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Fractional Differential Equations and Their Applications

Zlomkové diferenciální rovnice a jejich aplikace

Diploma Thesis
Diplomová práce

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Vedoucí práce

BRNO 2008

Abstract

In this thesis we discuss standard approaches to the problem of fractional derivatives and fractional integrals (simply called differintegrals), namely the Riemann-Liouville, the Caputo and the sequential approaches. We prove the basic properties of differintegrals including the rules for their compositions and the conditions for the equivalence of various definitions.

Further, we give a brief survey of the basic methods for the solving of linear fractional differential equations and mention limits of their usability. In particular, we formulate the theorem describing the structure of the initial-value problem for linear two-term equations.

Finally, we consider some physical applications, in particular fractional advection-dispersion equation and the viscoelasticity problem. Dealing with the the first issue we derive the fundamental solution in the form of Lévy α-stable distribution density and then we discuss relations between the generalized central limit theorem and the choice of the corresponding fractional model. In section concerning viscoelasticity we mention some typical models generalizing the standard ones via the replacement of the classical terms by the fractional terms. In particular, we derive the step response functions of those generalized systems.

Abstrakt

V této práci uvádíme nejběžnější přístupy k problematice zlomkových derivací a integrálů, a to přístupy Riemannův-Liouvillův, Caputův a metodu postupného derivování. Základní vlastnosti diferintegrálů potřebné pro jejich praktické užití, včetně pravidel pro jejich skládání a podmínek ekvivalence různých definic, jsou v textu dokázány.

Dále podáváme stručný přehled základních metod řešení lineárních zlomkových diferenciálních rovnic a vymezujeme rozsah jejich použití. Speciálně pro lineární rovnice s jedním diferenciálním členem je formulována a dokázána věta popisující strukturu řešení Cauchyova problému.

V poslední kapitole se věnujeme aplikačním problémům, zejména zlomkové advekční-disperzní rovnici a teorii viskoelasticity. Pro první zmíněný problém bylo odvozeno fundamentální řešení ve formě hustoty Lévyho α-stabilního rozdělení a následně diskutována souvislost zobecněného centrálního limitního teorémemu s oprávněností volby zlomkového modelu. V sekci o viskoelasticitě jsme uvádíme některé typické modely zobecněné záměnou klasických členů za zlomkové a zkoumáme odezvy těchto systémů na vstupní jednotkový skok.

Keywords

fractional calculus, fractional differential equations, fractional advection-dispersion equation, fractional viscoelasticity

Klíčová slova

zlomkový kalkulus, zlomkové diferenciální rovnice, zlomková advekční-disperzní rovnice, zlomková viskoelasticita

KISELA, T.: *Fractional Differential Equations and Their Applications.* Brno: Vysoké učení technické v Brně, Fakulta strojního inženýrství, 2008. 50 p. Vedoucí diplomové práce doc. RNDr. Jan Čermák, CSc.

Declaration

I wrote this diploma thesis all by myself under the direction of my supervisor, doc. RNDr. Jan Čermák, CSc. All sources I used are listed in the bibliography.

Prohlášení

Tuto diplomovou práci jsem vypracoval samostatně pod vedením doc. RNDr. Jana Čermáka, CSc. za použití zdrojů uvedených v seznamu literatury.

Acknowledgement

At first I would like to thank my supervisor, doc. RNDr. Jan Čermák, CSc. for the good advice about the choice of my diploma theme and for following suggestive and encouraging consultations which were practised only via e-mail due to the large geographical distance.

Next I thank Prof. Klaus Engel from Università degli Studi dell'Aquila for the help with the technical and stylistic revision of my thesis, Dr. Boris Baeumer from University of Otago for the good recommendation during searching the interesting applications of fractional calculus and Prof. Marco di Francesco from Università degli Studi dell'Aquila for the friendly discussion about transport equations.

The major part of this thesis was written in l'Aquila (Italy) while I was there on the Double diploma programme. Hence, I would like to say my thanks to the mathematical division chairman, Prof. Bruno Rubino for the great help and support whenever we met the administrative machinery or any other problem.

Poděkování

Především bych rád poděkoval vedoucímu svojí práce doc. RNDr. Janu Čermákovi, CSc. za dobrou radu při volbě jejího tématu a za následné přínosné a povzbudivé konzultace probíhající vinou velké geografické vzdálenosti většinou jen formou e-mailové korespondence.

Dále děkuji Prof. Klausi Engelovi z Università degli Studi dell'Aquila za pomoc s úpravami formální a jazykové stránky práce, Dr. Borisi Baeumerovi z University of Otago za správné nasměrování při hledání vhodných aplikačních témat a Prof. Marcovi di Francescovi z Università degli Studi dell'Aquila za přátelskou diskusi o transportních rovnicích.

Podstatná část této práce vznikla v italské l'Aquile během mého studia v rámci programu Double diploma, a proto bych rád vyjádřil svůj dík vedoucímu zdejší matematické sekce Prof. Brunovi Rubinovi za velkou pomoc při zařizování pobytu a všech věcí s ním spojených.

Contents

Chapter 1

Introduction

Fractional calculus is a mathematical branch investigating the properties of derivatives and integrals of non-integer orders (called fractional derivatives and integrals, briefly differintegrals). In particular, this discipline involves the notion and methods of solving of differential equations involving fractional derivatives of the unknown function (called fractional differential equations). The history of fractional calculus started almost at the same time when classical calculus was established. It was first mentioned in Leibniz's letter to l'Hospital in 1695, where the idea of semiderivative was suggested. During time fractional calculus was built on formal foundations by many famous mathematicians, e.g. Liouville, Grünwald, Riemann, Euler, Lagrange, Heaviside, Fourier, Abel etc. A lot of them proposed original approaches, which can be found chronologically in [10]. The theory of fractional calculus includes even complex orders of differintegrals and left and right differintegrals (analogously to left and right derivatives).

The fact, that the differintegral is an operator which includes both integer-order derivatives and integrals as special cases, is the reason why in present fractional calculus becomes very popular and many applications arise. The fractional integral may be used e.g. for better describing the cumulation of some quantity, when the order of integration is unknown, it can be determined as a parameter of a regression model as Podlubný presents in [1]. Analogously the fractional derivative is sometimes used for describing damping.

Other applications occur in the following fields: fluid flow, viscoelasticity, control theory of dynamical systems, diffusive transport akin to diffusion, electrical networks, probability and statistics, dynamical processes in self-similar and porous structures, electrochemistry of corrosion, optics and signal processing, rheology etc.

In this thesis we consider only the most common definitions named after Riemann and Liouville, Caputo, Miller and Ross which will be introduced in chapter 3. For the present let us only note that we use the name "differintegral" which can mean both derivative and integral of arbitrary order. Due to simplicity we will work only differintegrals of real order.

After this Chapter 1: *Introduction*, the thesis is organized as follows. In Chapter 2: *Preliminaries* we remind some techniques and special functions which are necessary for the understanding of the fractional calculus's rules. Chapter 3: *Basic Fractional Calculus* gives the definitions of differintegrals, their most important properties, composition rules, as well as Laplace and Fourier transforms. At the end we give several differintegrals of simple functions. Then we start our study of differential equations containing fractional derivatives, so called fractional differential equations (FDEs). We restrict ourselves to linear FDEs because there is a more compact theory. In Chapter 4: *Existence and Uniqueness Theorem* we give conditions for existence and uniqueness of solutions for linear initial-value problems. In Chapter 5: *LFDEs and Their Solutions* we investigate the main methods of solving for linear FDEs and illustrate them on several examples. Finally in Chapter 6: *Applications of Fractional Calculus* we discuss some concrete problems like the tautochrone problem, advection-dispersion equation, oscillations with fractional damping and fractional models of viscoelasticity.

This thesis tries to be self-contained, however if you find a part which is not perfectly clear, all answers are surely included in one of the book listed in the *Bibliography*.

Chapter 2

Preliminaries

At the beginning of this chapter we remind two facts from elementary mathematical analysis, i.e. the change of order of integration in two dimensions and the derivative of integrals depending on a parameter. Let us point out that we will use the Lebesgue integral in whole thesis and that if the Riemann integral of a function exists, both types of integral correspond.

Then we will introduce some important functions which are used in connection with fractional calculus. The Gamma function plays the role of the generalized factorial, the Beta function is necessary to compute fractional derivatives of power functions; the Mittag-Leffler functions and the Wright functions appear in the solution of linear FDEs. More information about these functions can be found in [1], [2],[10] or [14].

In the last part we are going to present some basic facts about Laplace transform and its properties. More details can be found, e.g. in [3].

2.1 Some Requisites from Ordinary Calculus

In this section we recall two procedures which are very useful and important to keep in mind during reading. In particular, the second one is, in some sense, fundamental for fractional calculus as we will see later.

Change of Order of Integration

Change of order of integration is a trick which we will use e.g. during the calculation of the fractional integral of the power function. We point out that this process does not impose any new condition for the integrated function, it is only a different view at the area we integrate over.

There are known more general versions (e.g. Fubini theorem), but for us the case of triangular areas is sufficient. The following formula (2.1) holds for all functions $f(t,\tau,\xi)$ integrable w.r.t. τ and ξ.

$$\int_a^t \int_a^\tau f(t,\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau = \int_a^t \int_\xi^t f(t,\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi \tag{2.1}$$

The geometrical idea of this formula will become clear from figure 2.1.

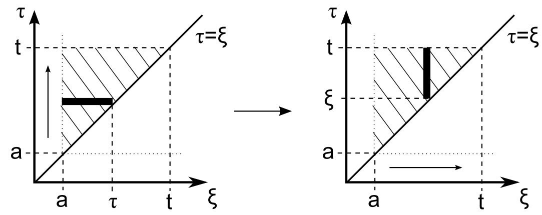


Figure 2.1: Geometrical illustration - change of order of integration.

Derivative of Integrals Depending on a Parameter

As we will see later, differintegrals are mostly given in a form of an integral depending on a parameter which is equal to the upper limit of integration. Hence it is very important to know the rule for the derivative w.r.t. this parameter.

It can be proven that the following formula holds when the integrated function $g(t,\tau)$ is integrable w.r.t. the second variable, its derivative $\frac{\partial}{\partial t}g(t,\tau)$ is continuous and $g(t,\tau)$ is defined in all points (t,t).

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t g(t,\tau)\,\mathrm{d}\tau = g(t,t) + \int_a^t \frac{\partial}{\partial t}g(t,\tau)\,\mathrm{d}\tau$$

In fractional calculus we often work with functions of the type $g(t,\tau) = (t-\tau)^r f(\tau)$ for some $r \geq 0$, thus let us look at the result in such situation. The case $r = 0$ is quite trivial (we simply obtain $f(t)$), otherwise we get the formula (2.2) bellow since in this case $g(t,t) = 0$ for all t.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t (t-\tau)^r f(\tau)\,\mathrm{d}\tau = r\int_a^t (t-\tau)^{r-1} f(\tau)\,\mathrm{d}\tau \tag{2.2}$$

2.2 The Gamma Function

In the integer-order calculus the factorial plays an important role because it is one of the most fundamental combinatorial tools. The Gamma function has the same importance in the fractional-order calculus and it is basically given by integral

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t} t^{z-1}\,\mathrm{d}t. \tag{2.3}$$

The exponential provides the convergence of this integral in ∞, the convergence at zero obviously occurs for all complex z from the right half of the complex plane ($\mathfrak{Re}(z) > 0$).

Other generalizations for values in the left half of the complex plane can be obtained in following way (see [1]). If in (2.3) we substitute e^{-t} by the well-known limit

$$\mathrm{e}^{-t} = \lim_{n\to\infty}\left(1 - \frac{t}{n}\right)^n$$

and then use n-times integration by parts, we obtain the following limit definition of the Gamma function (2.4).

$$\Gamma(z) = \lim_{n\to\infty}\frac{n!\,n^z}{z(z+1)\dots(z+n)} \tag{2.4}$$

Even if this expression was derived for $\mathfrak{Re}(z) > 0$, it is possible to use it as well as a definition of the Gamma function at points with negative real part except negative integer numbers. So now the Gamma function is defined for all $z \in \mathbb{C} - \{0, -1, -2, \dots\}$. Moreover in the sense of complex analysis the negative integers are simple poles of $\Gamma(z)$ (proof in [1]). For a better understanding the graph of $\Gamma(x)$ for real values of x is given in figure 2.2.

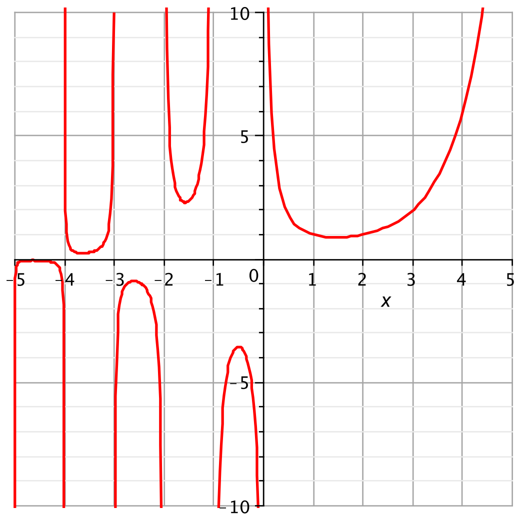


Figure 2.2: Graph of the Gamma function $\Gamma(x)$ in a real domain.

In many formulas the reciprocal Gamma function occurs, so it is reasonable to define it simply by (2.5). In this way we also avoided the problem in negative integers, i.e. the function $\frac{1}{\Gamma(z)}$ is defined for all complex z (especially for real values see figure 2.3 bellow).

$$\frac{1}{\Gamma(z)} = \lim_{n\to\infty} \frac{z(z+1)\dots(z+n)}{n!\, n^z} \tag{2.5}$$

The main property of the factorial is $(n+1)n! = (n+1)!$. Of course an analogous rule holds for the Gamma function. In fact it can be proved from the definition (2.3) by integrating by parts that

$$\Gamma(z+1) = z\,\Gamma(z). \tag{2.6}$$

Despite we derived (2.6) only for points in the right half of the complex plane, it follows from (2.4) that this formula holds more generally even for points z for which $-m < \mathfrak{Re}(z) \leq -m+1$ where $m \in \mathbb{N}$ since

$$\Gamma(z) = \frac{\Gamma(z+m)}{z(z+1)\dots(z+m-1)}.$$

This formula immediately implies that it is possible to calculate all values of the Gamma function if we know its values e.g. at the interval $(0, 1\rangle$.

It is natural to expect a connection between the Gamma function and the factorial. This is provided by the formula (2.6) and by the fact that $\Gamma(1) = 1$:

$$\Gamma(n+1) = n! \quad \text{for} \quad n \in \mathbb{N}_0. \tag{2.7}$$

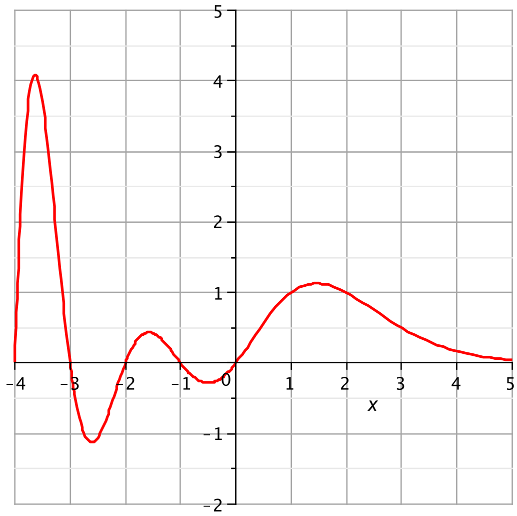


Figure 2.3: Graph of the reciprocal Gamma function $\frac{1}{\Gamma(x)}$ in a real domain.

2.3 The Beta Function

The Beta function is very important for the computation of the fractional derivatives of the power function. It is defined by the two-parameter integral

$$B(z,w) = \int_0^1 \tau^{z-1}(1-\tau)^{w-1}\,\mathrm{d}\tau \tag{2.8}$$

for z, w satisfying $\mathfrak{Re}(z) > 0$ and $\mathfrak{Re}(w) > 0$.

