roles

We rank the level of expertise in Six Sigma. For this purpose is used the raking from martial arts. Experts in Six Sigma are given belts. Starting with the Yellow Belt, continuing with the Green and Black Best, to the Master Black Belt. Over the Master Black Belt is the Champion, see [22], [17].

Yellow Belt

Doesn't need to have any previous experience or education. He has got a small role with the need of developing only foundational knowledge. He understands in a limited context how to apply, implement, perform and interpret Six Sigma. He understands the PDCA (Plan, Do, Check, Act) methodology.

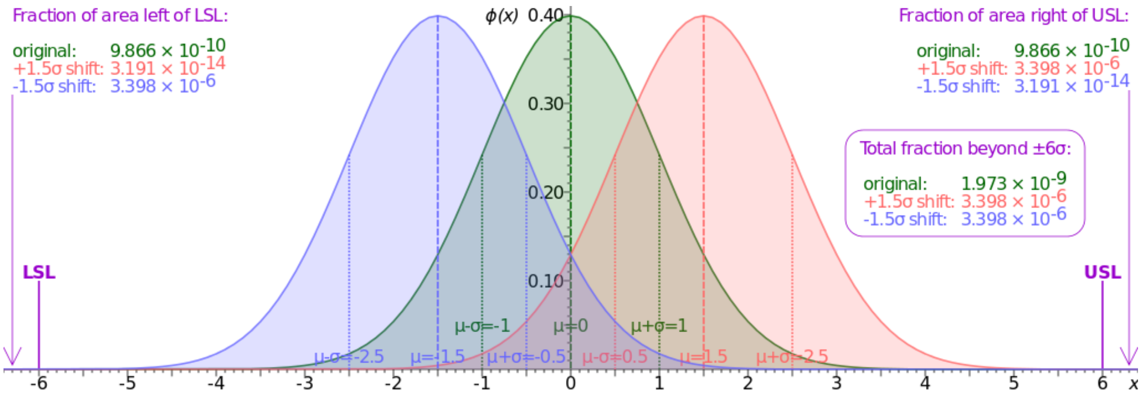


Figure 2.2: Graphical illustration of the role of the 1.5 σ shift

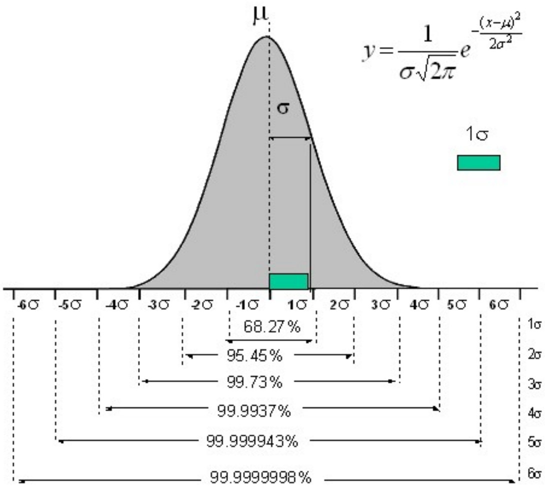


Figure 2.3: The normal distribution

Green Belt

A green belt operates under the guidance of a black belt. He should have at least 3 years of work experience. He spends 20-30% with Six Sigma projects. He is involved in quality improvement projects, but has no led. The Green Belt has an overview of Six Sigma and the DMAIC Methodology. He understands the impact of Six Sigma in the organization and can see the financial benefits of it.

Black Belt

The Black Belt spends 100% of his time with Six Sigma projects. He is the leader of the projects. He manages the team and organizes the Six Sigma projects. The Black Belt also teaches and trains project teams. His knowledge in Six Sigma is well-

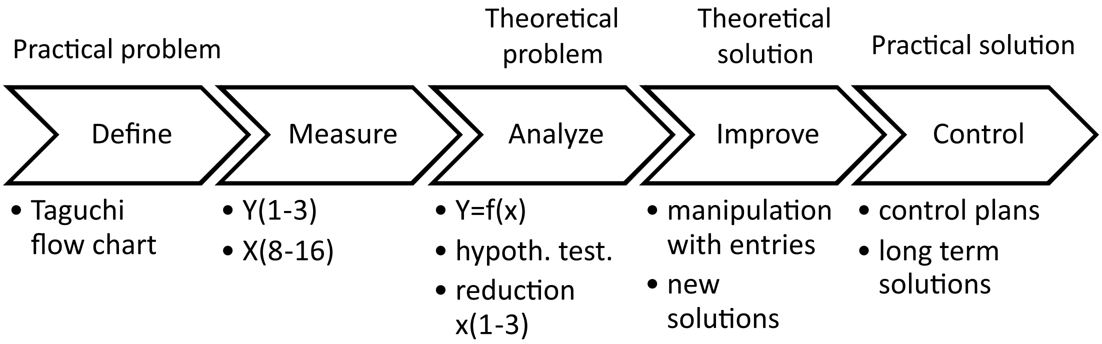


Figure 2.4: DMAIC

grounded. He identifies possible troubles in the project, whereas the Champion and Master Black Belt identify the projects. He understands all aspects of the DMAIC model. To become a Black Belt, 3 years of work experience are needed with one completed project or 2 completed Six Sigma projects.

Master Black Belt

The Master Black Belt is chosen by the champion. He has an overview about the company's goals and strategies. He is implementing Six Sigma across various functions and departments. He is networking with other Master Black Belts. He is required to know about advanced Six Sigma and have grounded knowledge in topics such as DFSS, Lean, Integration of initiatives, Cross-cultural project leadership, Strategic project selection and performance management. He has an overview about the project situation. He must at completed at least 10 Six Sigma projects or 5 years work experience as a Black Belt.

Champion

This is a position chosen from the upper management. He is responsible for the methodology Six Sigma in the company. He mentors the Black Belts. The role of a champion is to remove roadblocks. He is the intermediate piece between the Black Belts and the management. A Champion is supposed to have diplomatic skills as well as to be proficient in: Business and operations interface, Project selection, Pace mediation, Results implementation.

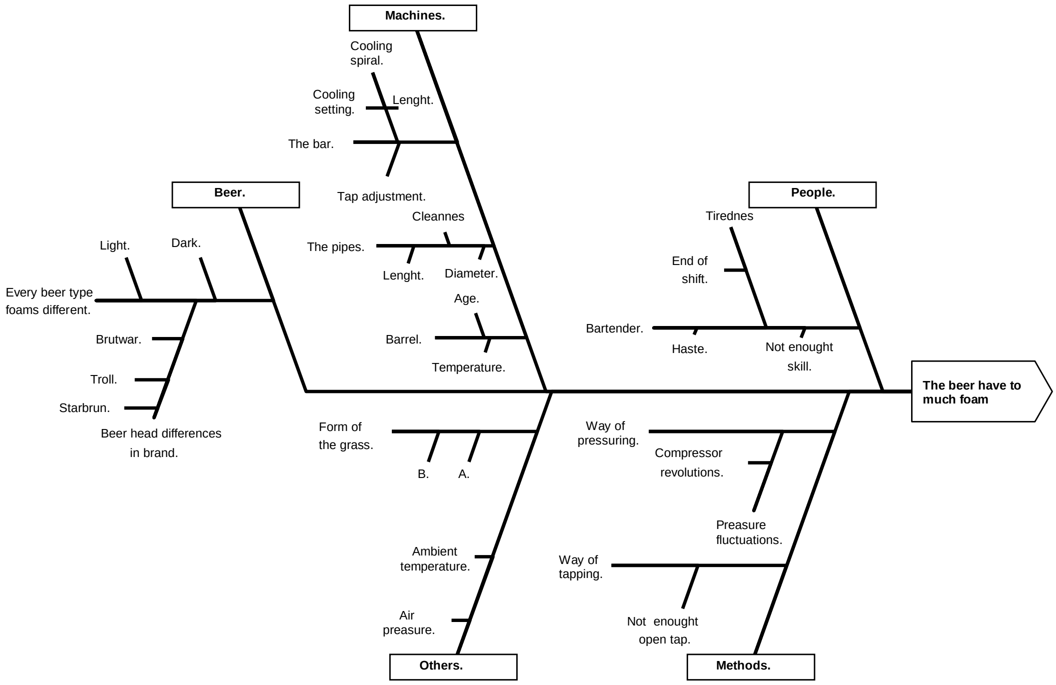


Figure 2.5: Ishikawa diagram

FTA & FMEA

FMEA (**F**aulure **M**ode and **E**ffect **A**nalysis) and FTA (**F**ault **T**ree **A**nalysis) are methods. These methods are used for evaluating risk and reliability in systems. Through, we are in this thesis mainly on reliability (durability) a single of a single component, we can make us an image, how it look when the components are put together. These methods are used extensively in the aviation industry. It find its place as well in production. Outside FTA and FMEA are many other variations of these methods and different approaches like FMECA (Failure Mode, Effect and Critical Analysis), RPN (Risk Priority Number). The main differences between FTA and FMEA is that in FTA we examine different combinations components and conditions, which lead to a single effect, whereas FMEA examines all single components and lists by every single one its range of effects. FMEA might be used more in processes and FTA is suitable in more difficult complex systems. Similar to these methods can be applied in urban planning the cross-impact analysis. With this method we predict the development of events in relationships with the surrounding factors. The relationships are visualized in a cross-impact matrix. On its basis the final development scenario is compiled. More can be found in [15].

FTA

FTA is a graphical representation analysis in risk management. It is constructed of *gates* and *events*. The main used events are a basic event, external event, undeveloped event and conditioning event [21], denoted in figure 2.6.

- basic event - it is the most used event, a failure or error in the system
- external event - an event that is expected to occur and is by itself not a fault
- undeveloped event - event with insufficient information
- conditioning event - a condition that has effect on the logic gate

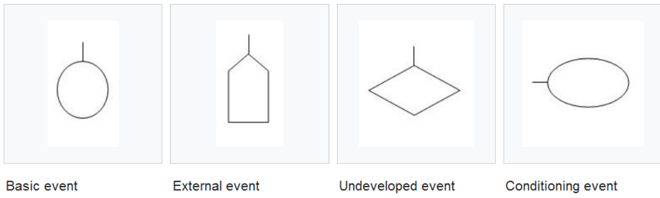


Figure 2.6: FTA events

Now, let's turn our attention at the gates. They characterize the relation between the input and output events. The main gates are the AND and OR gates, see 2.7. But other gates as an e.g. Exclusive OR gate or Priority AND gate are also used.

- AND gate - iff all events occur
- OR gate - at least one event occurs
- Priority AND gate - all events occur in a specified sequence
- Exclusive OR gate - exactly one event occur

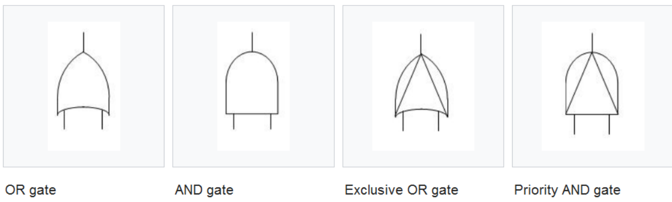


Figure 2.7: FTA gates

FMEA

There are different types of FMEA as functional, design, process, control of PFME. We can also extend FMEA to FMECA. FMEA looks at the distinct components of the system and analysis their different failure modes. By a valve for example can be the failure modes failure in opening the valve and failure in closing the valve.

Example

As an example, we take the electric circuit. Figure 2.8 is part of a seminar presentation to shown the FMEA and FTA methodologies.