If we use the Laplace transform for convolutions, see (2.21), we get a relation between the Beta function and the Gamma function (see [1]) which implies $B(z,w) = B(w,z)$.

$$B(z,w) = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)} \tag{2.9}$$

By the help of the Beta function some useful results about the Gamma function can be obtained (for the proof see [1]):

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{2.10}$$

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}\,2^{1-2z}\,\Gamma(2z) \tag{2.11}$$

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,(2n)!}{2^{2n}\,n!} \tag{2.12}$$

2.4 Mittag-Leffler Functions

The exponential function e^z is very important in the theory of integer-order differential equations. We can write it in a form of series:

$$\mathrm{e}^z = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)}.$$

The generalizations of this function, so called functions of the Mittag-Leffler type, play an important role in the theory of fractional differential equations (FDEs). First we introduce a two-parameter Mittag-Leffler function defined by formula (2.13).

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0,\ \beta \in \mathbb{R} \tag{2.13}$$

For special choices of the values of the parameters α, β we obtain well-known classical functions, e.g.:

$$\begin{aligned} E_{1,1}(z) &= \mathrm{e}^z, \\ E_{1,2}(z) &= \frac{\mathrm{e}^z - 1}{z}, \\ E_{2,1}(z^2) &= \cosh(z), \\ E_{2,2}(z^2) &= \frac{\sinh(z)}{z}, \\ E_{\frac{1}{2},1}(z) &= \mathrm{e}^{z^2}\operatorname{erfc}(-z), \quad \text{etc.} \end{aligned}$$

As we will see later, classical derivatives of the Mittag-Leffler function appear in solution of FDEs. Since the series (2.13) is uniformly convergent we may differentiate term by term and obtain

$$E^{(m)}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{t^k}{\Gamma(\alpha k + \alpha m + \beta)} \tag{2.14}$$

Sometimes other generalizations of the exponential are needed, so called three-parameter Mittag-Leffler function with parameters α, m, l satisfying the conditions $\alpha > 0$, $m > 0$ and $\alpha(jm + l) \notin \mathbb{Z}^-$.

$$E_{\alpha,m,l}(z) = 1 + \sum_{k=1}^{\infty} \left(\prod_{j=0}^{k-1} \frac{\Gamma(\alpha(jm+l)+1)}{\Gamma(\alpha(jm+l+1)+1)} \right) z^k \tag{2.15}$$

In this thesis the functions of the Mittag-Leffler type are sufficient and we will not need more general functions. However the two-parameter functions of the Mittag-Leffler type are the special case of Wright functions and some authors prefer to use rather them. So only for completeness we add following section.

2.5 Wright Functions

In this section we briefly introduce the Wright function and mention some relations to functions of the Mittag-Leffler type, for more information see [2].

The general Wright function is defined by series:

$${}_p\Psi_q \left[\begin{array}{c} (a_l, \alpha_l) \\ (b_j, \beta_j) \end{array} \middle| \, z \right] = \sum_{k=0}^{\infty} \frac{\prod_{l=1}^{p} \Gamma(a_l + \alpha_l k)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k)} \frac{z^k}{k!} \tag{2.16}$$

where $z, a_l, b_j \in \mathbb{R}$, $\alpha_l, \beta_j \in \mathbb{R}$. In particular, if the condition $\sum_{j=1}^{q} \beta_j - \sum_{l=1}^{p} \alpha_l > -1$ is satisfied, the series is convergent for any $z \in \mathbb{R}$.

The link between Wright functions and functions of the Mittag-Leffler type is as follows:

$$\begin{aligned} E_{\alpha,\beta}(z) &= {}_1\Psi_1 \left[\begin{array}{c} (1,1) \\ (\beta,\alpha) \end{array} \middle| z \right], \\ E_{\alpha,\beta}^{(m)}(z) &= {}_1\Psi_1 \left[\begin{array}{c} (m+1,1) \\ (\alpha m+\beta,\alpha) \end{array} \middle| z \right]. \end{aligned}$$

2.6 The Laplace Transform

The Laplace transform is a very useful tool for solving linear ODEs with constant coefficients since it converts linear differential equations to linear algebraic equations which can be solved easily. The final step, the inverse transform of the result, is usually the most complicated part of this approach. We will see in section 3.6, that the situation with linear FDEs with constant coefficients is completely analogous.

There are a lot of books about the Laplace transform so we introduce here only the most important properties which we will need later in this thesis. That is the reason why we do not mention, e.g. the formula for the inverse Laplace transform. The skipped facts or properties can be found in [1], [2] or in more precise way in [3].

Let $f(t)$ be a complex function of one real variable, such that the Lebesgue integral (2.17) converges at least for one complex s. Then the function $f(t)$ is called original and the function $F(s)$ defined by (2.17) is called Laplace image of the function $f(t)$ and we denote it by $F(s) = \mathcal{L}\{f(t), t, s\}$.

$$F(s) = \int_0^\infty f(t)\, \mathrm{e}^{-st}\, \mathrm{d}t \tag{2.17}$$

We will very often express the Laplace transform of a function simply by the corresponding capital letter.

Properties

All these properties follow directly from the formula (2.17) and their proofs can be found mostly in [3]. We that the suppose functions satisfy appropriate necessary conditions.

- Generalized linearity (if the series $\sum_{k=0}^{\infty} a_k f_k(t)$ is uniformly convergent)

$$\mathcal{L}\left\{ \sum_{k=0}^{\infty} a_k f_k(t), t, s \right\} = \sum_{k=0}^{\infty} a_k F_k(s). \tag{2.18}$$

- The image of derivatives

$$\mathcal{L}\left\{ \frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}, t, s \right\} = s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0). \tag{2.19}$$

- The image of integrals

$$\mathcal{L}\left\{\int_0^t f(\tau)\mathrm{d}\tau, t, s\right\} = \frac{F(s)}{s}. \tag{2.20}$$

- The image of convolutions

$$\mathcal{L}\left\{f(t) * g(t), t, s\right\} = F(s)\, G(s). \tag{2.21}$$

Here the convolution is defined by

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau \tag{2.22}$$

It is obvious that the convolution is commutative, associative and distributive w.r.t. summation. The principles of the Laplace transform will become clearer in sections 3.6 and 5.1 where we will make extensive use of it.

The Laplace Transform of Some Functions

Now we calculate the Laplace transforms of the functions we will need in the sequel, mainly for solving FDEs.

The most important function in the fractional calculus is the general power function. We calculate its Laplace image directly from the definition, where we assume that $\alpha > -1$.

$$\mathcal{L}\{t^\alpha, t, s\} = \int_0^\infty t^\alpha \mathrm{e}^{-st}\,\mathrm{d}t = \left| \begin{array}{c} st = r \\ \mathrm{d}t = \frac{\mathrm{d}r}{s} \end{array} \right| = \frac{1}{s^{\alpha+1}} \int_0^\infty r^{1+\alpha-1}\mathrm{e}^{-r}\,\mathrm{d}r = \frac{\Gamma(1+\alpha)}{s^{\alpha+1}} \tag{2.23}$$

Before we will derive the Laplace transform of the most important function in the theory of two-terms linear FDEs, we consider the following series.

$$\sum_{k=0}^\infty \frac{(k+m)!}{k!}\,x^k = \sum_{k=0}^\infty (m+k)\dots(k+1)\,x^k = \sum_{k=m}^\infty k(k-1)\dots(k-m+1)\,x^{k-m} =$$

$$= \frac{\mathrm{d}^m}{\mathrm{d}t^m}\sum_{k=m}^\infty x^k = \left| \begin{array}{c} \text{We can add the first } m \text{ terms to the sum} \\ \text{because they disappear after differentiation} \\ \text{so the result remains unchanged} \\ \text{(degrees of added terms are less than } m\text{)} \end{array} \right| = \frac{\mathrm{d}^m}{\mathrm{d}t^m}\sum_{k=0}^\infty x^k =$$

$$= \frac{\mathrm{d}^m}{\mathrm{d}t^m}\frac{1}{1-x} = \frac{m!}{(1-x)^{m+1}}$$

If we use this summation, linearity of the Laplace transform and the formula (2.23), we can derive (remember $\alpha, \beta > 0$ in (2.14)) for $\mathfrak{Re}(s) > |a|^{\frac{1}{\alpha}}$:

$$\mathcal{L}\left\{t^{\alpha m+\beta-1}E^{(m)}_{\alpha,\beta}(at^\alpha), t, s\right\} = \mathcal{L}\left\{t^{\alpha m+\beta-1}\sum_{k=0}^\infty \frac{(k+m)!}{k!}\frac{a^k t^{\alpha k}}{\Gamma(\alpha k+\alpha m+\beta)}, t, s\right\} =$$

$$= \sum_{k=0}^\infty \frac{(k+m)!\,a^k}{k!\,\Gamma(\alpha k+\alpha m+\beta)}\mathcal{L}\{t^{\alpha k+\alpha m+\beta-1}, t, s\} = \sum_{k=0}^\infty \frac{(k+m)!}{k!}\frac{a^k}{s^{\alpha k+\alpha m+\beta}} =$$

$$= s^{-\alpha m-\beta}\sum_{k=0}^\infty \frac{(k+m)!}{k!}\left(\frac{a}{s^\alpha}\right)^k = s^{-\alpha m-\beta}\frac{m!}{\left(1-\frac{a}{s^\alpha}\right)^{m+1}} = \frac{m!\,s^{\alpha-\beta}}{(s^\alpha-a)^{m+1}}. \tag{2.24}$$

2.7 The Fourier Transform

The Fourier transform is used, e.g. for solving partial differential equations. We will need it only in some applications of the fractional calculus so we only give the most important formulas. For further facts we recommend the same books like for the Laplace transform, i.e. [1], [2], [3].

Let $f(x)$ be a real function of one real variable, such that its Lebesgue integral over the real numbers converges, and such that $f(x)$ with its derivative are piecewise continuous. Then the function $f(x)$ is called original and the function $\hat{f}(k)$ defined by (2.25) Fourier image of the function $f(x)$ and we denote it by $\hat{f}(k) = \mathcal{F}\{f(x), x, k\}$.

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \tag{2.25}$$

For Fourier images we will use same letters like for the original function with hat and the variable k.

Main Properties

As we see from the definition, the Fourier transform is quite similar to the Laplace transform, thus they have many properties in common. We suppose functions to satisfy the appropriate necessary conditions.

- Linearity

$$\mathcal{F}\left\{a\,f(x) + b\,g(x), x, k\right\} = a\,\hat{f}(k) + b\,\hat{f}(k). \tag{2.26}$$

- The image of derivatives

$$\mathcal{F}\left\{\frac{\mathrm{d}^n f(x)}{\mathrm{d}t^n}, x, k\right\} = (\mathrm{i}k)^n \hat{f}(k). \tag{2.27}$$

- The image of convolutions

$$\mathcal{F}\left\{f(x) * g(x), x, k\right\} = \hat{f}(k)\,\hat{g}(k). \tag{2.28}$$

In the context of the Fourier transform, we usually intend by the convolution the expression

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(x-\xi) g(\xi)\,\mathrm{d}\xi. \tag{2.29}$$

Chapter 3

Basic Fractional Calculus

The main objects of classical calculus are derivatives and integrals of functions - these two operations are inverse to each other in some sense. If we start with a function $f(t)$ and put its derivatives on the left-hand side and on the right-hand side we continue with integrals, we obtain a both-side infinite sequence.

$$\dots \frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2},\ \frac{\mathrm{d}f(t)}{\mathrm{d}t},\ f(t),\ \int_a^t f(\tau)\,\mathrm{d}\tau,\ \int_a^t \int_a^{\tau_1} f(\tau)\,\mathrm{d}\tau\,\mathrm{d}\tau_1,\ \dots$$

Fractional calculus tries to interpolate this sequence so this operation unifies the classical derivatives and integrals and generalizes them for arbitrary order. We will usually speak of differintegral, but sometimes the name α-derivative (α is an arbitrary real number) which can mean also an integral if $\alpha < 0$, is also used, or we talk directly about fractional derivative and fractional integral.

There are many ways to define the differintegral and these approaches are called according to their authors. For example the Grünwald-Letnikov definition of differintegral starts from classical definitions of derivatives and integrals based on infinitesimal division and limit. The disadvantages of this approach are its technical difficulty of the computations and the proofs and large restrictions on functions. Fortunately there are other, more elegant approaches like the Riemann-Liouville definition which includes the results of the previous one as a special case.

In this thesis we will focus on the Riemann-Liouville, the Caputo and the Miller-Ross definitions since they are the most used ones in applications. We will formulate the conditions of their equivalence and derive the most important properties. Finally we will consider some examples of differintegrals for elementary functions.

3.1 The Riemann-Liouville Differintegral

The Riemann-Liouville approach is based on the Cauchy formula (3.1) for the n^{th} integral which uses only a simple integration so it provides a good basis for generalization.

$$I_a^n f(t) = \int_a^t \int_a^{\tau_{n-1}} \dots \int_a^{\tau_1} f(\tau)\,\mathrm{d}\tau\,\mathrm{d}\tau_1 \dots \mathrm{d}\tau_{n-1} = \frac{1}{(n-1)!}\int_a^t (t-\tau)^{n-1} f(\tau)\,\mathrm{d}\tau \tag{3.1}$$

Proof. The formula (3.1) can be proven by the help of mathematical induction. The case $n = 1$ is obviously fulfilled, so we show the case $n = 2$ which demonstrates the mechanism of the entire proof in a better way.

Let us substitute $n = 2$ into (3.1) and compute:

$$\frac{1}{1!}\int_a^t (t-\tau)f(\tau)\,\mathrm{d}\tau = \begin{vmatrix} u = t-\tau & u' = -1 \\ v' = f(\tau) & v = \int_a^\tau f(r)\,\mathrm{d}r \end{vmatrix} =$$
$$= \left[(t-\tau)\int_a^\tau f(r)\,\mathrm{d}r\right]_{\tau=a}^{\tau=t} + \int_a^t \int_a^\tau f(r)\,\mathrm{d}r = I_a^2 f(t).$$

The first term is zero because in the upper limit the polynomial is zero while in the lower one we integrate over a set of measure zero.

Now we suppose the formula holds for general n. Then we integrate it once more and see what we obtain:

$$\int_a^t I_a^n f(r)\,\mathrm{d}r = \int_a^t \frac{1}{(n-1)!}\int_a^r (r-\tau)^{n-1} f(\tau)\,\mathrm{d}\tau\,\mathrm{d}r = \begin{vmatrix} \text{change of order} \\ \text{of integration} \end{vmatrix} =$$
$$= \frac{1}{(n-1)!}\int_a^t f(\tau)\int_\tau^t (r-\tau)^{n-1}\,\mathrm{d}r\,\mathrm{d}\tau = \frac{1}{(n-1)!}\int_a^t f(\tau)\left[\frac{(r-\tau)^n}{n}\right]_\tau^t \mathrm{d}\tau =$$
$$= \frac{1}{n!}\int_a^t (t-\tau)^n f(\tau)\,\mathrm{d}\tau = I_a^{n+1} f(t).$$

This completes the proof of the Cauchy formula (3.1). □

Remark. The only property of the function $f(t)$ we used during the proof was its integrability. No other restrictions are imposed.

Now it is obvious how to get an integral of arbitrary order. We simply generalize the Cauchy formula (3.1) - the integer n is substituted by a positive real number α and the Gamma function is used instead of the factorial. Notice that the integrand is still integrable because $\alpha - 1 > -1$.

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau \tag{3.2}$$

This formula represents the integral of arbitrary order $\alpha > 0$, but does not permit order $\alpha = 0$ which formally corresponds to the identity operator. This expectation is fulfilled under certain reasonable assumptions at least if we consider the limit for $\alpha \to 0$ (see [1]). Hence, we extend the above definition by setting:

$$I_a^0 f(t) = f(t). \tag{3.3}$$

The definition of fractional integrals is very straightforward and there are no complications. A more difficult question is how to define a fractional derivative. There is no formula for the n^{th} derivative analogous to (3.1) so we have to generalize the derivatives through a fractional integral. First we perturb the integer order by a fractional integral according to (3.2) and then apply an appropriate number of classical derivatives. As we will see later (the formula (3.22)), we can always choose the order of perturbation less than 1.

The result of these ideas is the following ($\alpha > 0$):

$$\mathbf{D}_a^\alpha f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[I_a^{n-\alpha} f(t)\right] = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau, \tag{3.4}$$

where $n = [\alpha] + 1$. This formula includes even the integer order derivatives. If $\alpha = k$ and $k \in \mathbb{N}_0$ then $n = k + 1$ and we obtain:

$$\mathbf{D}_a^k f(t) = \frac{1}{\Gamma(1)} \frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}} \int_a^t f(\tau)\,\mathrm{d}\tau = \frac{\mathrm{d}^k f(t)}{\mathrm{d}t^k}.$$

We can see that classical derivatives are something like singularities among differintegrals because the integration disappears and so there is no dependence on the lower bound a anymore. In this sense the classical derivatives are the only differintegrals which do not depend on history, i.e. are local.

If we put $\mathbf{D}_a^{-\alpha} = I_a^\alpha$ and note that $f^{(0)}(t) = f(t)$, we can write both fractional integral and derivative by one expression and formulate the definition of the Riemann-Liouville differintegral.

Definition 3.1.1 (The Riemann-Liouville differintegral). Let a, T, α be real constants $(a < T)$, $n = \max(0, [\alpha] + 1)$ and $f(t)$ an integrable function on $\langle a, T)$. For $n > 0$ additional assume that $f(t)$ is n-times differentiable on $\langle a, T)$ except on a set of measure zero. Then the Riemann-Liouville differintegral is defined for $t \in \langle a, T)$ by the formula:

$$\mathbf{D}_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau. \tag{3.5}$$

Remark. In this thesis we will denote the differintegrals by various symbols according to the used approach. For the Riemann-Liouville approach the bold face capital letter $\mathbf{D}$ is reserved from now on.

3.2 The Caputo Differintegral

We will denote the Caputo differintegral by the capital letter with upper-left index ${}^{\mathrm{C}}\mathrm{D}$. The fractional integral is given by the same expression like before, so for $\alpha > 0$ we have

$${}^{\mathrm{C}}\mathrm{D}_a^{-\alpha} f(t) = \mathbf{D}_a^{-\alpha} f(t). \tag{3.6}$$

The difference occurs for fractional derivative. A non-integer-order derivative is again defined by the help of the fractional integral, but now we first differentiate $f(t)$ in the common sense and then go back by fractional integrating up to the required order. This idea leads to the following definition of the Caputo differintegral.

Definition 3.2.1 (The Caputo differintegral). Let a, T, α be real constants $(a < T)$, $n_c = \max(0, -[-\alpha])$ and $f(t)$ a function which is integrable on $\langle a, T)$ in case $n_c = 0$ and n_c-times differentiable on $\langle a, T)$ except on a set of measure zero in case $n_c > 0$. Then the Caputo differintegral is defined for $t \in \langle a, T)$ by formula:

$${}^{\mathrm{C}}\mathrm{D}_a^\alpha f(t) = I_a^{n_c-\alpha} \left[\frac{\mathrm{d}^{n_c} f(t)}{\mathrm{d}t^{n_c}} \right]. \tag{3.7}$$

Remark. For $\alpha > 0$, $\alpha \notin \mathbb{N}_0$, formula (3.7) is often written in the form:

$${}^{\mathrm{C}}\mathrm{D}_a^\alpha f(t) = \frac{1}{\Gamma(n_c-\alpha)} \int_a^t (t-\tau)^{n_c-\alpha-1} f^{(n_c)}(\tau)\,\mathrm{d}\tau. \tag{3.8}$$

The reason why n_c in the definition of the Caputo derivative is different from n introduced in the Riemann-Liouville case, is correspondence with integer-order derivatives. We cannot use n even in the Caputo definition because we would get wrong results for the k^{th} derivative of a function with zero $(k+1)^{\text{th}}$ derivative. This would be an effect of the paradox that we would need for the k^{th} derivative a $(k+1)$-times differentiable function.

On the contrary, we could use n_c with the Riemann-Liouville derivative, but will use n because then we do not need a limit relationship (3.3).

Anyway, the only difference between the values of n and n_c is for integers as we can see in figures 3.1 and 3.2. In addition, we know that both cases coincide with classical derivatives at those points, hence there should not be any problems.

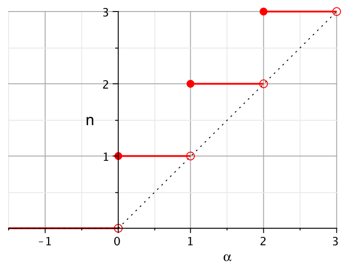


Figure 3.1: Function $n = [\alpha] + 1$ used for the Riemann-Liouville derivative.

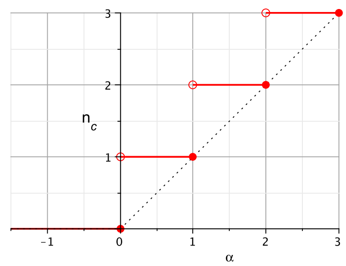


Figure 3.2: Function $n = -[-\alpha]$ used for the Caputo derivative.

Clearly, the Caputo derivative can also be written by the help of fractional integrals of the Riemann-Liouville type:

$$ {}^{\mathrm{C}}\mathrm{D}_a^{\alpha} f(t) = \mathbf{D}_a^{-(n_c-\alpha)} \left(\frac{\mathrm{d}^{n_c} f(t)}{\mathrm{d}t^{n_c}} \right). \tag{3.9} $$

Here we see that if we consider formula (3.3), the Caputo derivative of order $\alpha = n_c$ is equal to the classical n_c^{th} derivative.

The reasons which led to the definition of the Caputo derivative are mainly practical. As we will see in section 3.6, the Riemann-Liouville approach requires the initial conditions for differential equations in terms of non-integer derivatives which are hardly physical interpreted, whereas the Caputo approach uses integer-order initial conditions. Moreover, we sometimes also need fractional derivatives of constants to be zero. The Riemann-Liouville derivative with finite lower bound a does not satisfy this while the Caputo derivative does. More about the correspondence of these approaches can be found in subsection 3.5.2.

3.3 Sequential Fractional Derivatives

For sequential fractional derivatives, or so called Miller-Ross fractional derivative, we will use the calligraphic capital letter $\mathcal{D}$. The main idea of this type of derivative arises from the method of how the derivatives of higher orders are defined. In order to get the n^{th} derivative, we simply apply the first derivative n times, so the first derivative is the only one we really need.

Analogously we can obtain a fractional derivative by sequent application of differintegrals of other orders. Obviously we have to define this derivatives first. For this purpose we will use the Riemann-Liouville fractional derivatives. Of course there is infinite number of ways how to decompose a given order of derivative, so in this thesis we will think about the sequential derivative in the following form:

$$\mathcal{D}_a^{\sigma} f(t) \equiv \mathbf{D}_a^{-\alpha_0} \mathbf{D}_a^{\alpha_n} \mathbf{D}_a^{\alpha_{n-1}} \dots \mathbf{D}_a^{\alpha_1} f(t), \quad \sigma = \sum_{j=1}^{n} \alpha_j - \alpha_0 \text{ and } \alpha_j \in (0,1\rangle,\ \alpha_0 \in \langle 0,1). \tag{3.10}$$

Apparently it is possible to look at the previous two definitions of fractional derivatives in a sequential language. The natural sequence for the Riemann-Liouville derivative is

$$\mathbf{D}_a^{\alpha} f(t) = \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}t}\dots\frac{\mathrm{d}}{\mathrm{d}t}}_{n} \mathbf{D}_a^{-(n-\alpha)} f(t),$$

but it disagrees with our conditions on orders in sequence (namely the last term is the integral). So also for practical purposes we write the Riemann-Liouville fractional derivative in this equivalent way:

$$\mathbf{D}_a^{\alpha} f(t) = \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}t}\dots\frac{\mathrm{d}}{\mathrm{d}t}}_{n-1} \mathbf{D}_a^{\alpha-n+1} f(t). \tag{3.11}$$

The sequence for the Caputo fractional derivative also uses integer-order derivatives but at the other side. Here the integral is first in the sequence so there is no problem with our definition:

$${}^{\mathrm{C}}\mathrm{D}_a^{\alpha} f(t) = \mathbf{D}_a^{-(n-\alpha)} \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}t}\dots\frac{\mathrm{d}}{\mathrm{d}t}}_{n=n_c} f(t). \tag{3.12}$$

Consequently the sequential derivative can be used for contemporary describing the Riemann-Liouville and the Caputo derivatives (see subsection 5.1.1). In practice the sequential derivative appears naturally when we substitute one expression containing a derivative into another one.

Remark. We do not consider sequences of integrals (i.e. $\sigma < 0$) because its result is independent on concrete choice of the sequence, in fact it depends only on the order of whole integration as we will see in subsection 3.5.3.

3.4 The Right Differintegral

All definitions given in the previous sections were so called left differintegrals. The origin of this name is clear because we calculate the value of the differintegral at point t by the help of points on the left of it. If t means time, it seems to be logical since we use in fact the history of the function $f(t)$ and the future does not need to be known yet. On the other side, if t plays the role of a spatial variable, there is no reason why events on the left should be more important than those on the right.

In this thesis we may usually consider t to be time, so mostly we will not need right differintegrals. The only exception occurs only in the chapter about applications where we will use the right Riemann-Liouville derivative according to the spatial variable.

This is the reason why we do not introduce right differintegrals for all approaches, but only the following formula for the right Riemann-Liouville derivative (we will denote the right fractional derivative by left bottom index $-$), $n = [\alpha] + 1$:

$$
{}_{-}\mathbf{D}_b^\alpha f(t) = \frac{1}{n-\alpha} \left(-\frac{\mathrm{d}}{\mathrm{d}t} \right)^n \int_t^b (\tau - t)^{n-\alpha-1} f(\tau) \,\mathrm{d}\tau. \tag{3.13}
$$

3.5 Basic Properties of Differintegrals

In this section we will discuss the linearity of fractional derivatives and derive some rules for the composition of fractional derivatives and integrals. At the end we will mention a problem of the equivalence of the approaches and continuity with respect to the order of derivative.

In spite of their existence, we will not consider here the fractional versions of the Leibniz rule (the derivative of the product of functions) and the formula for the derivative of composite functions, because these relations are complicated and we do not need them in this thesis. Both of this formulas can be found in [1].

3.5.1 Linearity

All definitions considered in this thesis use only convolutions and derivatives. Because both of these operations are linear, differintegrals are also linear in all approaches.

Let λ, μ, α, a be real constants and $f(t)$, $g(t)$ arbitrary functions for which the needed operations are defined. Then the linearity can be express by the formula:

$$
\mathfrak{D}_a^\alpha (\lambda f(t) + \mu g(t)) = \lambda \mathfrak{D}_a^\alpha f(t) + \mu \mathfrak{D}_a^\alpha g(t), \tag{3.14}
$$

where $\mathfrak{D}$ indicates one of the differintegrals introduced above.

Now the natural question arises, whether differintegrals can be applied to an infinite series term by term. All differintegrals are in fact combinations of convolutions and integer-order derivatives, so we can use the theory of integer-order calculus. For better understanding let us prove it for the Riemann-Liouville derivative ($\alpha > 0$, $n = [\alpha] + 1$). Necessary conditions will grow up during computation.

$$
\begin{aligned}
\mathbf{D}_a^\alpha \sum_{k=0}^{\infty} f_k(t) &= \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t (t-\tau)^{n-\alpha-1} \sum_{k=0}^{\infty} f_k(\tau) \,\mathrm{d}\tau = \left| \begin{array}{c} \text{power term plays role} \\ \text{of a constant with} \\ \text{respect to the sum} \end{array} \right| = \\
&= \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t \sum_{k=0}^{\infty} (t-\tau)^{n-\alpha-1} f_k(\tau) \,\mathrm{d}\tau = \left| \begin{array}{c} \text{if original series was} \\ \text{uniformly convergent,} \\ \text{this one has same property} \end{array} \right| = \\
&= \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \sum_{k=0}^{\infty} \int_a^t (t-\tau)^{n-\alpha-1} f_k(\tau) \,\mathrm{d}\tau = \left| \begin{array}{c} \text{if this new sum and sum of } n^{\text{th}} \\ \text{derivatives of its terms are} \\ \text{both uniformly convergent} \end{array} \right| = \\
&= \frac{1}{\Gamma(n-\alpha)} \sum_{k=0}^{\infty} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t (t-\tau)^{n-\alpha-1} f_k(\tau) \,\mathrm{d}\tau = \sum_{k=0}^{\infty} \mathbf{D}_a^\alpha f_k(t)
\end{aligned}
$$

Similarly we can derive a general formula (3.15), where the fundamental condition is uniform convergence of original series. However other conditions depend on the concrete construction of the operator $\mathfrak{D}_a^\alpha$. The idea is that we have to deal with uniformly convergent series all the time.

$$\mathfrak{D}_a^\alpha \sum_{k=0}^{\infty} f_k(t) = \sum_{k=0}^{\infty} \mathfrak{D}_a^\alpha f_k(t) \tag{3.15}$$

Especially for finite lower bound a we have to care only about the uniform convergence connected with derivatives, because integrals stay uniformly convergent automatically.

3.5.2 Equivalence of the Approaches

As we will show in subsection 3.5.3, the operation of fractional differentiation is not commutative. It seems that this subsection is only a special case of the following one, but the equivalence of the Riemann-Liouville and the Caputo approach is a fundamental question so we discuss it separately here.

We impose $f(t)$ to be $(n-1)$-times continuously differentiable and $f^{(n)}(t)$ to be integrable, because we use the integration by parts. We suppose as usual $\alpha > 0$, $n = [\alpha] + 1$, but $\alpha \neq n - 1$ so $n = n_c$.

$$\begin{aligned}
\mathbf{D}_a^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau = \left| \begin{array}{cc} u = f(\tau) & u' = f'(\tau) \\ v' = (t-\tau)^{n-\alpha-1} & v = -\frac{(t-\tau)^{n-\alpha}}{n-\alpha} \end{array} \right| = \\
&= \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left[\frac{(t-a)^{n-\alpha} f(a)}{n-\alpha} + \int_a^t \frac{(t-\tau)^{n-\alpha}}{n-\alpha} f'(\tau)\,\mathrm{d}\tau \right] = \left| \begin{array}{c} n-1 \text{ times} \\ \text{integration} \\ \text{by parts} \end{array} \right| = \\
&= \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left[\sum_{k=0}^{n-1} \frac{(t-a)^{n+k-\alpha} f^{(k)}(a)}{\Gamma(n+k-\alpha+1)} + \frac{1}{\Gamma(2n-\alpha)} \int_a^t (t-\tau)^{2n-\alpha-1} f^{(n)}(\tau)\,\mathrm{d}\tau \right] = \\
&= \sum_{k=0}^{n-1} \frac{(t-a)^{k-\alpha} f^{(k)}(a)}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\,\mathrm{d}\tau = \\
&= \sum_{k=0}^{n-1} \frac{(t-a)^{k-\alpha} f^{(k)}(a)}{\Gamma(k-\alpha+1)} + {}^{\mathrm{C}}\mathrm{D}_a^\alpha f(t)
\end{aligned} \tag{3.16}$$

So we see that there occur additional terms in the sum that do not lead to the perfect equivalence of these two approaches. Of course for a special function for which $f^{(k)}(a) = 0$ for all $k = 0, \ldots, n-1$ the sum disappears, but in general we have to specify α or a.

The order of the derivative removes all terms only if $\alpha \in \mathbb{N}$, that is already known. Indeed this situation was omitted from the derivation at the beginning, but it is contained here as a limit case.

A more interesting situation occurs when $a \to -\infty$ because due to $k - \alpha < 0$ for all k, the power functions are zero for all values α and we obtain:

$$\mathbf{D}_{-\infty}^\alpha f(t) = {}^{\mathrm{C}}\mathrm{D}_{-\infty}^\alpha f(t). \tag{3.17}$$

So in general the Caputo and the Riemann-Liouville derivatives are unequal, however if we send the lower bound a to minus infinity, both approaches coincide. The disadvantage of this way is the fact that then a function $f(t)$ has to be integrable on a semi-infinite interval. On the contrary if a is a finite real number then for functions without $f^{(k)}(a) = 0$ for all $k = 0, \dots, n-1$ at least one approach leads to an infinite value of the fractional derivative in a neighborhood of the point a. For integer values of α both approaches coincide with the classical derivative as we mentioned before.