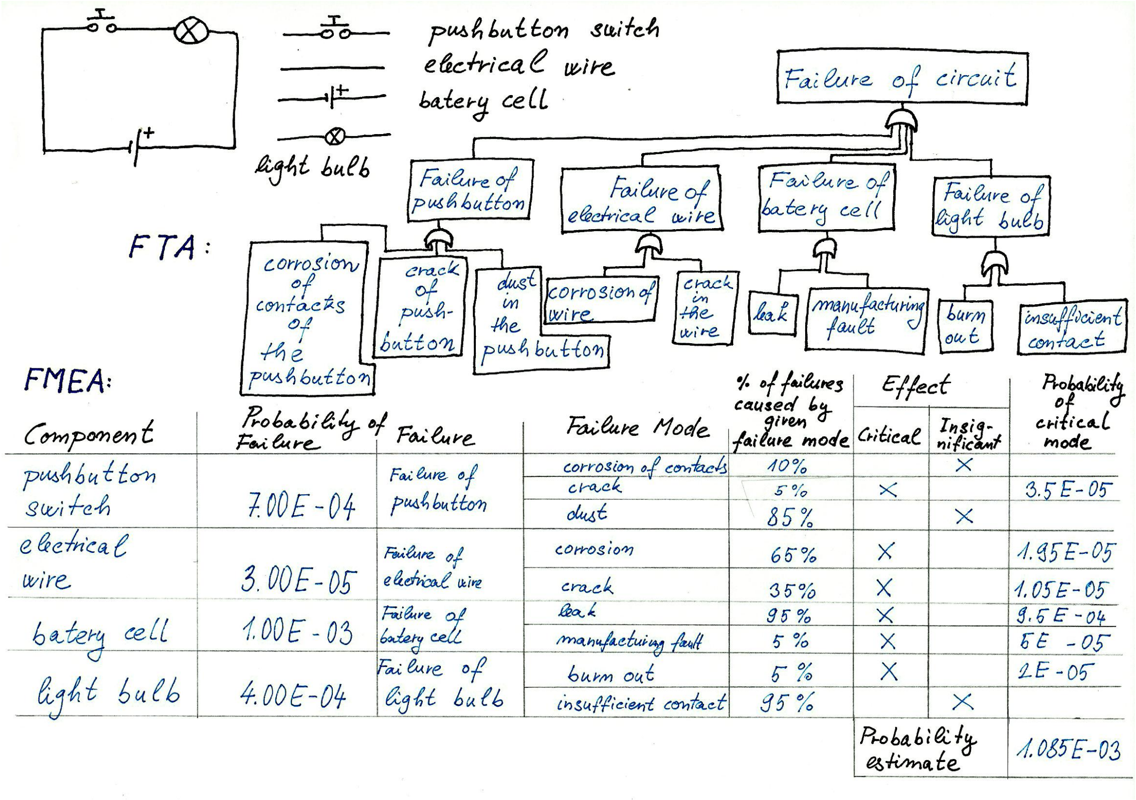


Figure 2.8: circuit example

2.3 DFSS & Design for Reliability

DFSS

Design for Six Sigma (DFSS) is a business-process management method related to Six Sigma [5]. Contrary to the classical Six Sigma DMAIC approach, it doesn't focus on improving existing processes, but the designing of products. Whereas Six Sigma used the DMAIC logic, Design for Six Sigma is familiar with DMADV (**D**efine, **M**easure, **A**nalyze, **V**erify) or IDOV (**I**dentify, **D**esign, **O**ptimize, **V**erify) logic. DFSS doesn't have an underlying process to work with. It generates a new one, o replaces the old one.

DfR

Design for Reliability (DfR) is a set of tools that support process and product design [19]. It encompasses all stages from early design stage to the aging stage. It gives a tutorial throughout the design cycle where to use which tools and what to be careful about for achieving reliability. Understanding the reliability form different aspects and seeing particular risks becomes even more important in complex systems. However, in this thesis our focus is given more on simple cases.

Comparison: DFSS & DfR

Whereas DFSS (focused on quality) have the objective that the product will basically work, DfR (focused on reliability) is interested in how long will the product work under specified conditions. Both approaches try avoiding defects. DFSS looks that the product doesn't have defects and the reliability is low, whereas DfR tries to have a reliable product on the long-term. Both methodologies have their similarities and differences, as we can see on figure 2.9. On figure 2.10 we can se the illustration of *the rule of ten*. The cost of a failure is ten times lower, when it is detected in the previous stage. Therefore it is a good investment to focus on reliability and early failure detection.

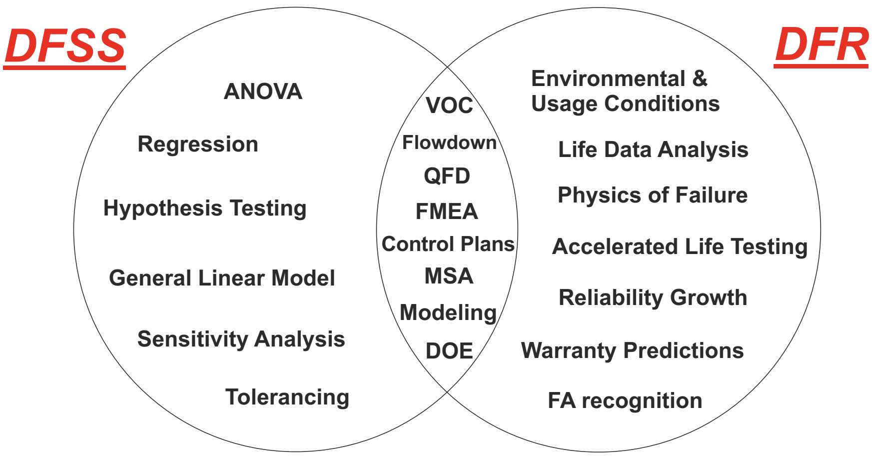


Figure 2.9: Tools used in DFSS and DFR [19]

2.4 Accelerated Lifetime Tests

The reliability tests are used not to seek product weaknesses, but also to demonstrate improvement and the ability to meet the demands of the customer. There are a

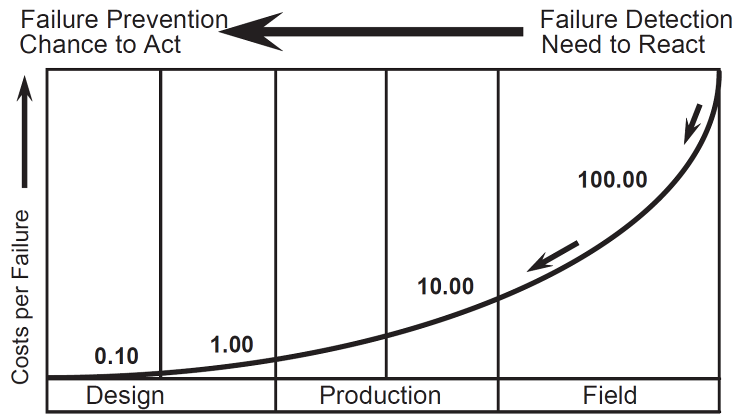


Figure 2.10: Failure prevention rule of 10

lot of testing techniques, see [5]. Let's look at the more important ones. On the assumption on physically founded models, we deal in this section the lifetime under operational conditions, can be by data obtained under high load levels.

Time-Acceleration Factor

If the Wöhler curve is linear, we model the relationship between the normal operation conditions and the lifetime in an accelerated test by the time-acceleration factor AF. Let's have 6 elastic bands. We examine, whether after stretching them a certain length they return into the starting position. In the table 2.1 we have the entries for how many times (cycles) they were able to stretch, before failing. We have 3 different levels of stretching length, i.e. 3 stress levels.

$$AF = \frac{t}{t_{acc}} \tag{2.1}$$

We can see that the acceleration factor for 36 cm with respect to 9 cm is 13.

No.	9 cm	18 cm	36 cm
1	58	16	4
2	63	17	5
3	65	18	5
4	72	21	5.5
5	78	22	6
6	86	23	6.5

Table 2.1: Sample of survival data from [1]

$$AF_{36cm} = \frac{t_{9cm}}{t_{36cm}} = \frac{74.85}{5.72} = 13$$

Using some similar data, we might continue in our thoughts that the factor is in a relationship with the stress level. This describes the Wöhler curve. It connects means of the failure distributions for different stress parameters. I.e., for every stress level, there is are different parameters, mostly of the Weibull distribution. The figure 2.11 illustrates a failure rate of a component under different pressure. The higher the pressure, the shorter the failure time. This concept has got practical use. Let's imagine an experiment would last under normal conditions year. Under increased stress conditions the components fail within days. With the Wöhler curve (accelerated factor), we estimate how would the experiment behave under normal stress conditions

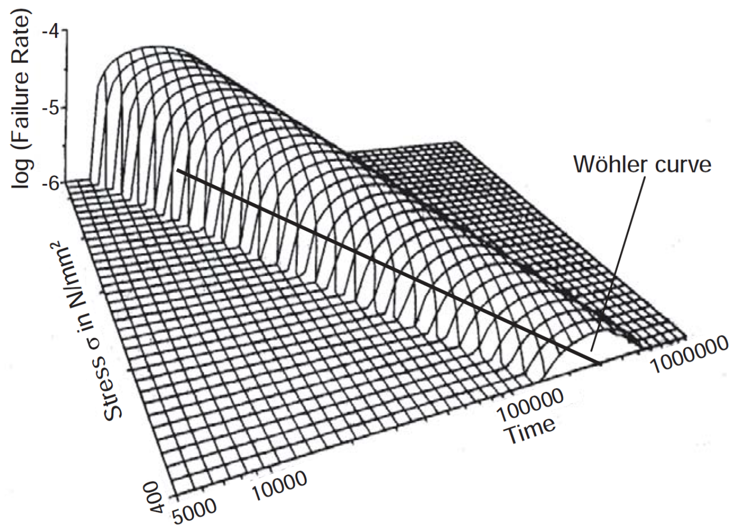


Figure 2.11: Ilustration of the Wöhler curve, retrived from [1].

Accelerated Life Testing (ALT)

This test is described in the situation shown in figure 2.11. An example might be letting the inert gas in the light bulb. A physical relationship describes the accelerated factor. It is a test to proof that the product meets the customer demands in a reasonable time period. Using a relationship between the accelerated and normal conditions, we can conduct tests in a reasonable time period.

Highly Accelerated Life Testing (HALT)

A reliability test used to find failure modes during the product development phase of the design cycle. Contrary to the Accelerated Life Testing, it uses 2 or more stress factors (e.g. temperature and vibration) in order to find failure modes as quickly as possible. A failure mode is a kind of manner that causes the component to fail (or work properly), see [16].

Environmental Stress Screening (ESS)

A stress test that uses stress conditions that are common in the customer environment. Basically, no special improvements are implemented. The objective of ESS is to accelerate early failures such that repair is accomplished at the most cost-effective stage, see [16] and [16].

Highly Accelerated Stress Screening (HASS)

HASS is focused on improving the products infant mortality. It uses higher stress levels as ESS, but not as aggressive as HAST.

Highly Accelerated Stress Testing (HAST)

An aggressive technique that uses an additional stress factor in order to highlight the remaining ones. For example if we cant to examine condensation on a device, which is dependent on temperature and humidity in normal conditions, we introduce the stress condition pressure to shorten the testing as explains [5].

Step-Stress Testing (SST)

A test designed to expose the failure distribution under stress condition. E.g. vibration in a car. The components are in a short time examined for failures and the effect of stress on them. The stress level is throughout the test constantly step-wise increasing and the impact in observed.