3.5.3 Composition

In this subsection we will derive rules for composition of differintegrals. You can expect some problems due to the definition of the sequential derivative because differences between various sequences of Riemann-Liouville derivatives are its essence.

First let us look at the composition of integrals because they are defined in the same way in both approaches. To point out the independence of the approach we will use the symbol $\mathfrak{D}$. We choose $a \in \mathbb{R}$, $\alpha, \beta > 0$, $f(t)$ an integrable function. During the computation we use the change of order of integration and the Beta function (2.8).

$$\begin{aligned}
\mathfrak{D}_a^{-\alpha}\left(\mathfrak{D}_a^{-\beta} f(t)\right) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\xi)^{\alpha-1} \left(\frac{1}{\Gamma(\beta)} \int_a^\xi (\xi-\tau)^{\beta-1} f(\tau)\,\mathrm{d}\tau \right) \mathrm{d}\xi = \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \int_a^\xi (t-\xi)^{\alpha-1} (\xi-\tau)^{\beta-1} f(\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi = \left| \begin{array}{c} \text{change of order} \\ \text{of integration} \end{array} \right| = \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t f(\tau) \int_\tau^t (t-\xi)^{\alpha-1} (\xi-\tau)^{\beta-1}\,\mathrm{d}\xi\,\mathrm{d}\tau = \left| \begin{array}{c} \frac{\xi-\tau}{t-\tau} = z \\ \mathrm{d}\tau = (t-\tau)\,\mathrm{d}z \\ z: 0 \to 1 \end{array} \right| = \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t f(\tau)(t-\tau)^{\alpha+\beta-1} B(\alpha,\beta)\,\mathrm{d}\tau = \left| \begin{array}{c} \text{according to (2.9)} \\ B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}) \end{array} \right| = \\
&= \frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-\tau)^{\alpha+\beta-1} f(\tau)\,\mathrm{d}\tau = \mathfrak{D}_a^{-(\alpha+\beta)} f(t)
\end{aligned}$$

So we just proved that fractional integrals are commutative (exactly the same result we have in ordinary calculus), i.e.,

$$\mathfrak{D}_a^{-\alpha}\left(\mathfrak{D}_a^{-\beta} f(t)\right) = \mathfrak{D}_a^{-(\alpha+\beta)} f(t), \quad a \in \mathbb{R},\ \alpha, \beta > 0. \tag{3.18}$$

Riemann-Liouville derivatives

Now we will study various compositions of Riemann-Liouville differintegrals. It is clear directly from the definition of the Riemann-Liouville differintegral that

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m} \mathbf{D}_a^\alpha f(t) = \mathbf{D}_a^{\alpha+m} f(t) \quad \text{for } \alpha \in \mathbb{R},\ m \in \mathbb{N}_0. \tag{3.19}$$

The next very useful formula says that the Riemann-Liouville differentiation operator is a left inverse to the Riemann-Liouville integration operator of the same order.

$$\mathbf{D}_a^\alpha\left(\mathbf{D}_a^{-\alpha} f(t)\right) = f(t), \quad \alpha > 0 \tag{3.20}$$

For the proof we only need the definition of the Riemann-Liouville derivative (3.4) and formulas (3.18) and (3.19). The constant n is as usual $n = [\alpha] + 1$:

$$\mathbf{D}_a^\alpha\left(\mathbf{D}_a^{-\alpha} f(t)\right) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[\mathbf{D}_a^{-(n-\alpha)}\left(\mathbf{D}_a^{-\alpha} f(t)\right)\right] = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\mathbf{D}_a^{-n} f(t)\right) = f(t).$$

The inverse formula does not hold, because a sum of new terms occurs. As usual we choose $\alpha > 0$ and $n = [\alpha] + 1$:

$$\mathbf{D}_a^{-\alpha}\left(\mathbf{D}_a^{\alpha} f(t)\right) = f(t) - \sum_{k=1}^{n} \mathbf{D}_a^{\alpha-k} f(t)\big|_{t=a} \frac{(t-a)^{\alpha-k}}{\Gamma(\alpha-k+1)}. \tag{3.21}$$

The proof is a little more complicated so we skip a lot of steps. We mostly use formulas (3.18) and (3.19), the sum comes up using integration by parts (so we assume $f(t)$ sufficiently continuously differentiable).

$$\begin{aligned}
\mathbf{D}_a^{-\alpha}\left(\mathbf{D}_a^{\alpha} f(t)\right) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \mathbf{D}_a^{\alpha} f(\tau)\,\mathrm{d}\tau = \left| \begin{array}{c} \text{two formulas} \\ \text{(3.19)} \end{array} \right| = \\
&= \frac{1}{\Gamma(\alpha+1)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t (t-\tau)^{\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\tau^n} \left[\mathbf{D}_a^{-(n-\alpha)} f(\tau)\right] \mathrm{d}\tau = \left| \begin{array}{c} n \text{ times} \\ \text{integration} \\ \text{by parts} \end{array} \right| = \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{D}_a^{-(\alpha+1-n)} \left(\mathbf{D}_a^{-(n-\alpha)} f(t)\right) - \sum_{k=1}^{n} \frac{\mathrm{d}^{n-k}}{\mathrm{d}t^{n-k}} \mathbf{D}_a^{\alpha-n} f(t)\big|_{t=a} \frac{(t-a)^{\alpha-k}}{\Gamma(\alpha-k+1)} = \\
&= f(t) - \sum_{k=1}^{n} \mathbf{D}_a^{\alpha-k} f(t)\big|_{t=a} \frac{(t-a)^{\alpha-k}}{\Gamma(\alpha-k+1)}
\end{aligned}$$

With these preliminary relations we can solve more general problems. The first one is the fractional derivative of the fractional integral. Two situations may occur here:

- $\beta \geq \alpha \geq 0$: we need subsequently the relations (3.18) and (3.20)

$$\mathbf{D}_a^{\alpha}\left(\mathbf{D}_a^{-\beta} f(t)\right) = \mathbf{D}_a^{\alpha}\left[\mathbf{D}_a^{-\alpha}\left(\mathbf{D}_a^{-(\beta-\alpha)} f(t)\right)\right] = \mathbf{D}_a^{\alpha-\beta} f(t)$$

- $\alpha > \beta \geq 0$: here we introduce $n = [\alpha] + 1$ and then use the formulas (3.4), (3.18) and (3.19)

$$\mathbf{D}_a^{\alpha}\left(\mathbf{D}_a^{-\beta} f(t)\right) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[\mathbf{D}_a^{-(n-\alpha)}\left(\mathbf{D}_a^{-\beta} f(t)\right)\right] = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\mathbf{D}_a^{\alpha-\beta-n} f(t)\right) = \mathbf{D}_a^{\alpha-\beta} f(t).$$

Both cases lead to the same consequence and so we can summarize:

$$\mathbf{D}_a^{\alpha}\left(\mathbf{D}_a^{-\beta} f(t)\right) = \mathbf{D}_a^{\alpha-\beta} f(t), \quad \alpha, \beta \geq 0. \tag{3.22}$$

Next we consider the fractional integration of a fractional derivative. We should again split the computation into the two cases $\beta \geq \alpha \geq 0$ and $\alpha > \beta \geq 0$ but the process is in both cases exactly the same. The difference is only in justifying the first step - in one case we use (3.18), in second case (3.22). Then we need the formula (3.21) and the fractional derivative of the power function (3.36) which will be derived in section 3.8. Finally we denote $m = [\beta] + 1$ and suppose that $f(t)$ has continuous derivatives of sufficient order.

$$\mathbf{D}_a^{-\alpha}\left(\mathbf{D}_a^{\beta} f(t)\right) = \mathbf{D}_a^{\beta-\alpha}\left[\mathbf{D}_a^{-\beta}\left(\mathbf{D}_a^{\beta} f(t)\right)\right] = \mathbf{D}_a^{-\alpha+\beta} f(t) - \sum_{k=1}^{m} \mathbf{D}_a^{\beta-k} f(t)\big|_{t=a} \frac{(t-a)^{\alpha-k}}{\Gamma(1+\alpha-k)} \tag{3.23}$$

The terms in the sum are obviously consequences of the lower bound a. We can understand this more easily in a special case of this formula with an integer-order integral ($\alpha \in \mathbb{N}$) - then we see that during every integration of an order less than β a new term $\mathbf{D}_a^{\beta-k} f(t)\big|_{t=a}$ arises and after it the power of an appropriate polynomial increases with all other integrations. The terms do not arise if we would apply the differintegral of order $-\alpha+\beta$ directly. They can be removed only if there exists a point a at which $\mathbf{D}_a^{\beta-k} f(t)\big|_{t=a} = 0$ for all $k = 1, \ldots, m$. Of course some terms disappear also for $\alpha \in \mathbb{N}$ and $\alpha < m$ due to the zeros of the reciprocal Gamma function, but not all of them (we suppose $\alpha > 0$).

The last possibility we did not discuss is the fractional derivative of a fractional derivative. Let $\alpha, \beta > 0$ and $n = [\alpha] + 1$, $m = [\beta] + 1$. Then by using (3.19) and (3.23) we can derive the rule:

$$\mathbf{D}_a^{\alpha}\left(\mathbf{D}_a^{\beta} f(t)\right) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[\mathbf{D}_a^{-(n-\alpha)}\left(\mathbf{D}_a^{\beta} f(t)\right)\right] = \left|\begin{array}{c}\text{we use (3.23) on}\\ \text{expression in square}\\ \text{brackets, then derivative}\end{array}\right| =$$

$$= \mathbf{D}_a^{\alpha+\beta} f(t) - \sum_{k=1}^{m} \mathbf{D}_a^{\beta-k} f(t)\big|_{t=a} \frac{(t-a)^{-\alpha-k}}{\Gamma(1-\alpha-k)}. \tag{3.24}$$

There are also some additional terms, but now there exist more ways how to remove them. First we can see that for $\alpha \in \mathbb{N}_0$ the entire sum disappears due to the reciprocal Gamma function and we obtain formula (3.19). It tells us that the integer-order derivatives do not depend on history which is represented by the lower bound a. It leads us to the idea to consider the limit $a \to -\infty$ which makes history far and unimportant. Due to $-\alpha - k < 0$ it really causes that the entire sum disappears for all positive values of α. The last possibility is to look for a point a at which $\mathbf{D}_a^{\beta-k} f(t)\big|_{t=a} = 0$ for all $k = 1, \ldots, m$.

It is a good point to say that if $f(t)$ is $(m-1)$-times continuously differentiable and $f^{(m)}(t)$ is integrable (the assumption of both previous formulas), then the conditions $\mathbf{D}_a^{\beta-k} f(t)\big|_{t=a} = 0$ for all $k = 1, \ldots, m$ are equivalent to the conditions $f^{(k)}(a) = 0$ for all $k = 0, \ldots, m-1$. Due to the assumptions above it is possible to write the Riemann-Liouville derivative after some integration by parts in the form ($\alpha > 0$, $n = [\alpha] + 1$):

$$\mathbf{D}_a^{\alpha} f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(1+k-\alpha)} + \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\,\mathrm{d}\tau. \tag{3.25}$$

Caputo derivatives

Next we study compositions of the Caputo derivatives. Thanks to formula (3.9) we can apply the results for compositions of Riemann-Liouville differintegrals.

If we consider all relations mentioned above, we can derive e.g. the case of integrated derivatives ($\alpha, \beta > 0$):

$${}^{\mathrm{C}}\mathrm{D}_a^{-\alpha}\left({}^{\mathrm{C}}\mathrm{D}_a^{\beta} f(t)\right) = {}^{\mathrm{C}}\mathrm{D}_a^{-\alpha+\beta} f(t). \tag{3.26}$$

Other combinations of derivatives and integrals are very complicated and we will not introduce them here, because we do not need them in the sequel.

3.5.4 Continuity with Respect to the Order of Derivation

In this subsection we will consider the function $g(\alpha) = \mathfrak{D}_a^\alpha f(t)$. We naturally expect that $g(\alpha)$ is a continuous function. It is clear that complications may occur only at points which represent the integer-order derivatives. We will work with function $f(t)$ which has a sufficient number of continuous derivatives.

If we look properly at the formulas (3.4) and (3.7), we find out that in the first case we have to calculate only the limit on the left, due to the fact that existence of the limit on the right is implied directly by the definition, in the second case we are interested in both limits.

The case of the Riemann-Liouville derivative ($\alpha > 0$, $n = [\alpha] + 1$):

$$\begin{aligned}
\lim_{\alpha\to n^-} \mathbf{D}_a^\alpha f(t) &= \lim_{\alpha\to n^-} \left\{ \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau \right\} = \left| \begin{array}{c} (n+1)\text{-times} \\ \text{integration} \\ \text{by parts} \end{array} \right| = \\
&= \lim_{\alpha\to n^-} \left\{ \sum_{k=0}^{n} \frac{(t-a)^{k-\alpha} f^{(k)}(a)}{\Gamma(k-\alpha+1)} + \int_a^t \frac{(t-\tau)^{n-\alpha} f^{(n+1)}(\tau)}{\Gamma(n-\alpha+1)}\,\mathrm{d}\tau \right\} = \\
&= f^{(n)}(a) + \int_a^t f^{(n+1)}(\tau)\,\mathrm{d}\tau = f^{(n)}(t).
\end{aligned}$$

The Riemann-Liouville fractional derivatives and integrals are continuously passed through integer-order derivatives and integrals, if the concrete function has a continuous derivatives of sufficient order. As we will see in section 3.8, this does not hold in points of the discontinuity of some derivative involved.

For the Caputo derivative the calculation of the left limit is similar and even more easy, there is only one integration by parts ($\alpha > 0$, $n_c = -[-\alpha]$). In fact we prove a special case of formula (3.3).

$$\begin{aligned}
\lim_{\alpha\to n_c^-} {}^{\mathrm{C}}\mathrm{D}_a^\alpha f(t) &= \lim_{\alpha\to n_c^-} \left\{ \frac{1}{\Gamma(n_c-\alpha)} \int_a^t (t-\tau)^{n_c-\alpha-1} f^{(n_c)}(\tau)\,\mathrm{d}\tau \right\} = \left| \begin{array}{c} \text{integration} \\ \text{by parts} \end{array} \right| = \\
&= \lim_{\alpha\to n_c^-} \left\{ \frac{(t-a)^{n_c-\alpha} f^{(n_c)}(a)}{\Gamma(n_c-\alpha+1)} + \int_a^t \frac{(t-\tau)^{n_c-\alpha} f^{(n_c+1)}(\tau)}{\Gamma(n_c-\alpha+1)}\,\mathrm{d}\tau \right\} = \\
&= f^{(n_c)}(a) + \int_a^t f^{(n_c+1)}(\tau)\,\mathrm{d}\tau = f^{(n_c)}(t).
\end{aligned}$$

The right limit for the Caputo derivative is as follows:

$$\lim_{\alpha\to(n_c-1)^+} {}^{\mathrm{C}}\mathrm{D}_a^\alpha f(t) = \int_a^t f^{(n_c)}(\tau)\,\mathrm{d}\tau = f^{(n_c-1)}(t) - f^{(n_c-1)}(a).$$

This result destroys our hope in the continuity of the Caputo derivative w.r.t. α. The function $f(t)$ would have to fulfill $f^{(n_c-1)}(a) = 0$ and it seems like a very strong restriction. Anyway, most functions used in fractional calculus satisfy this condition so it is not such a big complication.

In addition we could expect this result because it coincides with one of the requirement for the Caputo derivative - the zero value of all derivatives of a constant function. For example, if we think about the function $f(t) = t^2$ and its derivatives with lower bound

$a = 0$, we observe this situation for the 2-derivative. If α approaches the to number 2 from the left, the α-derivatives (see derivatives of power function (3.37)) tend to the second derivative (constant function) $f^{(2)}(t) = 2$. But all derivatives of order $\alpha > 2$ ($n_c \geq 3$) are equal to zero as we demand from the Caputo derivative, hence the right limit is zero too. Obviously $f^{(2)}(t) - f^{(2)}(0) = 0$ so we just confirmed the limit relationship above.

3.6 Laplace Transforms of Differintegrals

In this section we will use the fact that the Riemann-Liouville and the Caputo derivatives are special cases of the sequential derivative. Due to the definition of the Laplace transform we put $a = 0$ in this section. First of all we calculate the Laplace transform of the fractional integral of order α ($\alpha > 0$). To this end we need formulas for the Laplace transform of the power function (2.23) and the convolution (2.21).

$$\mathcal{L}\{\mathbf{D}_0^{-\alpha} f(t), t, s\} = \mathcal{L}\left\{\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, t, s\right\} = \frac{F(s)}{s^\alpha} \tag{3.27}$$

This result corresponds with the classical formula (2.20) as we expected. Moreover the case $\alpha = 0$ can be included in this formula in spite of the process of computation excludes it.

Due to the construction of the sequential derivative in (3.10) we are interested in the Laplace transform of the Riemann-Liouville derivative of order $\alpha \in (0, 1\rangle$. We can use for it the formula (3.4) and then (3.27) and (2.19).

$$\begin{aligned}\mathcal{L}\{\mathbf{D}_0^{\alpha} f(t), t, s\} &= \mathcal{L}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{D}_0^{\alpha-1} f(t), t, s\right\} = s\mathcal{L}\{\mathbf{D}_0^{\alpha-1} f(t), t, s\} - \mathbf{D}_0^{\alpha-1} f(t)\big|_{t=0} = \\ &= s^\alpha F(s) - \mathbf{D}_0^{\alpha-1} f(t)\big|_{t=0}\end{aligned} \tag{3.28}$$

Now we have collected all important relations for the transformation of the sequential derivative. First we use the formula for the image of the integral (3.27) and then apply rule (3.28) step by step. We keep the notation $\sigma = -\alpha_0 + \sum_{k=1}^{n} \alpha_k$ and $\sigma_m = \sum_{k=1}^{m} \alpha_k$, moreover the symbol $\mathcal{D}_0^{\sigma_k - 1}$ means the sequential derivative defined by

$$\mathcal{D}_0^{\sigma_k-1} \equiv \mathcal{D}_0^{\alpha_k-1}\mathcal{D}_0^{\alpha_{k-1}} \ldots \mathcal{D}_0^{\alpha_1}.$$

With these assumptions we obtain

$$\begin{aligned}\mathcal{L}\{\mathcal{D}_0^{\sigma} f(t), t, s\} &= \mathcal{L}\{\mathbf{D}_0^{-\alpha_0}\mathbf{D}_0^{\alpha_n} \ldots \mathbf{D}_0^{\alpha_1} f(t), t, s\} = s^{-\alpha_0}\mathcal{L}\{\mathbf{D}_0^{\alpha_n} \ldots \mathbf{D}_0^{\alpha_1} f(t), t, s\} = \\ &= s^{-\alpha_0+\alpha_n}\mathcal{L}\{\mathbf{D}_0^{\alpha_{n-1}} \ldots \mathbf{D}_0^{\alpha_1} f(t), t, s\} - s^{-\alpha_0}\mathbf{D}_0^{\alpha_n-1} f(t)\big|_{t=0} = \\ &= s^\sigma F(s) - \sum_{k=1}^{n} s^{\sigma-\sigma_k}\mathbf{D}_0^{\sigma_k-1} f(t)\big|_{t=0}\end{aligned} \tag{3.29}$$

The expression for the transform of the sequential derivative is known so we can substitute formulas (3.11) and (3.12) into (3.29) and calculate the transforms of the Riemann-Liouville and the Caputo derivatives respectively.

$$\mathcal{L}\{\mathbf{D}_0^{\alpha} f(t), t, s\} = s^\alpha F(s) - \sum_{k=1}^{n} s^{n-k}\mathbf{D}_0^{\alpha-n+k-1} f(t)\big|_{t=0} \tag{3.30}$$

$$\mathcal{L}\left\{{}^{\mathrm{C}}\mathrm{D}_0^{\alpha} f(t), t, s\right\} = s^\alpha F(s) - \sum_{k=1}^{n_c} s^{\alpha-k} f^{(k-1)}(0) \tag{3.31}$$

Here we can see directly that the Riemann-Liouville derivative needs initial conditions with non-integer derivatives while the Caputo derivative uses the values of the integer-order derivatives. The formulas (3.29), (3.30) and (3.31) will play a very important role in section 5.1.

If we look closely at the formula (3.30), we see that for the calculation of the k-derivative ($k \in \mathbb{N}_0$) there are $k+1$ initial conditions needed. This effect is a price for the weaker restriction in the definition of the Riemann-Liouville derivative caused by the choice $n = [\alpha] + 1$ instead of $n = -[-\alpha]$ as we discussed in subsection 3.2. Anyway, the additional condition has to be zero, because it is a first-order integral over a set of zero measure. That is the reason why in chapter 5 we often use $n = -[-\alpha]$ even for the Riemann-Liouville derivative.

3.7 Fourier Transform of Riemann-Liouville Differintegrals

Due to the definition of the Fourier transform (2.25) it is clear that the Fourier transform makes sense just for the differintegral with lower bound in minus infinity or for the right differintegral with upper bound in plus infinity. Here we will consider only the Riemann-Liouville differintegrals because we do not need the others in sequel.

Let us compute the Fourier transform of the Riemann-Liouville derivative with lower bound at minus infinity, $n = [\alpha] + 1$.

$$\begin{aligned}
\mathcal{F}\{\mathbf{D}^{\alpha}_{-\infty} f(t), t, k\} &= \mathcal{F}\left\{\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{-\infty}^{t}(t-\tau)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau, t, k\right\} = \\
&= \left|\begin{array}{c} t-\tau = r \\ \mathrm{d}\tau = -\mathrm{d}r \\ r: \infty \to 0 \end{array}\right| = \mathcal{F}\left\{\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{0}^{\infty} r^{n-\alpha-1} f(t-r)\,\mathrm{d}r, t, k\right\} = \\
&= \mathcal{F}\left\{\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{-\infty}^{\infty} H(r)\, r^{n-\alpha-1} f(t-r)\,\mathrm{d}r, t, k\right\} = \\
&= \mathcal{F}\left\{\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[\frac{H(t)\, t^{n-\alpha-1}}{\Gamma(n-\alpha)} * f(t)\right], t, k\right\}
\end{aligned}$$

By $H(t)$ we denote the Heaviside unit-step function which is one for $t > 0$ and zero for $t < 0$. The Fourier transform of the first term in the convolution above can be calculated using the definition and the result is

$$\mathcal{F}\left\{\frac{H(t)\, t^{\gamma-1}}{\Gamma(\gamma)}, t, k\right\} = (\mathrm{i}k)^{\gamma}$$

for $\gamma > 0$. If we use in addition the relations (2.28) and (2.27), we get the final formula:

$$\mathcal{F}\{\mathbf{D}^{\alpha}_{-\infty} f(t), t, k\} = (\mathrm{i}k)^{\alpha} \hat{f}(k). \tag{3.32}$$

Analogously we can obtain the expression for the Fourier transform of the right Riemann-Liouville derivative with infinite upper limit:

$$\mathcal{F}\{{}_{-}\mathbf{D}^{\alpha}_{\infty} f(t), t, k\} = (-\mathrm{i}k)^{\alpha} \hat{f}(k). \tag{3.33}$$

Let us note that some authors define the fractional derivative directly by formula (3.32) due to the correspondence with the Fourier transform of classical derivatives (2.27).

3.8 Examples

In this section we will derive the derivatives of functions we will need later, like the power function and functions of the Mittag-Leffler type (which both fulfill the continuity condition of the Caputo derivative w.r.t. its order). Moreover we will look at the exponential function and at the end we will consider an example of a discontinuous function and see what happens with the continuity of the Riemann-Liouville derivative w.r.t. its order.

3.8.1 The Power Function

The power function is one of the most important functions because many functions can be defined by an infinite power series. For simplicity we choose the lower bound a equal to zero point of the power function.

First we calculate its fractional integral of order $\alpha > 0$ for which we need the definition of the Beta function (2.8).

$$\begin{aligned}
\mathbf{D}_a^{-\alpha}(t-a)^\beta &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}(\tau-a)^\beta \,\mathrm{d}\tau = \left| \begin{array}{c} \frac{\tau-a}{t-a}=\xi \\ \mathrm{d}\tau=(t-a)\mathrm{d}\xi \\ \xi: 0\to 1 \end{array} \right| = \\
&= \frac{(t-a)^{\beta+\alpha}}{\Gamma(\alpha)}\int_0^1 (1-\xi)^{\alpha-1}\xi^\beta \,\mathrm{d}\tau = \frac{(t-a)^{\beta+\alpha}}{\Gamma(\alpha)}B(\alpha,\beta+1) = \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-a)^{\beta+\alpha}, \qquad \beta > -1 \qquad (3.34)
\end{aligned}$$

The condition $\beta > -1$ arises naturally as a product of the required integrability.

Now we can calculate the Riemann-Liouville derivative of the power function by using the formula (3.4) and just derived relation (3.34), where as usual $\alpha > 0$, $n = [\alpha] + 1$.

$$\begin{aligned}
\mathbf{D}_a^{\alpha}(t-a)^\beta &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathbf{D}_a^{-(n-\alpha)}(t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+n-\alpha+1)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}(t-a)^{\beta+n-\alpha} = \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}, \qquad \beta > -1 \qquad (3.35)
\end{aligned}$$

We see that the results of (3.34) and (3.35) are formally the same, the condition on β is also unchanged, so we may write generally

$$\mathbf{D}_a^{\alpha}(t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}, \qquad \alpha\in\mathbb{R},\ \beta > -1. \qquad (3.36)$$

A similar formula can be derived for the Caputo derivative, where we use analogically (3.7) and (3.34). But there is a big difference, because due to integrability we have to restrict the power β even more ($\alpha > 0$, $n_c = -[-\alpha]$).

$$\begin{aligned}
{}^{\mathrm{C}}\mathrm{D}_a^{\alpha}(t-a)^\beta &= \mathbf{D}_a^{-(n_c-\alpha)}\frac{\mathrm{d}^{n_c}}{\mathrm{d}t^{n_c}}(t-a)^\beta = \beta(\beta-1)\dots(\beta-n_c+1)\mathbf{D}_a^{-(n_c-\alpha)}(t-a)^{\beta-n_c} = \\
&= \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha} & \beta > n_c-1 \\ 0 & \beta\in\{0,1,\dots,n_c-1\} \\ \text{undefined} & \text{otherwise} \end{cases} \qquad (3.37)
\end{aligned}$$

For values $\beta \in \{0, 1, \ldots, n_c - 1\}$ the result is zero after differentiation, so any problems with integrability do not occur. The relation (3.37) shows the curious behaviour of the Caputo derivative, because from a certain point it is possible to calculate the integer-order derivatives but not the fractional derivatives between them.

That is because we need the function $f(t)$ to be even integrable up to its n_c^{th} derivative for the calculation of the fractional derivatives whereas for the classical derivatives the differentiability of a sufficient order is enough.

The particular case of the formula (3.37) is the derivative of a constant and we see that it is zero. On the other hand the Riemann-Liouville derivative of a constant is a non-zero function which is implied by (3.36).

3.8.2 Functions of the Mittag-Leffler Type

Now we will study differintegrals of Mittag-Leffler functions given by (2.13) and multiplied by a suitable power function. Those functions are solutions of homogeneous linear FDEs. For the computation we can use the generalized linearity of the differintegral (3.15) because Mittag-Leffler functions fulfill all needed conditions. Hence the problem is reduced to the calculation of the differintegral of the power function which we derived in the previous subsection (we consider $\lambda \neq 0$).

First we will compute the Riemann-Liouville differintegral for $\alpha \in \mathbb{R}$ so we will use the formula (3.36). The requirement $\beta > 0$ is the common consequence of the condition from (3.36) applied on all differintegrals in the sum.

$$\begin{aligned} \mathbf{D}_0^\alpha \left(t^{\beta-1} E_{\mu,\beta}(\lambda t^\mu)\right) &= \mathbf{D}_0^\alpha \sum_{k=0}^{\infty} \frac{\lambda^k t^{\mu k+\beta-1}}{\Gamma(\mu k+\beta)} = \sum_{k=0}^{\infty} \frac{\lambda^k t^{\mu k+\beta-\alpha-1}}{\Gamma(\mu k+\beta-\alpha)} = \\ &= t^{\beta-\alpha-1} E_{\mu,\beta-\alpha}(\lambda t^\mu), \quad \beta > 0 \end{aligned} \tag{3.38}$$

A more complicated situation occurs in case of the Caputo derivative. Problems are caused by the branched formula (3.37) and by the presence of two parameters in the derivate function. If we think about the conditions, the result splits up into four parts. If we put $n = -[-\alpha]$, $p = \left[\frac{n-\beta}{\mu}\right]$ and $\gamma = \mu + \beta - \alpha$, we get after some calculation the relationship

$$ {}^{\mathrm{C}}\mathrm{D}_0^\alpha \left(t^{\beta-1} E_{\mu,\beta}(\lambda t^\mu)\right) = \begin{cases} t^{\beta-\alpha-1} E_{\mu,\beta-\alpha}(\lambda t^\mu) & \beta > n_c \\ \lambda^{p+1} t^{\mu p+\gamma-1} E_{\mu,\mu p+\gamma}(\lambda t^\mu) & \beta \in \{1, 2, \ldots n_c\} \text{ and} \\ & \mu \in \{1, \ldots, n_c - \beta\} \cup (n_c - \beta; \infty) \\ \text{undefined} & \text{otherwise} \end{cases} . \tag{3.39}$$

The formula on the second row joins two cases which differ by the condition on μ - the relationship for $\mu > n_c - \beta$ is included because in this case, we obtain $p = 0$ and get it out from formula.

Let us note that the Caputo derivative of this function is undefined for some combinations of α, μ and β.

3.8.3 The Exponential Function

The exponential function is one of the most important functions in the integer-order calculus so we look at it separately. Anyway, here it is only a special case of the function introduced in previous subsection with $\beta = 1$ and $\mu = 1$, thus we check it quickly only in the Riemann-Liouville approach.

$$\mathbf{D}_0^\alpha \mathrm{e}^{\lambda t} = \mathbf{D}_0^\alpha E_{1,1}(\lambda t) = t^{-\alpha} E_{1,1-\alpha}(\lambda t) \tag{3.40}$$

The form of the result is not very familiar from integer-order calculus, but if we rewrite it by the help of sum, we can recognize there e.g. the first integral of exponential function with lower bound 0.

$$\frac{t^{-\alpha}}{\lambda t}\left[\sum_{k=0}^{\infty}\frac{\lambda^k t^k}{\Gamma(k-\alpha)} - \frac{1}{\Gamma(-\alpha)}\right] \xrightarrow{\alpha\to -1} \frac{1}{\lambda}\left(\mathrm{e}^{\lambda t} - 1\right)$$

This is only a confirmation of the facts about continuity with respect to the order of derivation presented in subsection 3.5.4.

If we want to obtain the fractionalized version of well-known formula $\lambda^{\pm n}\mathrm{e}^{\lambda t}$ with "+" for the integer-order derivatives and with "−" for the integer-order integrals, we have to set the lower bound a to minus infinity. The calculation for the fractional integral of the exponential function is as follows:

$$\mathbf{D}_{-\infty}^{-\alpha}\,\mathrm{e}^{\lambda t} = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}(t-\tau)^{\alpha-1}\mathrm{e}^{\lambda\tau}\,\mathrm{d}\tau = \left|\begin{array}{c}\lambda(t-\tau)=\xi\\ -\lambda\mathrm{d}\tau = \mathrm{d}\xi\\ \xi:\ \infty\to 0\end{array}\right| =$$
$$= \frac{-1}{\Gamma(\alpha)}\int_{\infty}^{0}\left(\frac{\xi}{\lambda}\right)^{\alpha-1}\mathrm{e}^{\lambda t-\xi}\,\frac{\mathrm{d}\xi}{\lambda} = \frac{\mathrm{e}^{\lambda t}}{\lambda^\alpha\,\Gamma(\alpha)}\int_0^\infty \xi^{\alpha-1}\mathrm{e}^{-\xi}\,\mathrm{d}\xi = \lambda^{-\alpha}\mathrm{e}^{\lambda t}.$$

Now we can compute the Riemann-Liouville derivative as usually and then we obtain the general formula we were looking for.

$$\mathbf{D}_{-\infty}^{\alpha}\,\mathrm{e}^{\lambda t} = \lambda^\alpha \mathrm{e}^{\lambda t}, \quad \alpha\in\mathbb{R} \tag{3.41}$$

Recalling the relation (3.17), it is clear that we would get the same result even in the Caputo sense.