3 PARAMETRIC METHODS

The parametric methods are based on the assumption that the data follow a known distribution. The goal is to find that distribution and estimate it's parameters.

3.1 Life data analysis

The basic concept is the life data analysis (commonly refered as *Weibull analysis*, see [23]). While looking for the optimal distribution, we examine probability plots and the Anderson-Darling statistic, subsection 0.2. We are looking for a distribution that expresses the best the nature of the collected data. This analysis requires to:

- gather life data
- select a lifetime distribution
- fit the distribution to the data by estimating parameters
- generate plots and results to express the life characteristics of the product

Let's look a the example at the end of this chapter, A reliability example with Minitab in section 3.4. We are interested in at which time half.

A more specific approach is to interfere the data with a bayesian hierarchical model. For this topic are dedicated the following sections INLA and MCMC.

3.2 INLA

Approximate simulation free Bayesian inference using integrated nested Laplace approximations. The technique is commonly used on spacial or latent Gaussian models, see [2]. It's application is also in reliability.

3.2.1 The INLA idea

We take the posterior distribution

$$\pi(\mathbf{x}, \boldsymbol{\theta}|\mathbf{y}) \propto \pi(\mathbf{y}|\mathbf{x}, \boldsymbol{\theta})\pi(\mathbf{x}|\boldsymbol{\theta})\pi(\boldsymbol{\theta}) \tag{3.1}$$

and approximate the posterior marginals

$$\pi(x_i|\mathbf{y}) \quad \text{and} \quad \pi(\theta_i|\mathbf{y}) \tag{3.2}$$

directly.

Stage 1: The data

The data are represented in the relationship 3.1 by the likelihood $\pi(\mathbf{y}|\mathbf{x}, \boldsymbol{\theta})$. It is the highest part of the hierarchical model. They are generated from the underlying components $\mathbf{x}$ and the hyperparameters $\boldsymbol{\theta}$. The main usage of INLA is in data with Gaussian responses (temperature, people infected with a disease in each area). It can be also good used in point data (air pollution measured at fix stations), binary data (binary image) or as in our example in survival data.

Stage 2: The latent model

The latent model is build up by latent (unobserved) components $\mathbf{x}$. They can represent structured random effects (AR(1), regional effects), unstructured random effects (individual effects, group effects), in general covariates (predictor variable). The latent components are linked to the likelihood through linear predictors.

Stage 3: Hyperparamether

The crucial part of Bayesian inference is to set a good prior distribution. The hyperparamether is a term in the prior distribution. They are usually the precisions of the covariate, spacial or unstructured effect. Each hyperparamether must be given a prior.

GMRF

A random vector $x = (x_1, ..., x_n)^t$ is called a GMRF (Gaussian Markov Random Field)with respect to a labeled graph G=(V,ϵ) with mean vector μ and precision matrix (inverse covariance matrix) $Q > 0$, if it's density has the form

$$\pi(x) = (2\pi)^{-n/2}|Q|^{1/2}exp(-1/2(x-\mu)^t Q(x-\mu))$$

and $Q \neq 0 <=> i, j, \in \epsilon$ for all $i \neq j$.

Note: Any normal distribution with a symmetric positive definite covariance matrix is also a GMRF and vice versa.

3.3 Markov Chain Monte Carlo

The Markow Chain Monte Carlo Method is a strategy for drawing samples from the target density f. In Bayesian analysis, the posterior moments can be written in an integral form, but practically cannot be analytically evaluated. We deal with this problem so that we simulate random draws $\boldsymbol{X}_1, ..., \boldsymbol{X}_n$ from the target distribution. On this samples, we perform *Monte Carlo Integration*. This is the most frequent approach of processing the samples. This a statistical estimation of the integral by the evaluation of the integral in random draws, which are drawn from the distribution and include the hole range on integration, see [9]. Let's say for example, we want to calculate $E\{h(\boldsymbol{X})\}$. Having drawn $\boldsymbol{X}_1, ..., \boldsymbol{X}_n$ i.i.d.(independent and identically distributed) random samples from f, we use Monte Carlo integration in formula 3.3 to evaluate the integral. We are going to discuss a specific strategy of Monte Carlo Integration called Markov Chain Monte Carlo.

$$\hat{\mu}_{MC} = \frac{1}{n}\sum_{i=1}^{n} h(\boldsymbol{X_i}) \to \int h(\boldsymbol{x})f(\boldsymbol{x})d\boldsymbol{x} = \mu \tag{3.3}$$

A Markov chain is a discrete-time stochastic process $\{X_i\}_{i=0}^{\infty}$, $X_i \in S$, where given the present state, past and future states are independent (Markov assumption 3.4) [12]. Note that a stochastic process is a collection of random variables $\boldsymbol{X} = \{X_t : t \in T\}$. We simulate a Markov chain $X_1, ..., X_n$ in a way that it converges to the target posterior distribution f. After convergence, we can estimate the posterior properties. The samples are typically dependent. The central algorithms we will use for this purpose is the Gibbs sampling algorithm and the Metropolis-Hastings algorithm.

$$P(X_{i+1} = x_{i+1}|X_0 = x_0, X_1 = x_1, ..., X_i = x_i) = P(X_{i+1} = x_{i+1}|X_i = x_i) \tag{3.4}$$

3.3.1 Metropolis algorithm

Because of absence of strong prior information in the problem discussed in chapter 5, we use the Metropolis algorithm 1. The efficiancy of the algorithm depends on the relative frequency of acceptance. A too large acceptance rate yealds a slow exploration of the target density. A too small acceptance rate causes large proposed moves, but rarely accepted. For independence proposals a hight acceptance rate is desired, which indicates that the proposal density is close to the target density. For random walk proposals, an acceptance rate between 20% and 50% is recommended. It is achieved by changing the variance of the proposal density, as stated in [12]. In the acceptance probability (see code line 6) $\min\left(1, \frac{\pi(x^*)}{\pi(x_{i-1})}\right)$ the fraction is not explicitly multiplied by the proposal ratio $\frac{Q(x_{i-1}|x^*)}{Q(x^*|x_{i-1})}$ since it is symmetric

Algorithm 1 Metropolis algorithm

procedure METROPOLIS($N, x0, Q(x), \pi(x)$) ▷ Initial value $x0 \sim \pi(x_0)$
  $x_0 \leftarrow x0$
  for $i = 1, 2, ..., N$ **do**
    Generate a proposal $x^* \sim Q(x^*|x_{i-1})$
    $u \sim \mathrm{U}(0,1)$
    if $u < \min\left(1, \frac{\pi(x^*)}{\pi(x_{i-1})}\right)$ **then** ▷ If less than the accep. prob. α
      $x_i \leftarrow x^*$
    else
      $x_i \leftarrow x_{i-1}$
    end if
  end for

$Q(x_{i-1}|x^*) = Q(x^*|x_{i-1})$. This is the difference to the Metropolis-Hastings algorithm.

3.3.2 Gibbs sampling algorithm

The Gibbs sampler is a Markov chain who samples univariate conditional distributions. The sampled stationary distribution becomes the target distribution f, see [3]. We sample repeatedly according to formula 3.5. $|\cdot$ denotes conditioning on the most recent updates. As an example, a Gibbs sampler for the normal distribution 3.6 in figure 3.1.

$$\begin{aligned}
x_1^{(i+1)}|\cdot &\sim \pi(x_1|x_1^{(i)}, \cdots, x_n^{(i)}) \\
x_2^{(i+1)}|\cdot &\sim \pi(x_2|x_1^{(i+1)}, x_3^{(i)}, \cdots, x_n^{(i)}) \\
&\vdots \\
x_{n-1}^{(i+1)}|\cdot &\sim \pi(x_n|x_1^{(i+1)}, x_2^{(i+1)}, \cdots, x_{n-2}^{(i+1)}, x_n^{(i+1)}) \\
x_n^{(i+1)}|\cdot &\sim \pi(x_n|x_1^{(i+1)}, \cdots, x_{n-1}^{(i+1)})
\end{aligned} \tag{3.5}$$

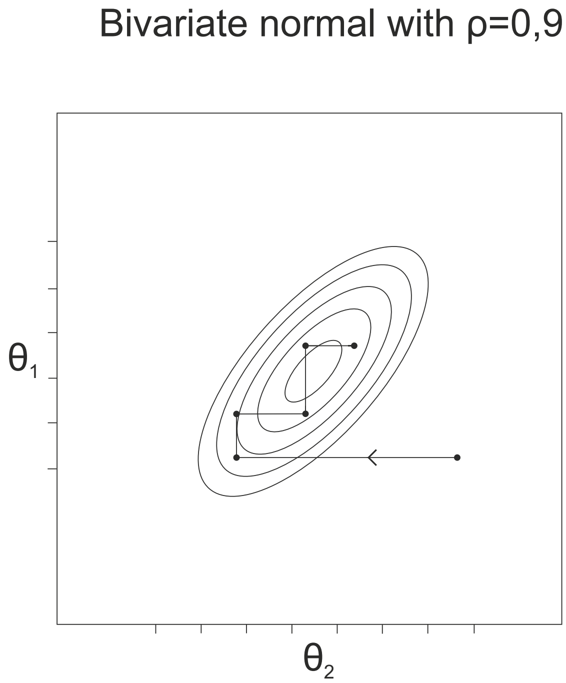


Figure 3.1: Gibbs sampler for normal distribution

$$
\begin{aligned}
\theta \sim & N_2\left(0, \Sigma\right), \Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \\
\theta_1|\theta_2 \sim & \; N\left(\rho\theta_2, \left[1-\rho^2\right]\right) \\
\theta_2|\theta_1 \sim & \; N\left(\rho\theta_1, \left[1-\rho^2\right]\right)
\end{aligned}
\tag{3.6}
$$

3.4 A reliability example with Minitab

Let's have data of engine windings in an electro motor. Sometimes a short-circuit happens and the winding fails. We use the under two temperature, 80 °C (50 observations) and 100 °C (40 observations). Let's be interested in:

- at which times fails 10% of the windings
- whether the survival curves differ

The rule of thumb says: *do not to trust the data.* Can we trust them, who did collect them. Did something change during the collection process? Which units are we using? Can we see a typing error? In our first ideas, we shouldn't thing over, which method we are going to use. This we can do always. On the first place, we should be cautious with the data.

We have a first sight at the right censored data on the Dotplot 5.1.

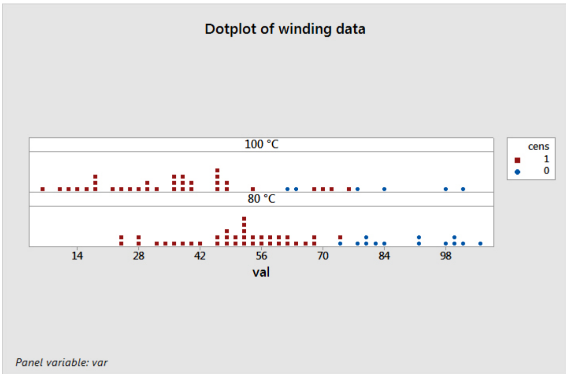


Figure 3.2: Dotplot

The Parametric Method

In the next stage we evaluate which distribution the data follow. We create for this purpose a distribution ID plot 3.4. In this case the lognormal distribution wins. We see that the data follow good the plot and in both cases the Adreson-Darling statistic in listings 3.1 and 3.2 is the lowest. With a look on the probability plots 3.3, we clearly select the lognormal distribution. Having selected the distribution, we have a closer look on the data under they are lognormaly distributed. Figure 3.4 provides us with visual information about the general and survival properties.