3.8.4 A Discontinuous Function

All conclusions of subsection 3.5.4 depend on continuous differentiability of a sufficient order due to the comfortable use of integration by parts. Now we introduce an example where this condition is not fulfilled. Let us consider the following function with one point of discontinuity.

$$f(t) = \begin{cases} t & t<1 \\ 1-t & t\geq 1\end{cases}$$

There is no problem in case $t < 1$ (we use formula (3.36)), but we can easily see the influence of the discontinuity during calculation of the α-integral of $f(t)$ for $t \geq 1$. There is a term which is not a convolution but only an integral dependent on a parameter.

$$
\begin{aligned}
\mathbf{D}_0^{-\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau = \frac{1}{\Gamma(\alpha)} \int_0^1 (t-\tau)^{\alpha-1} \tau \,\mathrm{d}\tau + \mathbf{D}_1^{-\alpha}(1-t) = \\
&= \left| \begin{array}{c} t-\tau=\xi \\ -\mathrm{d}\tau = \mathrm{d}\xi \\ \xi: \ t \to t-1 \end{array} \right| = \frac{1}{\Gamma(\alpha)} \int_{t-1}^t \xi^{\alpha-1}(t-\xi)\,\mathrm{d}\xi - \mathbf{D}_1^{-\alpha}(t-1) = \\
&= \frac{1}{\Gamma(\alpha)} \left[t\frac{\xi^\alpha}{\alpha} - \frac{\xi^{\alpha+1}}{\alpha+1} \right]_{t-1}^t - \frac{\Gamma(2)}{\Gamma(2+\alpha)}(t-1)^{1+\alpha} = \\
&= \frac{1}{\Gamma(\alpha)} \left(\frac{t^{1+\alpha}}{\alpha} - \frac{t^{1+\alpha}}{1+\alpha} - \frac{t(t-1)^\alpha}{\alpha} + \frac{(t-1)^{1+\alpha}}{1+\alpha} \right) - \frac{(t-1)^{1+\alpha}}{\Gamma(2+\alpha)} = \\
&= \frac{t^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{(t-1)^\alpha}{\Gamma(2+\alpha)}(\alpha(t-1)-(1+\alpha)t) - \frac{(t-1)^{1+\alpha}}{\Gamma(2+\alpha)} = \\
&= \frac{t^{1+\alpha}}{\Gamma(2+\alpha)} - \frac{(t-1)^\alpha}{\Gamma(1+\alpha)} - 2\frac{(t-1)^{1+\alpha}}{\Gamma(2+\alpha)}, \quad t \geq 1
\end{aligned} \tag{3.42}
$$

If we use the formula (3.42) for deriving the fractional derivative, we can write all Riemann-Liouville differ-integrals in compact form

$$
\mathbf{D}_0^{\alpha} f(t) = \begin{cases} \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} & t<1 \\ \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{(t-1)^{-\alpha}}{\Gamma(1-\alpha)} - 2\frac{(t-1)^{1-\alpha}}{\Gamma(2-\alpha)} & t \geq 1 \end{cases}. \tag{3.43}
$$

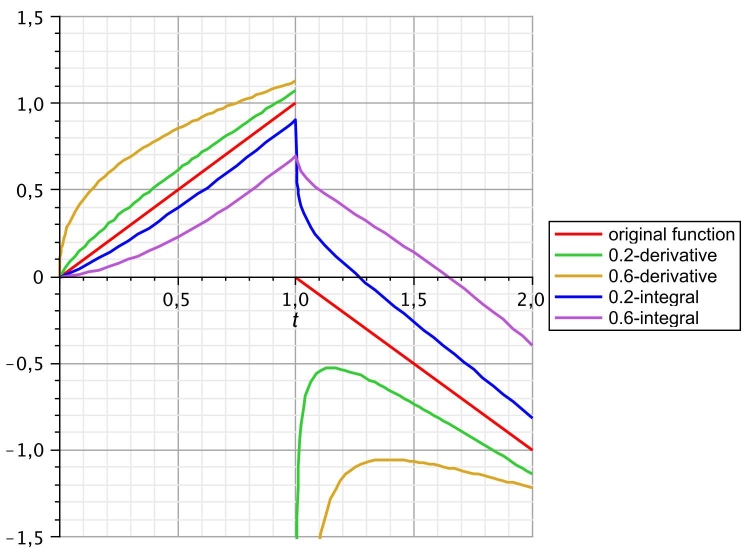


Figure 3.3: Low-order differintegrals of the discontinuous function $f(t)$.

In figure 3.3 there are some Riemann-Liouville differintegrals of order closed to zero. We see that there is a continuity w.r.t. order of the differintegral at all points except the point of discontinuity. Next, fractional integrals have a smoothing effect like we expect from classical integrals. The fractional derivatives are unbounded in a neighborhood of the discontinuity point, this corresponds with the first derivative which is undefined in this point. The unboundedness occurs on the right-hand side of the discontinuity point because we use left differintegrals, so first we have to pass the point of discontinuity. If we are on the left, the differintegral "does not know" about a problem ahead.

The behaviour described above may be observed in better way in following figures. Figure 3.4 shows the smoothing effect of the integral in detail, figure 3.5 continuous transition from the first integral to the second one.

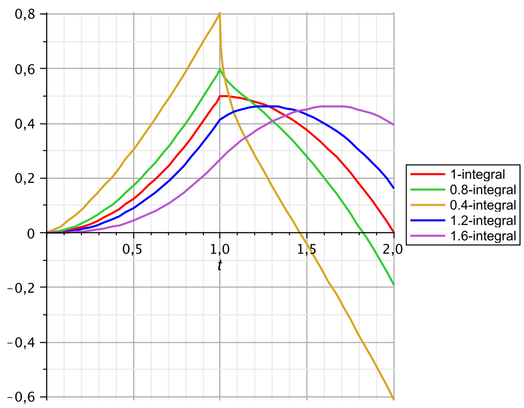


Figure 3.4: The fractional integrals of orders closed to 1.

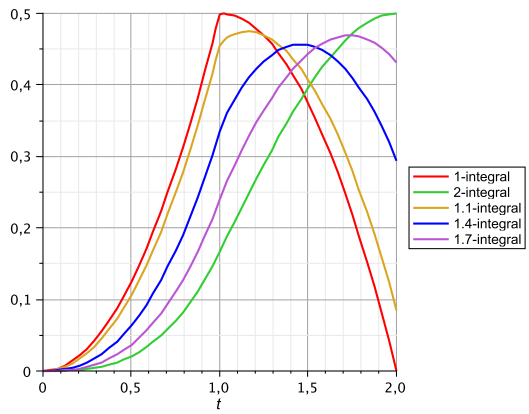


Figure 3.5: The fractional integrals of orders between 1 and 2.

Figure 3.6 represents the approach of α-derivatives to 1-derivative, figure 3.7 shows a transition from the first derivative to the second one (which is zero except at the point 1 where is undefined). We can see that α-derivatives of higher orders are functions of a similar quality but they change the side of approaching the integer-order derivatives. This fact is caused by the various signs of the Gamma function with a negative argument (see figure 2.2).

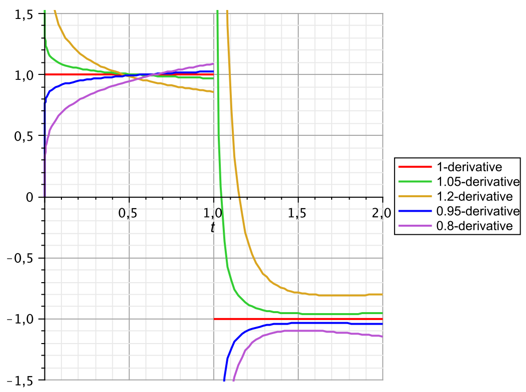


Figure 3.6: The fractional derivatives of orders closed to 1.

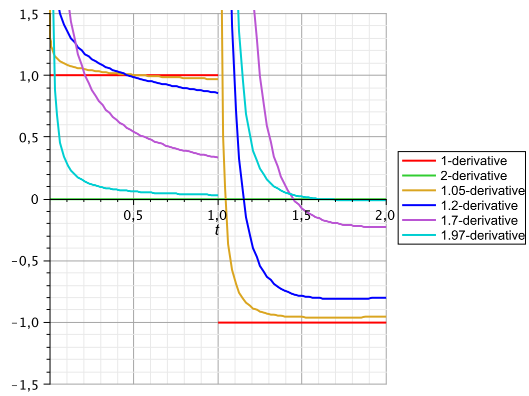


Figure 3.7: The fractional derivatives of orders between 1 and 2.

If we consider the Caputo derivative, the results are the same as (3.43) for $\alpha \leq 1$, but for all α-derivatives with $\alpha > 1$ we obtain the zero function at all point $t \geq 0$ (so the Caputo derivative removes the point of discontinuity).

Chapter 4

The Existence and Uniqueness Theorem

Before we will solve some fractional differential equations, we will give sufficient conditions for existence and uniqueness of solutions. There are many variations of the existence and uniqueness theorem for various spaces of functions and for various types of differintegrals and in general this problem is still open for nonlinear equations and some types of fractional derivatives.

We will introduce only the existence and uniqueness theorem for a continuous case of general linear fractional differential equations (LFDEs) with a sequential derivative. As we will see this theorem is very similar to the one in the theory of ODEs. We will not present an exact proof, only outline its main ideas. For more information, even about nonlinear cases, see [1] and [2].

Linear Fractional Differential Equations

Let us consider the initial-value problem of the form

$$\mathcal{D}_0^{\sigma_m} y(t) + \sum_{k=1}^{m-1} p_k(t) \mathcal{D}_0^{\sigma_k} y(t) + p_0(t) y(t) = f(t),$$
$$\mathcal{D}_0^{\sigma_k - 1} y(t)\big|_{t=0} = b_k, \tag{4.1}$$

where $k = 1, \ldots, m$ and $0 < t < T < \infty$. The construction of the sequential derivative follows (3.10) and the function $f(t)$ is bounded on the interval $\langle 0, T\rangle$.

The conditions for existence and uniqueness of a solution are described by the following theorem.

Theorem 4.0.1 (Existence and Uniqueness for LFDEs). *If $f(t)$ is bounded on $\langle 0, T\rangle$ and $p_k(t)$ for $k \in \{0, \ldots, m-1\}$ are continuous functions in the closed interval $\langle 0, T\rangle$, then the initial-value problem (4.1) has the unique solution $y(t) \in L_1(0, T)$.*

Proof. The first step is the proof of existence and uniqueness of the solution for the case $p_k(t) \equiv 0$ for $k \in \{0, \ldots, m-1\}$. If we consider $f(t)$ bounded on $\langle 0, T\rangle$, we can show e.g. by using the Laplace transform that the solution $y(t) \in L_1(0, T)$ exists and is given by the formula

$$y(t) = \frac{1}{\Gamma(\sigma_m)} \int_0^t (t-\tau)^{\sigma_m - 1} f(\tau)\, \mathrm{d}\tau + \sum_{k=1}^{m} \frac{b_k}{\Gamma(\sigma_k)} t^{\sigma_k - 1}. \tag{4.2}$$

Then we assume that the solution of the general problem (4.1) exists, and denote

$$\mathcal{D}_0^{\sigma_m} y(t) = \varphi(t).$$

Now we use the relation (4.2) for this equation (with $f(t)$ replaced by $\varphi(t)$) and substitute it into the equation (4.1). Finally we obtain the Volterra integral equation of the second kind for the function $\varphi(t)$:

$$\varphi(t) + \int_0^t K(t,\tau)\varphi(\tau)\,\mathrm{d}\tau = g(t), \tag{4.3}$$

where the kernel $K(t,\tau)$ and the right-hand side $g(t)$ are given by the expressions:

$$\begin{aligned}
K(t,\tau) &= p_0(t)\frac{(t-\tau)^{\sigma_m-1}}{\Gamma(\sigma_m)} + \sum_{k=1}^{m-1} p_k(t)\frac{(t-\tau)^{\sigma_m-\sigma_k-1}}{\Gamma(\sigma_m-\sigma_k)},\\
g(t) &= f(t) - p_0(t)\sum_{j=1}^{m} b_j\frac{t^{\sigma_j-1}}{\Gamma(\sigma_j)} - \sum_{k=1}^{m-1} p_k(t)\sum_{j=k+1}^{m} b_j\frac{t^{\sigma_j-\sigma_k-1}}{\Gamma(\sigma_j-\sigma_k)}.
\end{aligned}$$

The existence and uniqueness of the function $\varphi(t) \in L_1(0,T)$ is implied by the theory of Volterra integral equations (necessary conditions on the kernel and the right-hand side are satisfied here). Hence, according to (4.2), there is the unique solution $y(t) \in L_1(0,T)$ even for the initial-value problem (4.1).

More details can be found in [1]. □

Remark. In addition, if $f(t)$ is a continuous function in the closed interval $\langle 0,T\rangle$, then the unique solution $y(t)$ of the initial-value problem (4.1) is continuous in $\langle 0,T\rangle$ too. We chose the zero lower bound, but in a general case the theorem is completely analogous.

Remark. The technique used for proving the existence and uniqueness of the solution for Volterra integral equations is often called the successive approximation method. If some restrictions are fulfilled, it provides a recipe how to construct a solution of FDE (if one term with derivative occurs) in a limit form by the recursion formula (for lower bound a):

$$\begin{aligned}
y_0(t) &= \sum_{k=1}^{n} \frac{b_k}{\Gamma(\alpha-k+1)}(t-a)^{\alpha-k}\\
y_j(t) &= y_0(t) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau, y_{j-1}(\tau))\,\mathrm{d}\tau,
\end{aligned}$$

where n is the number of initial conditions and $f(t,y)$ is the right-hand side of the equation

$$\mathbf{D}_0^{\alpha} y(t) = f(t, y(t)). \tag{4.4}$$

Especially for linear FDEs we would get the solution in closed form by this method, this procedure will be discussed in more detail in subsection 5.2.

Chapter 5

LFDEs and Their Solutions

In this chapter we will introduce some of the most important methods of solving linear fractional differential equations:

$$\mathfrak{D}_a^{\sigma_m} y(t) + \sum_{k=1}^{m-1} p_k(t)\mathfrak{D}_a^{\sigma_k} y(t) + p_0(t)y(t) = f(t), \tag{5.1}$$

where $\mathfrak{D}_a^{\sigma}$ represents a general fractional derivative and $p_k(t)$, $f(t)$ are functions we do not specify now.

Mostly we will consider an initial-value problem composed of the equation (5.1) and appropriate initial conditions, which depend on a construction of all operators $\mathfrak{D}_a^{\sigma_k}$ for $k \in \{1, 2, \dots m\}$. A kind of this dependence will be shown during this chapter.

Like for ordinary differential equations the Laplace transform is a very useful tool here. Another method is the reduction to a Volterra integral equation which is implied by the proof of the existence and uniqueness theorem in chapter 4. It can also be used for nonlinear equations (the solution is not in closed form generally). Finally, for some equations it is convenient to use tricks like the power series method, the compositional method and the method of the transformation to an ODE.

5.1 The Laplace Transform Method

The Laplace transform method is one of the most powerful methods of solving LFDEs with constant coefficients. On the other hand it is useless for LFDEs with general variable coefficients or for nonlinear FDEs.

Obviously, in this section we will study equations of the form

$$\mathfrak{D}_a^{\sigma_m} y(t) + \sum_{k=1}^{m-1} A_k\mathfrak{D}_a^{\sigma_k} y(t) + A_0 y(t) = f(t) \tag{5.2}$$

with appropriate initial conditions which are given in the point $a = 0$.

At the beginning of this section we will study in detail two-term equations in order to understand the spirit of LFDEs. We will show that the solution of LFDEs is compound of the solution of a homogeneous equation and of the particular solution like in the theory of LODEs. This property is implied by the linearity so it holds even for equations with more terms.

For this reason more general cases will be considered only with zero right-hand side and mostly examples. The particular solution problem for non-homogeneous equations will be discussed in the last subsection.

5.1.1 The Two-Term Equation

Here we will consider the initial-value problem with the sequential fractional derivative:

$$\begin{aligned} \mathcal{D}_0^{\sigma} y(t) - \lambda y(t) &= f(t) \\ \mathcal{D}_0^{\sigma_k - 1} y(t)\big|_{t=0} &= b_k \end{aligned} \tag{5.3}$$

where λ, b_k are real constants, $k \in \{1, \ldots, m\}$ and the sequential derivative is composed like described before in the formula (3.10):

$$\mathcal{D}_0^{\sigma} \equiv \mathbf{D}_0^{-\alpha_0} \mathbf{D}_0^{\alpha_m} \ldots \mathbf{D}_0^{\alpha_1}, \quad \sigma = \sum_{j=1}^{m} \alpha_j - \alpha_0,\ \alpha_j \in (0;1\rangle,\ \alpha_0 \in \langle 0;1) \tag{5.4}$$

$$\mathcal{D}_0^{\sigma_k} \equiv \mathbf{D}_0^{\alpha_k} \mathbf{D}_0^{\alpha_{k-1}} \ldots \mathbf{D}_0^{\alpha_1}, \quad \sigma_k = \sum_{j=1}^{k} \alpha_j. \tag{5.5}$$

Let us note that in case $\alpha_0 = 0$ we have $\sigma = \sigma_m$. Anyway, we still keep the notation established in subsection 3.6:

$$\mathbf{D}_0^{\sigma_k - 1} \equiv \mathbf{D}_0^{\alpha_k - 1} \mathbf{D}_0^{\alpha_{k-1}} \ldots \mathbf{D}_0^{\alpha_1}.$$

For better understanding the idea of the method let us consider the simple example.

Example 5.1.1. Solve the following initial-value problem.

$$\begin{aligned} \mathcal{D}_0^{\frac{4}{3}} y(t) - \lambda y(t) &= t^2 \\ \mathcal{D}_0^{\frac{1}{3}} y(t)\big|_{t=0} &= 1 \\ \mathcal{D}_0^{-\frac{1}{6}} y(t)\big|_{t=0} &= 2 \end{aligned}$$

We can easily see that in this case there are $\alpha_1 = \frac{5}{6}$ and $\alpha_2 = \frac{1}{2}$. First we apply the Laplace transform on both sides of the equation and we obtain:

$$\begin{aligned} s^{\frac{4}{3}} Y(s) - 1 - 2s^{\frac{1}{2}} - \lambda Y(s) &= \frac{2}{s^3} \\ Y(s) &= \frac{1}{s^{\frac{4}{3}} - \lambda} + \frac{2s^{\frac{1}{2}}}{s^{\frac{4}{3}} - \lambda} + \frac{2s^{-3}}{s^{\frac{4}{3}} - \lambda}. \end{aligned}$$

Then it is easy to compute the inverse Laplace transform according to the formula (2.24):

$$y(t) = t^{\frac{1}{3}} E_{\frac{4}{3},\frac{4}{3}}(\lambda t^{\frac{4}{3}}) + 2t^{-\frac{1}{6}} E_{\frac{4}{3},\frac{5}{6}}(\lambda t^{\frac{4}{3}}) + 2t^{\frac{10}{3}} E_{\frac{4}{3},\frac{13}{3}}(\lambda t^{\frac{4}{3}})$$

Now it is a good point to show the interesting property of FDEs which is different from its analogy in the theory of ODEs. The number of initial conditions is not given by the order of the equation exactly. In case of a sequential derivative there exists the minimum $[\sigma_m] + 1$, but the real count is equal to m which depends on values α_k and not only on the order σ, so in fact there is no upper limit.

For the Riemann-Liouville derivative the number of initial conditions is unique but depends on all orders of derivatives appearing in the equation as we will see later. The case of the Caputo derivative coincides with the theory of ODE because its initial conditions are given for integer-order derivatives.

Let us demonstrate this by the previous example. In the original problem the order of the equation $\left(\frac{4}{3}\right)$ was constructed as $\frac{1}{2} + \frac{5}{6}$. Now we use the different decomposition: $\alpha_1 = \frac{1}{6}$, $\alpha_2 = \frac{2}{3}$ and $\alpha_3 = \frac{1}{2}$. The initial conditions get the form, e.g.:

$$\begin{aligned} \mathcal{D}_0^{\frac{1}{3}} y(t)\big|_{t=0} &= 1 \\ \mathcal{D}_0^{-\frac{1}{6}} y(t)\big|_{t=0} &= 2 \\ \mathcal{D}_0^{-\frac{5}{6}} y(t)\big|_{t=0} &= 3 \end{aligned}$$

With this modification the Laplace transform of the equation is after some computations:

$$Y(s) = \frac{1}{s^{\frac{4}{3}} - \lambda} + \frac{2s^{\frac{1}{2}}}{s^{\frac{4}{3}} - \lambda} + \frac{3s^{\frac{7}{6}}}{s^{\frac{4}{3}} - \lambda} + \frac{2s^{-3}}{s^{\frac{4}{3}} - \lambda}$$

and the inverse Laplace transform gives the result:

$$y(t) = t^{\frac{1}{3}} E_{\frac{4}{3},\frac{4}{3}}(\lambda t^{\frac{4}{3}}) + 2t^{-\frac{1}{6}} E_{\frac{4}{3},\frac{5}{6}}(\lambda t^{\frac{4}{3}}) + 3t^{-\frac{5}{6}} E_{\frac{4}{3},\frac{1}{6}}(\lambda t^{\frac{4}{3}}) + 2t^{\frac{10}{3}} E_{\frac{4}{3},\frac{13}{3}}(\lambda t^{\frac{4}{3}})$$

There is only one difference in comparison with the previous initial-value problem. There is a new term which is obviously related to the added level of the differentiation. While in the first case we "stopped" on the levels $\frac{5}{6}$ and $\frac{4}{3}$, now we splitted the first step and got the new differentiation level $\frac{1}{6}$.

The number of independent solutions seems to depend on the number of initial conditions. It is not very complicated to verify that the last term is the particular solution of the original equation and the others solve only its homogeneous part. So it is quite similar situation like in the theory of linear ODE with constant coefficients.

General Case

Now let us find the solution of a general two-term initial-value problem defined by (5.3). The procedure is equivalent to the previous example:

$$\begin{aligned} s^\sigma Y(s) - \sum_{k=1}^{m} s^{\sigma-\sigma_k} \mathcal{D}_0^{\sigma_k - 1} y(t)\big|_{t=0} - \lambda Y(s) &= F(s) \\ Y(s) &= \sum_{k=1}^{m} b_k \frac{s^{\sigma-\sigma_k}}{s^\sigma - \lambda} + \frac{1}{s^\sigma - \lambda} F(s) \end{aligned}$$

By using the inverse Laplace transform we get:

$$y(t) = \sum_{k=1}^{m} b_k t^{\sigma_k - 1} E_{\sigma,\sigma_k}(\lambda t^\sigma) + t^{\sigma-1} E_{\sigma,\sigma}(\lambda t^\sigma) * f(t) \tag{5.6}$$

On this base we can formulate the theorem about the solution of (5.3) and about the following two initial-value problems ($m \in \mathbb{N}$, $k \in \{1, \ldots, m\}$).

$$\begin{aligned} \mathcal{D}_0^{\sigma} y(t) - \lambda y(t) &= 0 \\ \mathcal{D}_0^{\sigma_k - 1} y(t)\big|_{t=0} &= b_k \end{aligned} \qquad (5.7) \qquad\qquad \begin{aligned} \mathcal{D}_0^{\sigma} y(t) - \lambda y(t) &= f(t) \\ \mathcal{D}_0^{\sigma_k - 1} y(t)\big|_{t=0} &= 0 \end{aligned} \qquad (5.8)$$

Theorem 5.1.2. *Let $f(t)$ be a real function bounded on the interval $\langle 0, T\rangle$. Next we introduce functions:*

$$y_k(t) = t^{\sigma_k - 1} E_{\sigma,\sigma_k}(\lambda t^{\sigma}), \tag{5.9}$$

$$y_p(t) = t^{\sigma - 1} E_{\sigma,\sigma}(\lambda t^{\sigma}) * f(t) \tag{5.10}$$

($\sigma_k \neq 0$ for all k, in case $\sigma_k = 0$ we simply put $y_k(t) = 0$).

Then the initial-value problem (5.3), where $\mathcal{D}_0^{\sigma}$ is constructed s.t. $\alpha_0 < \alpha_1$, has the unique solution:

$$y(t) = \sum_{k=1}^{m} b_k y_k(t) + y_p(t). \tag{5.11}$$

The sum of functions $b_k y_k(t)$ solves (5.7) and the non-zero functions y_k form the system of independent solutions of the appropriate equation. The function $y_p(t)$ solves (5.8) and in (5.3) plays a role of the particular solution known from the theory of ODEs.

We will prove this theorem directly by the definition of a sequential derivative. This is not the most simple way but it shows better the spirit of the problem.

Proof. The existence and uniqueness of the solution is implied by the theorem 4.0.1.

First let us prove the properties of $y_k(t)$ mentioned above. We are going to show that every $y_k(t)$ solves the homogeneous equation in (5.3). The case $y_k(t) = 0$ is trivial, so we consider $\sigma_k \neq 0$. We have to calculate all derivatives of non-zero functions $\mathcal{D}_0^{\sigma_r} y_k(t)$. We always need the definition of the Mittag-Leffler function (2.13) and the rule for fractional derivatives of the used functions (3.38).

- $r < k$:

$$\mathcal{D}_0^{\sigma_r} y_k(t) = \mathcal{D}_0^{\sigma_r}\left[t^{\sigma_k - 1} E_{\sigma,\sigma_k}(\lambda t^{\sigma})\right] = t^{\sigma_k - \sigma_r - 1} E_{\sigma,\sigma_k - \sigma_r}(\lambda t^{\sigma})$$

- $r = k$: Here we apply the formula $\frac{1}{\Gamma(0)} = 0$ and then we shift the index in the sum.

$$\begin{aligned} \mathcal{D}_0^{\sigma_k} y_k(t) &= t^{-1} E_{\sigma,0}(\lambda t^{\sigma}) = \sum_{j=0}^{\infty} \frac{\lambda^j t^{\sigma j - 1}}{\Gamma(\sigma j)} = \left|\ \text{shift}\ \right| = \lambda \sum_{j=0}^{\infty} \frac{\lambda^j t^{\sigma j + \sigma - 1}}{\Gamma(\sigma j + \sigma)} = \\ &= \lambda t^{\sigma - 1} E_{\sigma,\sigma}(\lambda t^{\sigma}) \end{aligned}$$

- $r > k$: W.r.t. the previous case it is necessary to use the sequential differentiation. If the condition $\sigma_r - \sigma_k < \sigma$ is not fulfilled for all r and k, we cannot continue differentiating because the function becomes non-integrable. This condition is equivalent to $\alpha_0 < \alpha_1$.

$$\mathcal{D}_0^{\sigma_r} y_k(t) = \mathcal{D}_0^{\sigma_r - \sigma_k}[\lambda t^{\sigma - 1} E_{\sigma,\sigma}(\lambda t^{\sigma})] = \left| \begin{array}{c} \sigma_r - \sigma_k < \sigma_m \text{ so} \\ \text{according to} \\ \text{the case } r < k \end{array} \right| = \lambda t^{\sigma - \sigma_r + \sigma_k - 1} E_{\sigma,\sigma - \sigma_r + \sigma_k}(\lambda t^{\sigma})$$

Especially for $\sigma_r = \sigma$ we get the result: $\mathcal{D}_0^{\sigma} y_k(t) = \lambda t^{\sigma_k - 1} E_{\sigma,\sigma_k}(\lambda t^{\sigma}) = \lambda y_k(t)$.

Now it is clear that every function y_k satisfies the homogeneous equation:

$$\mathcal{D}_0^{\sigma} y(t) - \lambda y(t) = 0. \tag{5.12}$$

It still remains to find the values $\mathcal{D}_0^{\sigma_r - 1} y_k(t)\big|_{t=0}$, so that we could verify the initial conditions are met. For this purpose we may partially use the results obtained above, the procedure is very similar.

- $r < k$:

$$\mathcal{D}_0^{\sigma_r - 1} y_k(t) = t^{\sigma_k - \sigma_r} E_{\sigma, \sigma_k - \sigma_r + 1}(\lambda t^{\sigma})$$

  Because $\sigma_k - \sigma_r > 0$ it is obvious that $\mathcal{D}_0^{\sigma_r - 1} y_k(t)\big|_{t=0} = 0$ for $r < k$.

- $r = k$: Due to $\sigma_k - 1 < \sigma_k$ we may use the previous formula.

$$\mathcal{D}_0^{\sigma_k - 1} y_k(t)\big|_{t=0} = E_{\sigma,1}(\lambda t^{\sigma})\big|_{t=0} = 1$$

- $r > k$: Here we again use the condition $\sigma_r - \sigma_k < \sigma$ which guarantees the zero value of the following expression.

$$\begin{aligned} \mathcal{D}_0^{\sigma_r - 1} y_k(t)\big|_{t=0} &= \mathcal{D}_0^{\sigma_r - 1 - \sigma_k} [\lambda t^{\sigma - 1} E_{\sigma,\sigma}(\lambda t^{\sigma})]\big|_{t=0} = \\ &= \lambda t^{\sigma - \sigma_r + \sigma_k} E_{\sigma, \sigma - \sigma_r + \sigma_k + 1}(\lambda t^{\sigma})\big|_{t=0} = 0 \end{aligned}$$

It is obvious that functions $y_k(t)$ generate the system of independent solutions of the equation (5.12), thus we may write every of its solutions in the form:

$$y_h(t) = \sum_{k=1}^{m} c_k y_k(t).$$

The fact that this sum solves an arbitrary initial-value problem defined by (5.7) may be shown via the generalized Wronskian. If the Wroskian is nonzero (in point 0 in our case), all combinations of initial conditions are feasible.

$$W(0) = \begin{vmatrix} \mathcal{D}_0^{\sigma_1 - 1} y_1(t) & \mathcal{D}_0^{\sigma_1 - 1} y_2(t) & \dots & \mathcal{D}_0^{\sigma_1 - 1} y_m(t) \\ \mathcal{D}_0^{\sigma_2 - 1} y_1(t) & \mathcal{D}_0^{\sigma_2 - 1} y_2(t) & \dots & \mathcal{D}_0^{\sigma_2 - 1} y_m(t) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{D}_0^{\sigma_m - 1} y_1(t) & \mathcal{D}_0^{\sigma_m - 1} y_2(t) & \dots & \mathcal{D}_0^{\sigma_m - 1} y_m(t) \end{vmatrix}_{t=0} = \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{vmatrix} = 1$$

We just showed not only the independence and sufficiency of the system of functions $y_k(t)$ but even the equality $c_k = b_k$. We proved that the initial-value problem (5.7) has the unique solution:

$$y_h(t) = \sum_{k=1}^{m} b_k y_k(t). \tag{5.13}$$

Now it remains to prove that $y_p(t)$ defined by (5.10) solves the initial-value problem (5.8). We start with the calculation of an α-integral of $y_p(t)$ because it will be very useful.