To determine, when 10% of the items will fail, we look at the table tables of percentiles in listing 3.3 and for 10%, we estimate the time to be at 32,1225 for 80 °C windings and 14,7606 for 100 °C windings. We reject the null hypothesis that the distributions are the same according to the statistic in listing 3.5 on the significance lever of 5%.

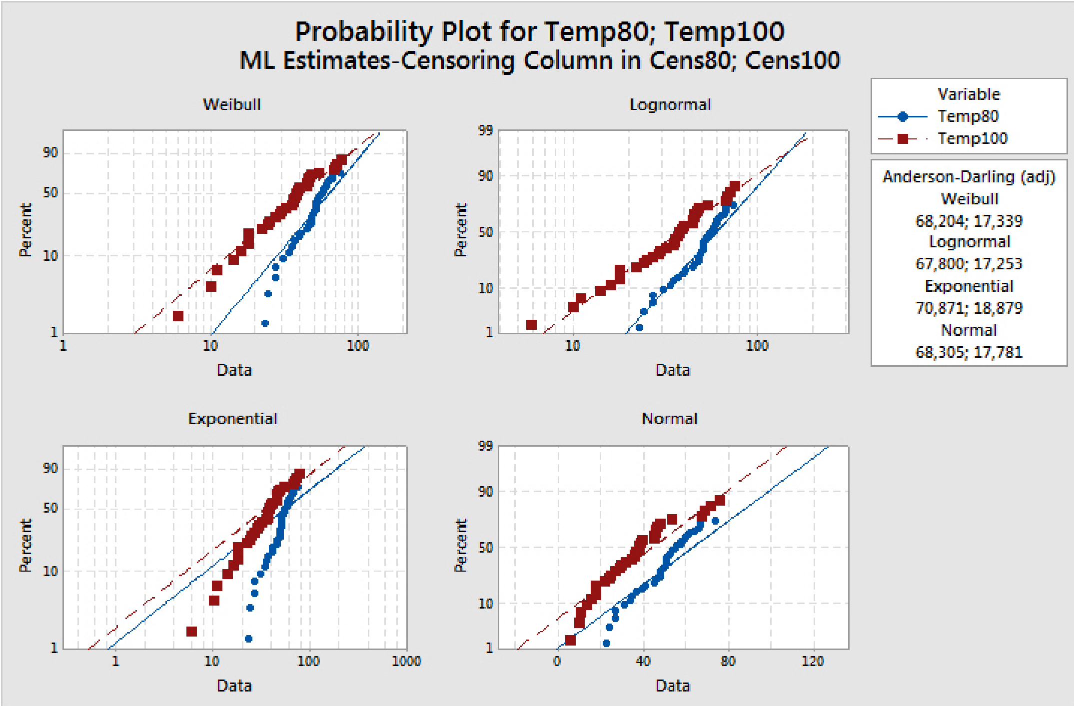


Figure 3.3: Probability plot

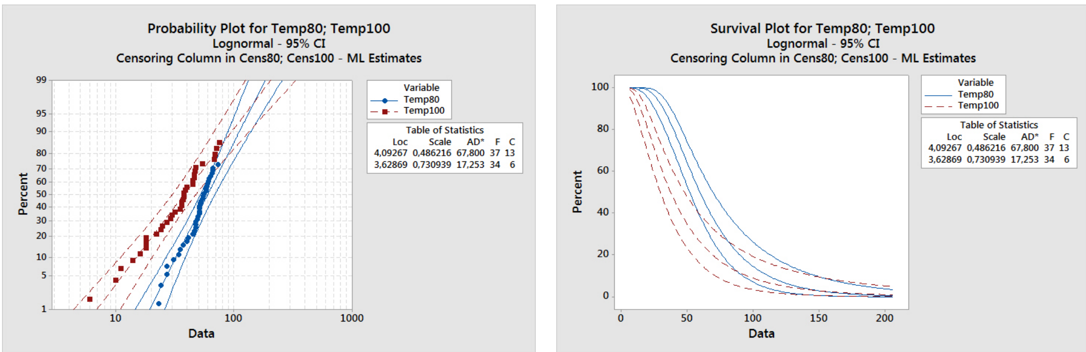


Figure 3.4: Probability and survival plot for the chosen lognormal distribution

Listing 3.1: Anderson-Darling statistic for Temp80

```
Results for variable: Temp80

Goodness-of-Fit

                 Anderson-Darling
Distribution                (adj)
Weibull                    68,204
Lognormal                  67,800
Exponential                70,871
Normal                     68,305
```

Listing 3.2: A-D for Temp100

```
Results for variable: Temp100

Goodness-of-Fit

                 Anderson-Darling
Distribution                (adj)
Weibull                    17,339
Lognormal                  17,253
Exponential                18,879
Normal
```

Listing 3.3: percentiles for Temp80

```
Variable: Temp80
Censoring Information  Count
Uncensored value          37
Right censored value      13

Table of Percentiles

                          95,0% Normal CI
Percent  Percentile     Lower     Upper
    0,1     13,3317   9,21026   19,2975
      1     19,3281   14,4953   25,7722
      2     22,0674   17,0178   28,6154
      3     24,0034   18,8304   30,5975
      4     25,5709   20,3126   32,1906
      5     26,9212   21,5978   33,5566
      6     28,1265   22,7506   34,7727
      7     29,2276   23,8074   35,8819
      8     30,2501   24,7910   36,9113
      9     31,2110   25,7170   37,8788
     10     32,1225   26,5962   38,7970
     20     39,7837   33,9646   46,5999
     30     46,4184   40,1936   53,6073
      .
      .
      .
     98     162,590   120,175   219,977
     99     185,634   133,271   258,570
```

Listing 3.4: p. for Temp100

```
Variable: Temp100
Censoring Information  Count
Uncensored value          34
Right censored value       6

Table of Percentiles
                          95,0% Normal CI
Percent  Percentile     Lower     Upper
    0,1     3,93505   2,19401   7,05767
      1     6,87764   4,33827   10,9034
      2     8,39410   5,52121   12,7619
      3     9,52528   6,42827   14,1144
      4     10,4756   7,20360   15,2338
      5     11,3181   7,89954   16,2162
      6     12,0884   8,54184   17,1076
      7     12,8069   9,14535   17,9343
      8     13,4863   9,71949   18,7129
      9     14,1354   10,2707   19,4544
     10     14,7606   10,8036   20,1667
     20     20,3589   15,6197   26,5362
     30     25,6717   20,1592   32,6916
      .
      .
      .
     98     168,993   107,427   265,843
     99     206,255   125,600   338,704
```

Listing 3.5: scale and location tests

```
Distribution Analysis: Temp80; Temp100
Test for Equal Scale and Location Parameters
Chi-Square  DF      P
   18,6468   2  0,000

Test for Equal Scale Parameters
Chi-Square  DF      P
   5,29599   1  0,021

Test for Equal Location Parameters
Chi-Square  DF      P
   11,2988   1  0,001
```

3.5 A reliability example with INLA

We consider data from [18], discussed in [20] or also [8]. Patients are placed catheters in 2 ways for the time of their hospitalization. Unfortunately sometimes an infection occurs. The variables variables are `time`, `event` and `placement`. The `time` represents the time spent in the hospital. If the hospitalization is without complications censoring occurs. If the patients get an infection, we observe the event of interest `event` (0, 1). As described in the R dokumentation, event, he status indicator, can acquire 1=observed event, 0=right censored event, 2=left censored event, 3=interval censored event. `Placement` represents 2 ways (1, 2) of placing the catheter. The data set is right censored. Here a sample of the data in table 3.1.

For the **exponential model** we have the following parts: The data are exponentially distributed exponential 3.7, where t_i is the i^{th} survival time following the λ_i distribution. The latent components are β_0 and β_1). The GMRF $\mathbf{x} = \boldsymbol{\beta} = (\beta_0, \beta_1)$ is represented in the latent field 3.8 by the latent linear model 3.9. β_1, β_2, must come from a Gaussian distribution that we can use INLA and the GMRF definition in subsection 3.2.1 holds. We assign the following priors 3.10. The priors have a small precision since we don't have any prior information about them. So the are set small. We give just a small hint that the mean could be at zero. No hyperparameter is used for this model.

$$t_i \sim Exp(\lambda_i) \tag{3.7}$$

$$\lambda_i = 1/exp(\eta_i) \tag{3.8}$$

$$\eta_i = \beta_0 + trt_i\beta_1 \tag{3.9}$$

$$\begin{aligned} \beta_0 &= N(0, 0.001) \\ \beta_1 &= N(0, 0.001) \end{aligned} \tag{3.10}$$

In the **Weibull model** we see time Weibull distributed in equation 3.11. The latent field is as by the previous model 3.12 with the latent linear model 3.13 and the priors 3.14. In this case the model has one hyperparameter (parameter controlling the distribution) α, assigned a prior 3.15.

$$t_i \sim \text{Weibull}(\alpha, \lambda_i) \tag{3.11}$$

$$\lambda_i = 1/exp(\eta_i) \tag{3.12}$$

$$\eta_i = \beta_0 + trt_i\beta_1 \tag{3.13}$$

$$\beta_0 = N(0, 0.001)$$
$$\beta_1 = N(0, 0.001) \tag{3.14}$$

$$\alpha \sim \text{Gamma}(1, 0.001) \tag{3.15}$$

We set up an R-code and process it by the software RStudio.

Survival data		
Time	Event	Placement
0.15	1	1
2.85	0	2
0.35	1	2
etc.		

Table 3.1: Sample of survival data from [20]

Given the code 3.6, we get the results for the *exponential model* in listing 3.7. $\boldsymbol{\beta} = (\beta_0, \beta_1) = (-0.6242, -0.5334)$. For the second placement we get $\eta_2 = -0,6242 + \mathbf{2} \cdot (-0,5334) = -1,39974828$ and $t_2 \sim Exp(4,05417932)$. Similarly for the first placement.

In the *Weibull model* the results are in listing 3.8. Here the for the second placement we obtain $\boldsymbol{\beta} = (\beta_0, \beta_1) = (-0.5923, -0.5438)$. Therefore $\lambda_2 = e^{-1,40969274} = 0.24421831014$ $t_2 \sim \text{Weibull}(0.9217, 4.094697075)$.

Listing 3.6: Survival INLA

```
#install.packages(("INLA", repos="http://www.math.ntnu.no/inla/R/testing")
#library(MASS)
#library(sp)
#setwd("~/R")
###########################
#Code for Exponential Model
###########################
data=read.table("Kidney-infec.txt", header = T)
# The routines in R-INLA work with objects of class "inla.surv",
# which is a data structure that combines times, censoring and
# truncation information Here we have right censored data and thus
# the time is represented in this way
inla.surv(data$time, data$event)
formula = inla.surv(time, event) ~ placement
result = inla(formula,family="exponentialsurv", data= data, verbose=TRUE)
summary(result)
################################
# Code for Weibull Model example
################################
data=read.table("Kideny-infec.txt", header = T)
inla.surv(data$time, data$event)
formula = inla.surv(time, event) ~ placement
model = inla(formula,family="weibullsurv", data= data, verbose=TRUE )
```

Listing 3.7: Result INLA exponential model

```
Time used:
```

```
 Pre−processing    Running inla  Post−processing          Total
         0.7510          0.4390           0.1190         1.3091

Fixed effects:
                mean      sd 0.025quant 0.5quant 0.975quant     mode kld
(Intercept) −0.6242 0.5979    −1.8391  −0.6098     0.5108 −0.5805   0
placement   −0.5334 0.3969    −1.3261  −0.5289     0.2336 −0.5197   0

The model has no random effects

The model has no hyperparameters

Expected number of effective parameters(std dev): 2.00(0.00)
Number of equivalent replicates : 59.50

Marginal log−Likelihood:  −67.49
```

Listing 3.8: Result INLA Weibull model

```
Time used:
 Pre−processing    Running inla  Post−processing          Total
         0.8260          0.6230           0.2200         1.6691

Fixed effects:
                mean      sd 0.025quant 0.5quant 0.975quant     mode kld
(Intercept) −0.5923 0.6000     −1.811  −0.5779     0.5470 −0.5489   0
placement   −0.5438 0.3972     −1.337  −0.5393     0.2237 −0.5302   0

The model has no random effects

Model hyperparameters:
                                  mean     sd 0.025quant 0.5quant 0.975quant   mode
alpha parameter for weibullsurv 0.9217 0.1128     0.7101   0.9182      1.151 0.9098

Expected number of effective parameters(std dev): 2.00(0.00)
Number of equivalent replicates : 59.50

Marginal log−Likelihood:  −67.75
```

4 NON-PARAMETRIC METHODS

In this chapter we will investigate the Kaplan-Meier method and the Actuarian Method. Whereas the Kaplan-Meier method is more useful in clinical studies, the Actuarian method is more of use in cases, where the data about survivals are not exact. The Kaplan-Meier assumes all censorings precedes failures of the component (we are not censoring already failed components). The Actuarian method precedes half of the components we are censoring, have failed at the end of the computed time interval.

4.1 Kaplan-Meier Method

Denote t_i the ith time of the actual death, d_i the number of death, n_i the corresponding number of patients. $d_i = n_i - n_{i+1}$ denotes failures (deaths) in age $\langle t_i, t_{i+1})$. We know $S(t_i) = P(t > t_i)$ is the probability of surviving beyond time t_i. For $t \in [t_1, t_2)$ we have the probability of survival in time interval $[0, t_1)$ times the probability of survival in time interval $[t_1, t_2)$ given the probability the previous survival (given you are still alive).

$$S(t_1) = P(T > t) = P(0 < T < t_1) \cdot P(t_1 < T < t_2 | 0 < T < t_1) \tag{4.1}$$

i.e,

$$\hat{S}(t_0) = \frac{n_0 - 0}{n_0} = 1 \tag{4.2}$$

$$\hat{S}(t_1) = \frac{n_0 - 0}{n_0} \cdot \frac{n_1 - d_1}{n_1} \tag{4.3}$$

simplified

$$\hat{S}(t) = 1 - \frac{d_1}{n_1} \tag{4.4}$$

subsequently we get in general $t \in [t_j, t_{j+1})$, $j \in \mathbb{N}$

$$\hat{S}(t) = \left(1 - \frac{d_1}{n_1}\right)\left(1 - \frac{d_2}{n_2}\right) \cdots \left(1 - \frac{d_j}{n_j}\right) = \prod_{i=1}^{j} \left(1 - \frac{d_i}{n_i}\right) \tag{4.5}$$

$$\hat{S}(t_i) = \prod_{j=1}^{i} \frac{n_j - r_j}{n_j}, i = 1, ..., m \tag{4.6}$$

where m is the total number of data points, n the total number of units. n_j is defined by

$$n_i = n - \sum_{j=0}^{i-1} c_j - \sum_{j=0}^{i-1} r_j \tag{4.7}$$

where r_j is the number of failures in the j^{th} data group and c_j the number of censorings in the j^{th} data group. The recurrent formula

$$\hat{S}(t_i) = \hat{S}(t_{i-1}) \cdot (1 - p_i) \tag{4.8}$$

, where $p_i = \frac{d_i}{n_i}$ holds.

4.2 Actuarian Method

This method also called the life-tables method is widely used in insurance mathematics for calculations with life-tables. We will denote a failure what is usually in insurance mathematics written as a death. The time is parted into equal sized intervals. Contrary to the Kaplan-Meier method the interval does not end at each failure, but at a given time step. Let's have a look at the method *without censoring*, as in the insurance. Let's denote:

x	age of the persons
n_x	number living on the beginning of the interval x^{th} time interval
d_x	number of failures observations in the x^{th} time interval, $d_x = n_x - n_{x+1}$
c_x	number of censorings observations in the x^{th} time interval, $d_x = n_x - n_{x+1}$
q_x	probability of failure from age x to age $x+1$ calculated as $\frac{d_x}{n_x}$
s_x	probability of survival from age x to $x+1$ calculated as $\frac{n_{x+1}}{n_x}$

Evidently, $n_x = c_x + d_x$. We will not work with cumulative numbers and other items, since it is in our case not needed. Further reading and extension might be found in [10]. The probability of survival in the i^{th} interval is

$$s_j = 1 - \frac{d_j}{n_j - \frac{1}{2}c_j} \tag{4.9}$$

We see that it is supposed that about half of the censorings happened before the failures and half after. The survival function is then:

$$\hat{S}(j) = \prod_{i=1}^{j} s_i \tag{4.10}$$

If there wouldn't be any censorings, the situation would be much more easy. We could just devide the $S(j) = \frac{n_j}{n_1}$

$S_j = 1 - \dfrac{d_j}{n_j - \frac{1}{2}c_j}$ denotes that censorings are equally distributed.

5 ANALYSIS OF TECHNICAL DATA

In this part we are going to apply the gained knowledge on industrial data. We are investigating the lifetime of of valves. Within process improvement, we would like to determine, whether we can use one valve type (type B) instead of the used (type A) without loss of reliability. The effort of this substitution is mainly motivated by the lower cost of the new proposed valve. The valves are working in an open/close mechanism. There was held an experiment, to determine the lifetime of the valves. First was type A tested and after some time was add type B. In the experiment the valves are opening and closing 100 times faster as normal. One such opening and closing we denote as one cycle. It is therefore considered an accelerated test. We request from the valves to hold at least 2 million cycles. Unfortunately, during the experiment the condition changed for a while, which caused some of the valves failed prematurely. We tested valve type A up to 6048 cycles and valve type B up to $5287,68$ cycles. This is given by the conditions of the test we started testing valve type A about 760 cycles earlier than the second type, which was added later to the test and after some time we stopped the test, which caused significant censoring by valve type B as can be seen on the dotplot 5.1.

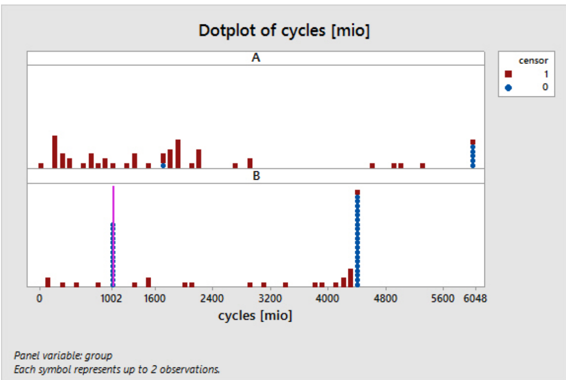


Figure 5.1: Dotplot of failures

5.1 Parametric analysis

A common methodology how to approach survival data is to assume that they come from a particular distribution and examine then the distribution. To find the optimal distribution to fit the data in, we compare with a probability plot the data with the proposed distribution as we can see on figures 5.2 and 5.3. While looking at the chart and the Anderson-Darling statistic in subsection 0.2, we see the performance is very poor and the data are not suitable for a serious parametric analysis. However, for academic purposes and with the hope that they might perform better in the next experiment, where would not happen a similar accident, we continue with the analysis. We see outlayers at the left tail of the distribution. In praxis the valves will be tested before use to lower later vacancy claims. This gives us a clarification for removing in valve A the 5^{th} and 99^{th} data entry with value 0. In valve B we remove the 5^{th}, 24^{th} and 56^{th} data entry with value $69, 12$. In the adjusted probability plot 5.4, the Weibull distribution perform as well the best. The next step is the Weibull analysis. The the main question is whether the valve B is better. The minor question or more a wish is whether enough valves get have a lifetime expectancy of more than 4000 cycles. In the figure 5.5, we see that the distributions might differ. This confirms also the test for equal share and scale as well as the partial ones, listing 5.1.

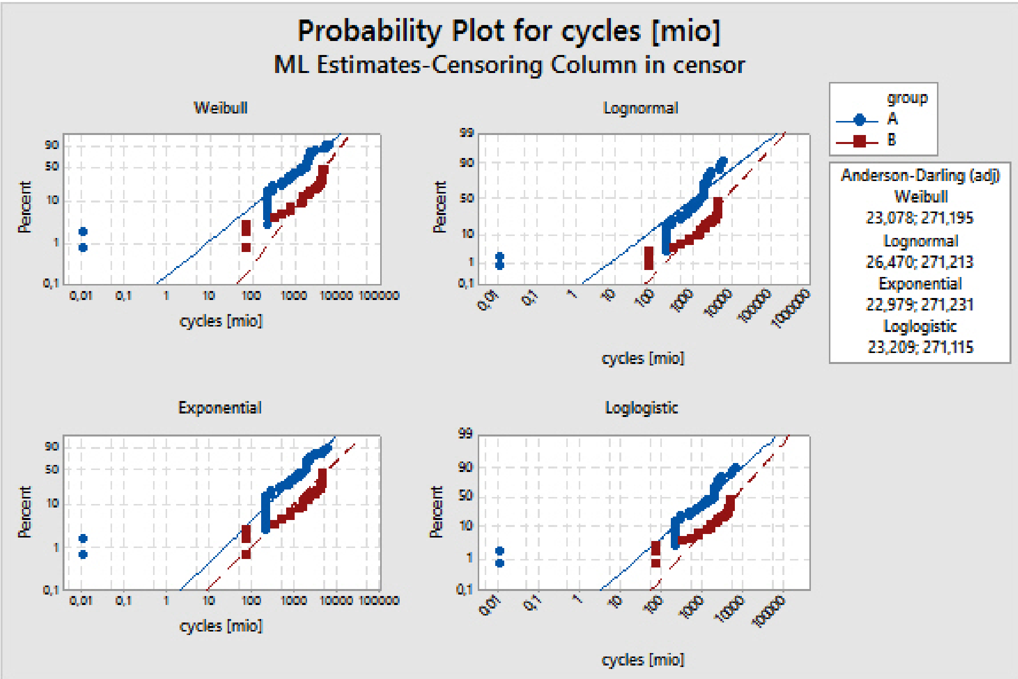


Figure 5.2: Probability plot for different distributions

Listing 5.1: Test for equal shape and scale in the Weibull distribution

```
Test for Equal Shape and Scale Parameters
Chi−Square  DF      P
   80,5942   2  0,000

Test for Equal Shape Parameters
Chi−Square  DF      P
   16,9594   1  0,000

Test for Equal Scale Parameters
Chi−Square  DF      P
   42,7504   1  0,000
```

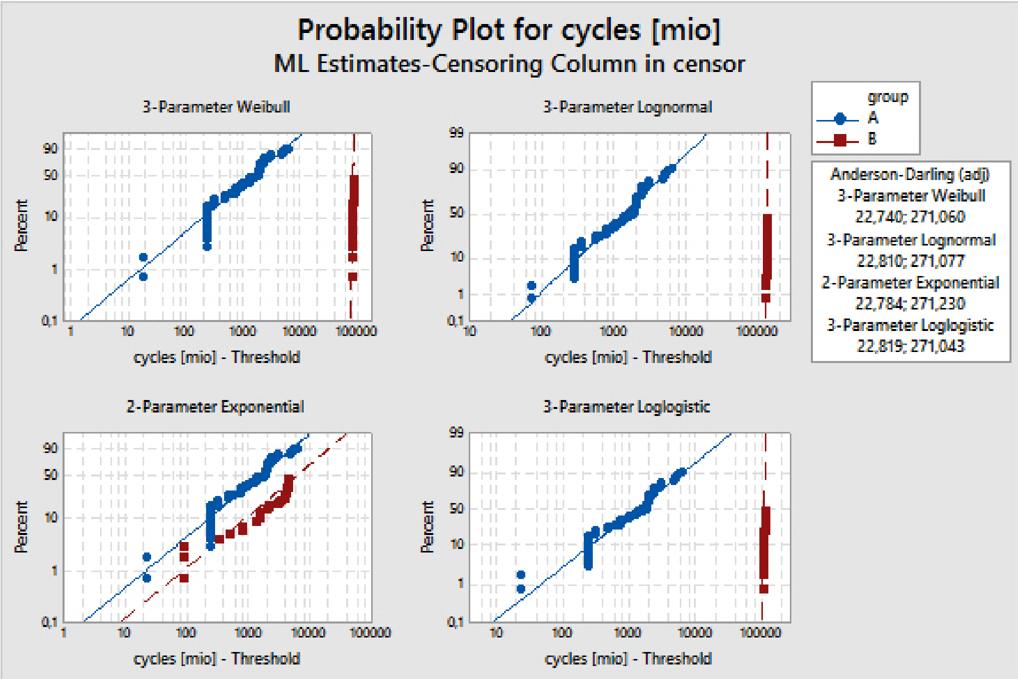


Figure 5.3: Second probability plot for different distributions

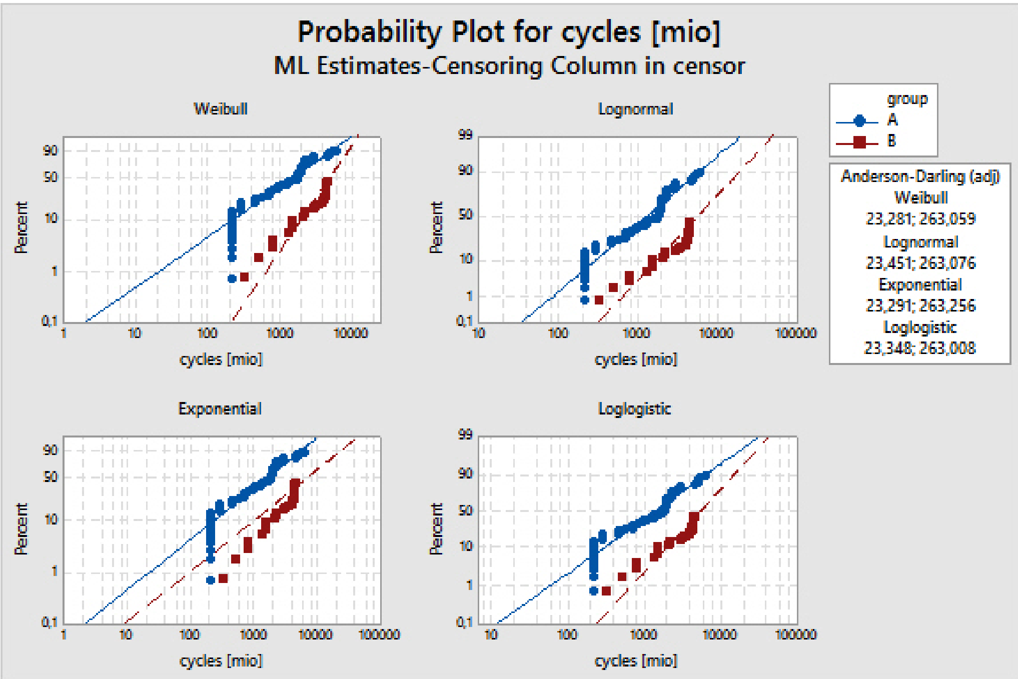


Figure 5.4: Adjusted probability plot for different distributions

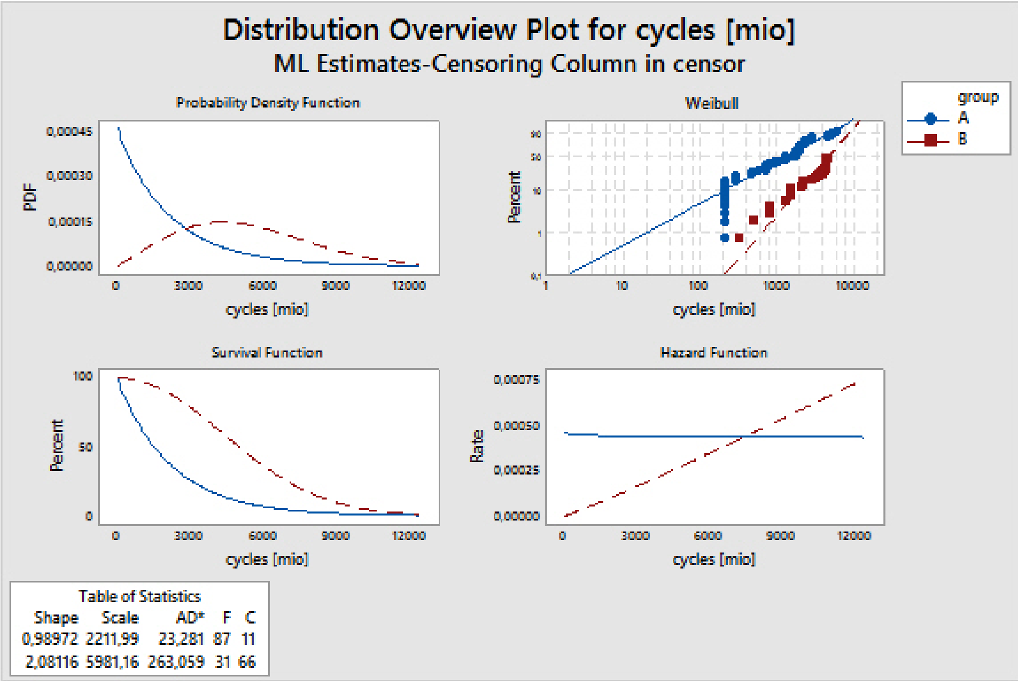


Figure 5.5: Weibull analysis overview chart

5.2 Nonparametric analysis

We examine the data with the Caplan-Mayer and Actuarian method. In the test statistic in listing 5.2 for comparing the two survival curves. For both statistics we reject the null hypothesis that survival curves don't differ. When we are looking the survival plot of the Actuarian Method, we notice that failure rate of group B is nicely constant. This constant failure rate would probably continue, if at about 2000 cycles the situation wouldn't change (accident in the experiment). At about changes the curve to a wear-out phase, which has surprisingly in this case a lower failure rate. We are there probably examining the remaining strong components. In contrast valve type B has clearly from the beginning a lower failure rate. In this case it can be seen better on the Kaplan-Meier Failure Plot 5.6. The wear-out phase begins at about 4300 cycles. Again with the remaining strong components with a lower failure rate.

Listing 5.2: Nonparametric comparison of the valve type A and B survival functions

```
Comparison of Survival Curves
Test Statistics
Method      Chi−Square  DF  P−Value
Log−Rank       51,1298   1    0,000
Wilcoxon       53,1759   1    0,000
```

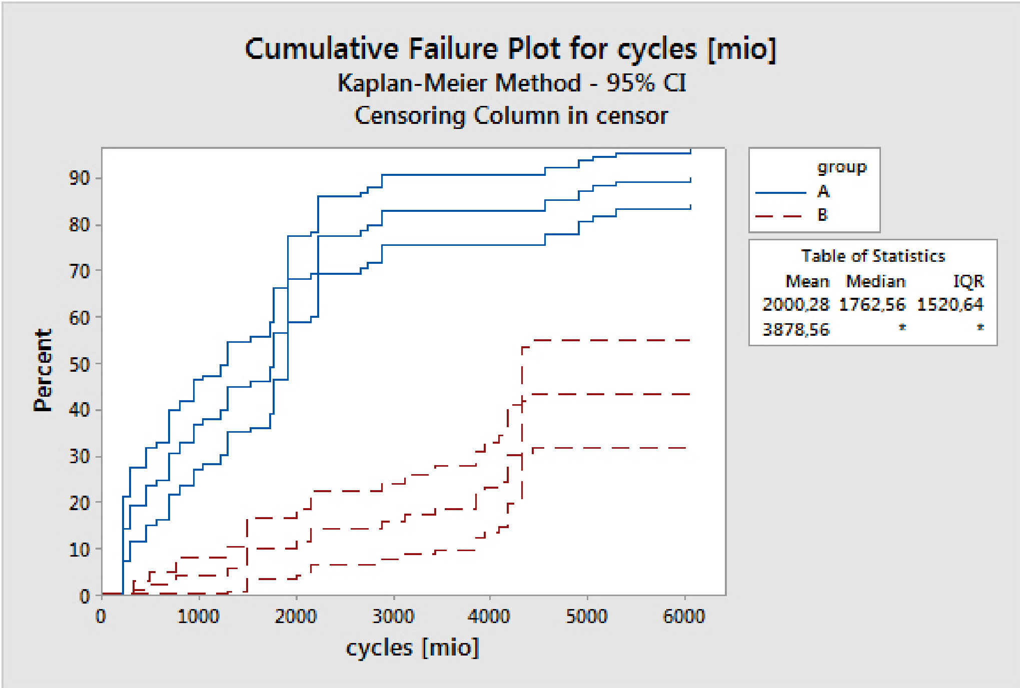


Figure 5.6: Cumulative failure plot with the Caplan-Meier Method

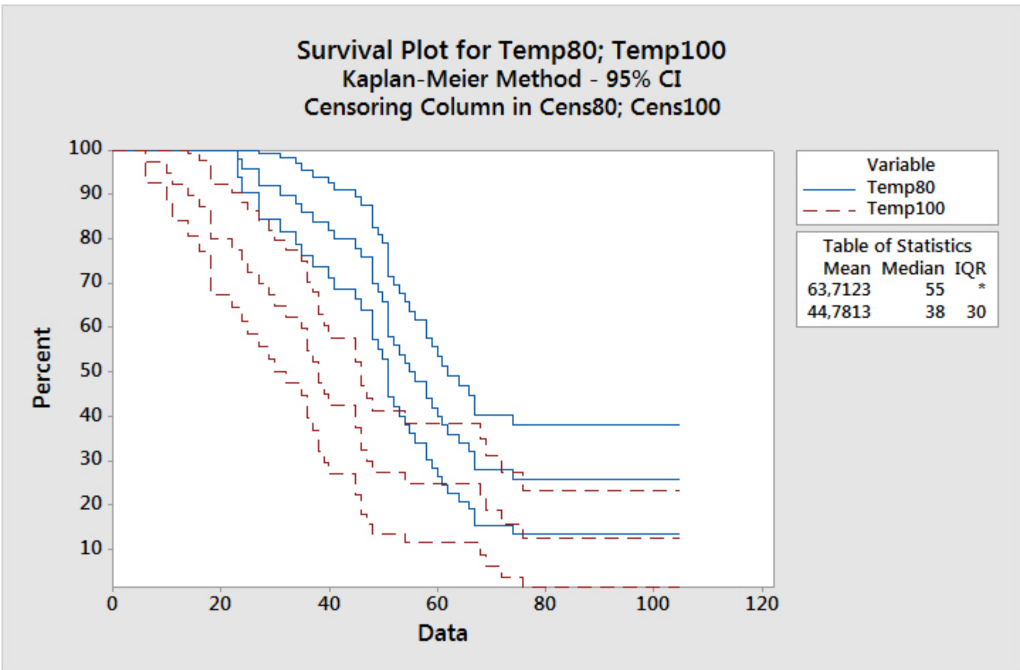


Figure 5.7: Survival plot with the Caplan-Meier Method

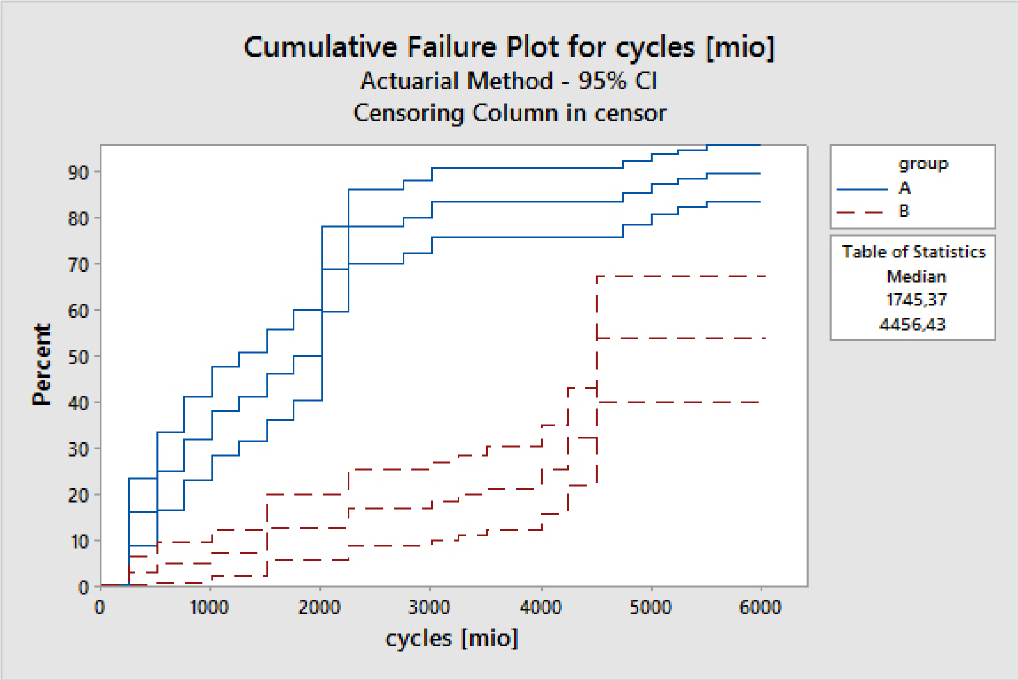


Figure 5.8: Cumulative failure plot with the Actuarian Method

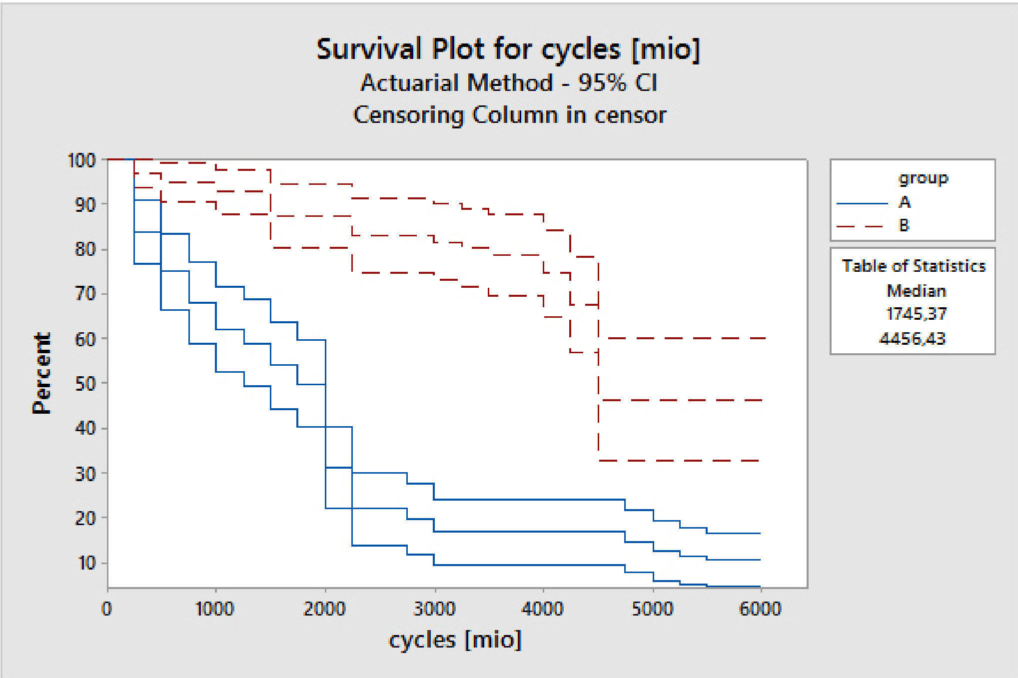


Figure 5.9: Survival plot with the Actuarian Method

5.3 INLA

In the way the example provided in the previous INLA Example, I have constructed an INLA method that solves the our technical data in this section. Let's model the data with the subsequent hierarchical model. The relevant `R` code is provided in listing 5.3. Note, that to be able to compute the result, we scale the data by the maximum value. The computed result is shown in listing 5.4. To get the relevant coefficients, we transform the data back as can be seen in formula 5.2. In comparism to the Weibul analysis we conclude that however η_B and β differ a bit, nearly the same parameters are calculated for $1/\lambda_A$.

$$\begin{aligned} t_i &\propto \text{Weibull}(\beta, \lambda_i) \\ \lambda_i &= exp(\mu_i) \\ \mu_i &= \psi_A + \mathbf{1}_B \cdot \psi_B \\ \psi_A &= N(0, 0.001) \\ \psi_B &= N(0, 0.001) \\ \beta &\sim \text{Gamma}(1, 0.001) \end{aligned} \tag{5.1}$$

data			
Valve	cycles	censor	group
1	4423,68	0	B
2	1002,24	0	B
3	4181,76	1	B
etc.			

Table 5.1: Data table sample

Listing 5.3: INLA code - Weibull model

```
library("MASS")
library("INLA")
data=read.table("valve.txt", header = T, dec=",")
data$cycles = data$cycles / max(data$cycles)
inla.surv(data$cycles, data$censor)
formula = inla.surv(cycles, censor) ~ group
model=inla(formula,family="weibullsurv", data= data, verbose=TRUE )
summary(model)
```

Listing 5.4: Calculated R result.

```
Time used: 3.8702
Fixed effects:
               mean     sd 0.025quant 0.5quant 0.975quant    mode
(Intercept)  1.0759 0.1192     0.8368   1.0776     1.3051  1.0812
groupB      -1.4757 0.2096    -1.8986  -1.4718    -1.0746 -1.4638

Model hyperparameters:
                                mean     sd 0.025quant 0.5quant 0.975quant mode
alpha parameter for weibullsurv 1.104 0.0708     0.9719    1.102      1.246     1.098
```

$$
\begin{aligned}
\beta &= 1,104 \\
1/\lambda_A &= \frac{1}{e^{1,0759}} \cdot \texttt{max(data\$cycles)} = 2062,311908 \\
1/\lambda_B &= \frac{1}{e^{(1,0759-1,4757)}} \cdot \texttt{max(data\$cycles)} = 9020,751441
\end{aligned}
\tag{5.2}
$$

5.4 MCMC

As a highlight of this work I created to the INLA model a MCMC algorithm. Let's have the stated model 5.1. My MCMC algorithm consists of a Metropolis algorithm within a Gibbs algorithm, see code 2. The marginalization will be performed in the likelihood function 5.3, see [4]. Since the full-conditionals don't belong to a known distribution, we introduce for each a Metropolis step using a univariate normal distribution. For easier calculations, we characterize the Weibull's rate paramether as $1/\lambda_i$. We define the variables 5.5. Since the sum of normal distributed random variables via the characteristic function is normaly distributed, see [25], we can write by the change of variable formula, theorem 1, the prior distributions 5.6. Now we derive the full-conditionals in equations 5.7. The the survival and hazard function 5.4 in the likelihood function 5.3 with the prior distributions 5.6 are combined. In the listing 5.5 the programmed `R` code. After removing the burn-in period 5.10, the trace plots are in figure 5.11. The resuts (means) have been calculated as the shape parameter $\beta = 0,821552$ and the paramethers $1/\lambda_A = 2186,976$, $1/\lambda_B = 4366,361$ and compared in table 5.2.

$$
L(\boldsymbol{x}, \boldsymbol{\theta}, \boldsymbol{c}) = \prod_{i=1}^{n} \{[h(x_i, \boldsymbol{\theta})]^{c_i} \cdot [S(x_i, \boldsymbol{\theta})]\} \tag{5.3}
$$

$$
\begin{aligned}
h(x_i|\beta, \lambda) &= \beta \lambda^\beta x_i^{\beta-1} \\
S(x_i|\beta, \lambda) &= e^{-(\lambda x_i)^\beta}
\end{aligned}
\tag{5.4}
$$

$$
\begin{aligned}
\lambda_A &= e^{\psi_A} \\
\lambda_B &= e^{\psi_A + \psi_B}
\end{aligned}
\tag{5.5}
$$

$$
\begin{aligned}
\pi(\lambda_A) = \tilde{f}_{\lambda_A}(\psi_A) &= e^{-\ln(\lambda_A)\frac{0,001}{2}} \cdot \frac{1}{\lambda_A} = e^{-\psi_A \frac{1}{2000}} \cdot \frac{1}{e^{\psi_A}} \\
\pi(\lambda_B) = \tilde{f}_{\lambda_B}(\psi_A + \psi_B) &= e^{-\frac{1}{2}\frac{\ln(\lambda_B)}{1000+1000}} \cdot \frac{1}{\lambda_A} = e^{-\frac{\psi_A+\psi_B}{4000}} \cdot \frac{1}{e^{\psi_A+\psi_B}}
\end{aligned}
\tag{5.6}
$$

Let us denote $C_1 = \{x_i \in X | c = 1\}$, $C_0 = \{x_i \in X | c = 0\}$, $\Lambda_A = \{x_i \in X | \lambda = \lambda_A\}$ and $\Lambda_B = \{x_i \in X | \lambda = \lambda_B\}$.

$$\begin{aligned}
\pi(\lambda_A|\cdot) &= e^{\psi_A \beta \cdot c_A} \cdot e^{-\sum_{\Lambda_B} \left(x_i e^{\psi_A}\right)^{\beta}} \times \pi(\lambda_A) \\
\pi(\lambda_B|\cdot) &= e^{(\psi_A+\psi_B)\beta \cdot c_B} \cdot e^{-\sum_{\Lambda_A} \left(x_i e^{\psi_A+\psi_B}\right)^{\beta}} \times \pi(\lambda_B) \\
\pi(\beta|\cdot) &= \beta^{c} e^{c\psi_A + c_B \psi_B} \prod_{C_1} x_i^{\beta-1} \cdot e^{-\left(\sum_{C_1}\left(x_i e^{\psi_A}\right) + \sum_{C_0}\left(x_i e^{\psi_A+\psi_B}\right)\right)^{\beta}} \times \pi(\beta)
\end{aligned} \tag{5.7}$$

$\psi^{(i)*}$ represents the possibilities $\psi_A^{(i)*}$, $\psi_B^{(i)*}$ and $\beta^{(i)*}$

Algorithm 2 WeibullMCMC

```
procedure MCMC(ψ_A, ψ, N)                          ▷ Initial values ψ_A, ψ
    ψ_A^(0) ← ψ_A; ψ^(0) ← ψ;
    ψ^(i)* ≜ N(ψ^(i−1), t)                         ▷ t is the relevant tuning parameter.
    while i ≤ N do
        ψ_A^(i) ← METROPOLIS(1, ψ_A^(i)*, π(λ_A | λ_B^(i−1), β^(i−1)))
        ψ^(i) ← METROPOLIS(1, ψ_B^(i)*, π(λ_B | λ_B^(i), β^(i−1)))
        β^(i) ← METROPOLIS(1, β_A^(i)*, π(β | λ_B^(i), β^(i)))
    end while
```

$\frac{\beta_{MCMC}}{\beta_{average-Minitab}}$	$\frac{1/\lambda_{A-MCMC}}{\lambda_{A-Minitab}}$	$\frac{1/\lambda_{B-MCMC}}{\lambda_{B-Minitab}}$
0,98869163	0,73001909	1,505446016

Table 5.2: Compared Minitab and MCMC results.

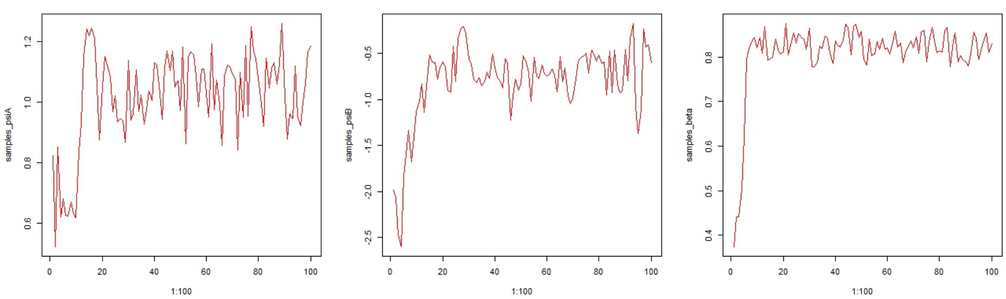


Figure 5.10: The burn-in period.

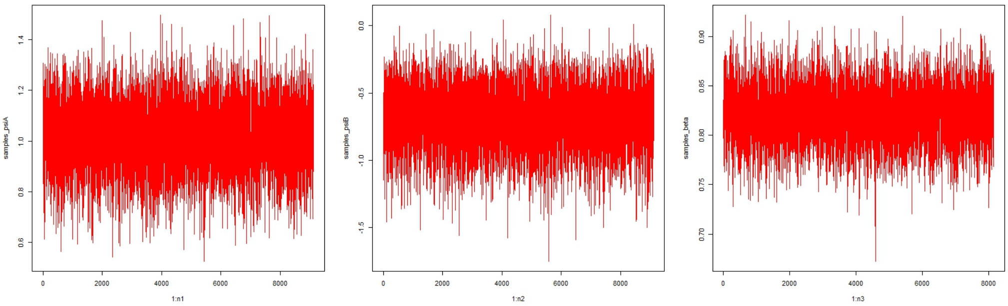


Figure 5.11: The trace plots.

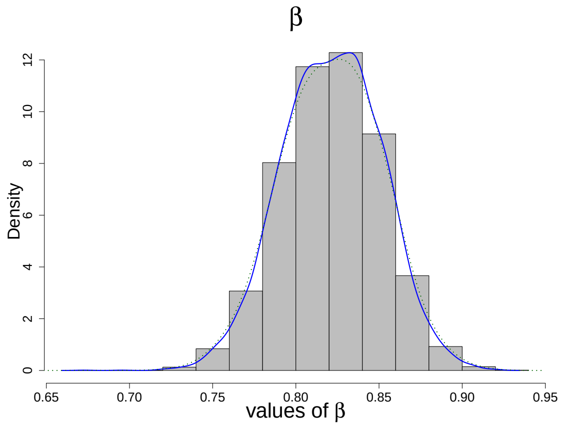


Figure 5.12: Parameter β.

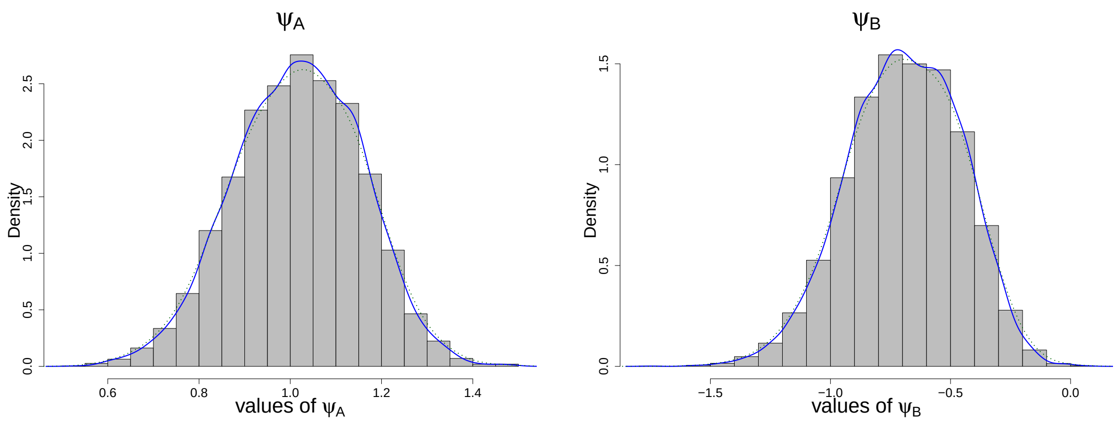


Figure 5.13: Parameters ψ_A and ψ_B.

Listing 5.5: Programmed Markov chain Monte Carlo algorithm

```
set.seed(11)
```

```
setwd("~/R/mcmc")
library("MASS")
#posterior sample size
M = 20000
#Load the data set
data=read.table("valve.txt", header = T, dec=",")
data$cycles = data$cycles / max(data$cycles)
#data$cycles = data$cycles / median(data$cycles)
data<-as.data.frame(data)
#initial values
beta=0.3; psiA=1; psiB=-2;
#given values
c=length(data$valve)
cA=length(which(data$censor == 1 & data$group == "A"))
cB=length(which(data$censor == 1 & data$group == "B"))
###########################################
#Functions used in the MCMC's iteration step
###########################################
#full conditional $\lambda_A$
psi_Af<-function(psiA, beta, cA){
  exp(psiA*beta*cA -sum( (data[which(data$group == "A"),]$cycles*
                                  exp(psiA))^beta )
  -psiA/4000-psiA)   #prior
}
#full conditional $\lambda_B$
psi_Bf<-function(psiB, beta, cB){
  exp(psiB*beta*cB -sum( (data[which(data$group == "B"),]$cycles* #problem
                                   exp(psiB))^beta )
  -psi/2000-psi)      #prior
}
#full conditional $\beta$
beta_f<-function(beta, psiA, psiB, c, cB){
  beta^c*prod( data[which(data$censor == 1),]$cycles )^(beta-1)*
    exp((c*psiA+cB*psiB)*beta-
  (sum( data[which(data$group == "A"),]$cycles*
         exp(psiA)) + sum(data[which(data$group == "B"),]$cycles*exp(psiB)))^beta-
    0.001*beta)                 #gamma prior
}
###########
#Metropolis
###########
#vector of samples
samples_psiA <- numeric(M)
samples_psiB<- numeric(M)
samples_beta <- numeric(M)
#indexes
acceptedA=0;acceptedB=0;acceptedbeta=0;i=0;
#tuning parameters
t1=0.3;t2=0.5;t3=0.07;
while ( i <= M ) {
#propose new value
psi_A_star=rnorm(1,psiA,t1)
psi_B_star=rnorm(1,psiB,t2)
beta_star=rnorm(1,beta,t3)
# acceptance step lambda_A
alpha_prob= min(1,exp(
  log(psi_Af(psi_A_star,beta,cA))-log(psi_Af(psiA,beta,cA))
))
u_alpha=runif(1)
  if (alpha_prob > u_alpha){
    psiA=psi_A_star
    acceptedA = acceptedA + 1
    samples_psiA[acceptedA] <- psiA
  }
# acceptance step lambda_B
alpha_prob= min(1,exp(
  log(psi_Bf(psi_B_star,beta,cB))-log(psi_Bf(psiB,beta,cB))
))
u_alpha=runif(1)
if (alpha_prob > u_alpha){
  psiB=psi_B_star
  acceptedB = acceptedB + 1
  samples_psiB[acceptedB] <- psiB
}
# acceptance step beta
alpha_prob= min(1,exp(
  log(beta_f(beta_star, psiA, psiB, c, cB))-
  log(beta_f(beta, psiA, psiB, c, cB))
))
```

```
u_alpha=runif(1)
if (alpha_prob > u_alpha){
  beta=beta_star
  acceptedbeta = acceptedbeta + 1
  samples_beta[acceptedbeta] <- beta
}
  i=i+1;
}
acceptrate_psiA=acceptedA/M
acceptrate_psiB=acceptedB/M
acceptrate_beta=acceptedbeta/M
samples_psiA <- samples_psiA[100:acceptedA]
samples_psiB <- samples_psiB[100:acceptedB]
samples_beta <- samples_beta[100:acceptedbeta]
acceptedpsiA=acceptedA
n1=length(samples_psiA)
n2=length(samples_psiB)
n3=length(samples_beta)
par(mfrow = c(1,3))
#Plots the generated kappa_u and kappa_v's versus iteration
plot(1:n1,samples_psiA, type = "l", col=2)
plot(1:n2,samples_psiB, type = "l", col=2) #type = "b", pch='.'
plot(1:n3,samples_beta, type = "l", col=2)
#variable means
result_psiA=mean(samples_psiA)
result_psiB=mean(samples_psiB)
result_beta=mean(samples_beta)
result_psiA
result_psiB
result_beta
#dev.off()
hist(samples_psiA,probability='TRUE', col="grey", main = expression(psi[A]),
     xlab=expression(paste("values of ",psi[A]) ))
lines(density(samples_psiA), col="blue", lwd=2)
lines(density(samples_psiA, adjust=2), lty="dotted", col="darkgreen", lwd=2)
```

Outcome

Nonparametric models showed themselves as a good alternative to parametric models, which need confidence, that the data follow a common distribution. Valve type A type had high failure rate during the 1. test interval. Valve type B performed better during the hole experiment and is recommend to substitute the used valve type B. Suitable appeared the usage of non-parametric methods, which distinctly proofed the different behavior of the valves and the durability of the 1. type is greater than the 2. type, see figure 5.9. We do recommend to do a burn in test and let the valves run 100 cycles, before we will use them. The calculation in the thesis were done under this situation.

6 CONCLUSION

In the introductory chapter we apprised with the basic definitions and concepts of the survival analysis. In the subsequent chapter, we described the main distributions, which we used for the parametric lifetime methods. In the chapter dealing with Six Sigma we showed methods that are related to the reliability problematic. In particular, we can use the methods provided in this thesis in the phase Analyse. The next chapter was dedicated to parametric methods. We examined closely the Weibull analysis and a new method called INLA. The subsequent chapter dealer in contrast to the previous with non-parametric methods. We treated the most use methods, the Kaplan-Meier Method and the Actuarian Method. To get a first impression with the practical use in Minitab, we provided in the next-to-last chapter a small example in Minitab. The last chapter was concentrated on solving an industrial reliability problem. A bayesian hierarchical model was derived and solved using integrated nested Laplace approximation. In comparism was programmed a Markov chain Monte Carlo algorithm.

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