During the computation we use the change of order of integration, the definition of the Beta function (2.8) and the transition of sum in front of integral (this is possible due to the uniform convergence of the Mittag-Leffler function).

$$\begin{aligned}
\mathbf{D}_0^{-\alpha} y_p(t) &= \mathbf{D}_0^{-\alpha}\left[\int_0^t (t-r)^{\sigma-1} E_{\sigma,\sigma}(t-r)^{\sigma} f(r)\,\mathrm{d}r\right] = \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\int_0^{\tau}(\tau-r)^{\sigma-1}E_{\sigma,\sigma}(\tau-r)^{\sigma} f(r)\,\mathrm{d}r\right)\mathrm{d}\tau = \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t\int_r^t (t-\tau)^{\alpha-1} f(r)\sum_{j=0}^{\infty}\frac{\lambda^j}{\Gamma(\sigma j+\sigma)}(\tau-r)^{\sigma j+\sigma-1}\,\mathrm{d}\tau\,\mathrm{d}r = \\
&= \left|\begin{array}{c} \tau = r + z(t-r) \\ \mathrm{d}\tau = (t-r)\mathrm{d}z \\ z:\ 0\to 1 \end{array}\right| = \frac{1}{\Gamma(\alpha)}\int_0^t \sum_{j=0}^{\infty}\frac{\lambda^j f(r)}{\Gamma(\sigma j+\sigma)}(t-r)^{\sigma j+\sigma+\alpha-1}\cdot \\
&\quad\cdot\int_0^1 (1-z)^{\alpha-1} z^{\sigma j+\sigma-1}\,\mathrm{d}z\,\mathrm{d}r = \int_0^t f(r)\sum_{j=0}^{\infty}\frac{\lambda^j (t-r)^{\sigma j+\sigma+\alpha-1}}{\Gamma(\sigma j+\sigma+\alpha)}\,\mathrm{d}r
\end{aligned}$$

Let us compute the values $\mathcal{D}_0^{\sigma_k-1} y_p(t)$. There is no reason to use sequential differentiation, because as we see, the differintegral works only with Green function and not with $f(t)$ (see [1]) and if we consider $\sigma > \sigma_k - 1$, it is clear that there cannot occur a problem like for $y_k(t)$. Hence, we will differentiate in the Riemann-Liouville sense, we put $n = [\sigma_k]$.

$$\begin{aligned}
\mathbf{D}_0^{\sigma_k-1} y_p(t) &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathbf{D}_0^{-(n+1-\sigma_k)} y_p(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t f(r)\sum_{j=0}^{\infty}\frac{\lambda^j (t-r)^{\sigma j+\sigma+n-\sigma_k}}{\Gamma(\sigma j+\sigma+n+1-\sigma_k)}\,\mathrm{d}r = \\
&= \int_0^t f(r)\sum_{j=0}^{\infty}\frac{\lambda^j (t-r)^{\sigma j+\sigma-\sigma_k}}{\Gamma(\sigma j+\sigma+1-\sigma_k)}\,\mathrm{d}r
\end{aligned}$$

Due to $\sigma j + \sigma - \sigma_k > -1$ for all j, k and boundedness of $f(t)$ on a needed domain, the initial conditions $\mathbf{D}_0^{\sigma_k-1} y_p(t)\big|_{t=0} = 0$ are satisfied for all k.

Now it remains to verify that we have the solution of the equation in (5.8). We calculate both terms on the left-hand side and then we subtract them. During the calculation we use the compositional property (3.22), $n = [\sigma] + 1$.

$$\mathbf{D}_0^{\sigma} y_p(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathbf{D}_0^{-(n-\sigma)} y_p(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t f(r)\sum_{j=0}^{\infty}\frac{\lambda^j (t-r)^{\sigma j+n-1}}{\Gamma(\sigma j+n)}\,\mathrm{d}r$$

$$\begin{aligned}
\lambda y_p(t) &= \lambda\int_0^t f(r)\sum_{j=0}^{\infty}\frac{\lambda^j (t-r)^{\sigma j+\sigma-1}}{\Gamma(\sigma j+\sigma)}\,\mathrm{d}r = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathbf{D}_0^{-n}\int_0^t f(r)\sum_{j=1}^{\infty}\frac{\lambda^j (t-r)^{\sigma j-1}}{\Gamma(\sigma j)}\,\mathrm{d}r = \\
&= \frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t f(r)\sum_{j=1}^{\infty}\frac{\lambda^j (t-r)^{\sigma j+n-1}}{\Gamma(\sigma j+n)}\,\mathrm{d}r
\end{aligned}$$

It is easy to see that:

$$\mathbf{D}_0^{\sigma} y_p(t) - \lambda y_p(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t f(r)\frac{(t-r)^{n-1}}{\Gamma(n)}\,\mathrm{d}r = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathbf{D}_0^{-n} f(t) = f(t).$$

We just proved that $y_p(t)$ solves the initial-value problem (5.8) and $y_h(t)$ given by (5.13) solves (5.7). According to the superposition principle (it is a linear FDE and fractional derivatives are linear too), the function

$$y(t) = y_h(t) + y_p(t)$$

solves the original initial-value problem (5.3). □

We showed how the solution of the initial-value problem (5.3) with the sequential fractional derivative looks like. Of course the cases of the Riemann-Liouville and the Caputo derivative are included.

The Riemann-Liouville derivative is given by the sequence (3.11), so there is no problem with the condition $\alpha_0 < \alpha_1$ (because $\alpha_0 = 0$), but the situation $\sigma_1 = 0$ may occur. Hence we obtain from (5.6) after some computations the following formula where $n = -[-\sigma]$:

$$y(t) = \sum_{k=1}^{n} b_k t^{\sigma-n+k-1} E_{\sigma,\sigma-n+k}(\lambda t^\sigma) + t^{\sigma-1} E_{\sigma,\sigma}(\lambda t^\sigma) * f(t).$$

For the Caputo derivative there is $\alpha_1 = 1$ so again no problem occurs with the condition $\alpha_0 < \alpha_1$. If we consider the sequence (3.12), we get from (5.6):

$$y(t) = \sum_{k=1}^{n_c} b_k t^{k-1} E_{\sigma,k}(\lambda t^\sigma) + t^{\sigma-1} E_{\sigma,\sigma}(\lambda t^\sigma) * f(t),$$

where $n_c = -[-\sigma]$.

5.1.2 Homogeneous equations with sequential fractional derivatives

In general a homogeneous LFDE with constant coefficients and with sequential derivatives has the form (with zero initial point)

$$\mathcal{D}_0^{\sigma_m} y(t) + \sum_{k=1}^{m-1} A_k \mathcal{D}_0^{\sigma_k} y(t) + A_0 y(t) = 0 \tag{5.14}$$

where the derivatives $\mathcal{D}_0^{\sigma_j}$ for $j = 1, \ldots, m$ may be composed of completely different sequences. It implies a large number of initial conditions in fact independent on m.

We will study the special case:

$$\begin{aligned} \mathcal{D}_0^{\sigma_m} y(t) + \sum_{k=1}^{m-1} A_k \mathcal{D}_0^{\sigma_k} y(t) + A_0 y(t) &= 0 \\ \mathcal{D}_0^{\sigma_k - 1} y(t)\big|_{t=0} &= b_k \end{aligned} \tag{5.15}$$

where all symbols have the same meaning as usual, and the sequential derivatives are calculated one after the other. This means that we keep the notation (5.5).

By applying the Laplace transform on (5.15) we obtain:

$$s^{\sigma_m}Y(s)-\sum_{j=1}^{m}s^{\sigma_m-\sigma_j}b_j+\sum_{k=1}^{m-1}\left[A_k s^{\sigma_k}Y(s)-\sum_{j=1}^{k}b_j s^{\sigma_k-\sigma_j}\right]+A_0Y(s)=0$$

$$Y(s)=\sum_{k=1}^{m}b_k\sum_{j=k}^{m}\frac{s^{\sigma_j-\sigma_k}}{s^{\sigma_m}+\sum_{i=1}^{m-1}A_i s^{\sigma_i}+A_0}$$

The calculation of the inverse Laplace transform is possible but we are not going to do it here in general. The technique of this calculation can be found in [1] on the pages 157-158.

Instead we will consider the example.

Example 5.1.3. Solve the following three-term initial-value problem in the sense we described above.

$$\mathcal{D}_0^{\frac{4}{3}}y(t)+2\mathcal{D}_0^{\frac{1}{2}}y(t)+y(t)=0$$
$$\mathcal{D}_0^{\frac{1}{3}}y(t)\big|_{t=0}=1$$
$$\mathcal{D}_0^{-\frac{1}{2}}y(t)\big|_{t=0}=2$$

The Laplace transform gives the formula:

$$s^{\frac{4}{3}}Y(s)-1-2s^{\frac{5}{6}}+2s^{\frac{1}{2}}Y(s)-4+Y(s)=0$$

and under the assumption $\left|\frac{1}{s^{\frac{4}{3}}+2s^{\frac{1}{2}}}\right|<1$ we can write:

$$Y(s)=\frac{2s^{\frac{5}{6}}+5}{s^{\frac{4}{3}}+2s^{\frac{1}{2}}+1}=\frac{2s^{\frac{5}{6}}+5}{s^{\frac{4}{3}}+2s^{\frac{1}{2}}}\cdot\frac{1}{1+\frac{1}{s^{\frac{4}{3}}+2s^{\frac{1}{2}}}}=\frac{2s^{\frac{5}{6}}+5}{s^{\frac{4}{3}}+2s^{\frac{1}{2}}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(s^{\frac{4}{3}}+2s^{\frac{1}{2}})^k}=$$
$$=\sum_{k=0}^{\infty}(-1)^k\left[\frac{2s^{-\frac{1}{2}k+\frac{1}{3}}}{(s^{\frac{4}{3}}+2s^{\frac{1}{2}})^{k+1}}+\frac{5s^{-\frac{1}{2}k-\frac{1}{2}}}{(s^{\frac{4}{3}}+2s^{\frac{1}{2}})^{k+1}}\right].$$

Then we use the term-by-term inversion and get the result:

$$y(t)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left[2t^{\frac{13}{6}k-\frac{4}{3}}E^{(k)}_{\frac{5}{6},\frac{4}{3}k-\frac{1}{3}}(-2t^{\frac{5}{6}})+5t^{\frac{13}{6}k-\frac{1}{2}}E^{(k)}_{\frac{5}{6},\frac{4}{3}k+\frac{1}{2}}(-2t^{\frac{5}{6}})\right].$$

Even in the general case the solution is given as an infinite series of the generalized Wright functions (derivatives of the Mittag-Leffler functions are special cases), but the coefficients are much more complicated.

5.1.3 Homogeneous equations with Riemann-Liouville derivatives

In this subsection we will study the homogeneous LFDE with a Riemann-Liouville derivative and appropriate initial conditions. Its general form is

$$\mathbf{D}_0^{\sigma_m}y(t)+\sum_{k=1}^{m-1}A_k\mathbf{D}_0^{\sigma_k}y(t)+A_0y(t)=0$$
$$\mathbf{D}_0^{\sigma_j-r}y(t)\big|_{t=0}=b_{jr} \tag{5.16}$$

where $j = 1, \ldots, m$ and $r = 1, \ldots, -[-\sigma_j]$. Due to the formula (3.11) it is in fact one of more difficult cases of (5.14) as we mentioned in the previous subsection. There are more initial conditions than in the initial-value problem (5.15), but the Laplace transform removes this problem because only integer-order powers of s occur in transformed derivatives next to initial conditions.

Let us explain this property in an example.

Example 5.1.4. Solve the following initial-value problem where $\alpha \in (1; 2\rangle$, $\beta \in (0; 1\rangle$ and A, B, b_1, b_2, b_3 are real constants.

$$\begin{aligned} \mathbf{D}_0^{\alpha} y(t) + A\,\mathbf{D}_0^{\beta} y(t) + By(t) &= 0 \\ \mathbf{D}_0^{\alpha-1} y(t)\big|_{t=0} &= b_1 \\ \mathbf{D}_0^{\alpha-2} y(t)\big|_{t=0} &= b_2 \\ \mathbf{D}_0^{\beta-1} y(t)\big|_{t=0} &= b_3 \end{aligned}$$

After the Laplace transform we have

$$s^{\alpha} Y(s) - b_2 s - b_1 + A\, s^{\beta} Y(s) - A\, b_3 + B\, Y(s) = 0$$

and we see that it is possible to combine the corresponding levels of the initial conditions. In this case it is only the first level - the initial conditions belonging to the first derivative of every differentiating term.

Denoting $D_1 = b_1 + A\, b_3$ and $D_2 = b_2$ we may write

$$Y(s) = \frac{D_2 s + D_1}{s^{\alpha} + A\, s^{\beta} + B} = \left| \begin{array}{c} \text{same process} \\ \text{like in the} \\ \text{previous example} \end{array} \right| = \sum_{k=0}^{\infty} (-B)^k \frac{D_2 s^{-\beta k - \beta + 1} + D_1 s^{-\beta k - \beta}}{(s^{\alpha-\beta} + A)^{k+1}}$$

which implies the result

$$y(t) = \sum_{k=0}^{\infty} \frac{(-B)^k}{k!} \left[D_2 t^{\alpha k + \alpha - 2} E^{(k)}_{\alpha-\beta, \alpha+\beta k - 1}(-A\, t^{\alpha-\beta}) + D_1 t^{\alpha k + \alpha - 1} E^{(k)}_{\alpha-\beta, \alpha+\beta k}(-A\, t^{\alpha-\beta}) \right].$$

It is clear that after the trick with the synthesis of the initial conditions the problem is very similar to the one solved in the previous subsection.

5.1.4 Homogeneous equations with Caputo derivatives

The Caputo derivative differs from the previous two cases a little bit because it has a fractional integral in its sequence and all derivatives in the sequence are of the same order. Hence, the number of initial conditions depends only on the maximal order of derivatives in the equation. Otherwise, due to various powers of s next to identical initial conditions in the Laplace transform of the equation, the solution is of a more complicated shape.

The general form of such equation is (5.17) where $j = 1, \ldots, -[-\sigma_m]$.

$$\begin{aligned} {}^{\mathrm{C}}\mathrm{D}_0^{\sigma_m} y(t) + \sum_{k=1}^{m-1} A_k\, {}^{\mathrm{C}}\mathrm{D}_0^{\sigma_k} y(t) + A_0 y(t) &= 0 \\ y^{j-1}(0) &= b_j \end{aligned} \tag{5.17}$$

Again we consider the example for this type of equations with $m = 2$.

Example 5.1.5. Solve the following initial-value problem, where $\alpha \in (1;2\rangle$, $\beta \in (0;1\rangle$, A, B, b_1, b_2 are real constants.

$$\begin{aligned} {}^{\mathrm{C}}\mathrm{D}_0^{\alpha} y(t) + A\,{}^{\mathrm{C}}\mathrm{D}_0^{\beta} y(t) + B\,y(t) &= 0 \\ y(0) &= b_1 \\ y'(0) &= b_2 \end{aligned}$$

As usual in this section we begin with the Laplace transform.

$$s^{\alpha} Y(s) - b_1 s^{\alpha-1} - b_2 s^{\alpha-2} + A\,s^{\beta} Y(s) - A\,b_1 s^{\beta-1} + B\,Y(s) = 0$$

$$\begin{aligned} Y(s) &= \frac{b_1(s^{\alpha-1} + A\,s^{\beta-1}) + b_2 s^{\alpha-2}}{s^{\alpha} + A\,s^{\beta} + B} = \left| \begin{array}{c} \text{same process} \\ \text{like in the} \\ \text{previous example} \end{array} \right| = \\ &= \sum_{k=0}^{\infty} (-B)^k \left[b_1 \frac{s^{\alpha-\beta-1-\beta k} + A\,s^{-1-\beta k}}{(s^{\alpha-\beta} + A)^{k+1}} + b_2 \frac{s^{\alpha-\beta-2-\beta k}}{(s^{\alpha-\beta} + A)^{k+1}} \right] \end{aligned}$$

The inverse Laplace transform gives term by term the final result.

$$\begin{aligned} y(t) = \sum_{k=0}^{\infty} \frac{(-B)^k}{k!} &\left[b_1 \left(t^{\alpha k} E^{(k)}_{\alpha-\beta,\beta k+1}(-A\,t^{\alpha-\beta}) + A\,t^{\alpha k+\alpha-\beta} E^{(k)}_{\alpha-\beta,\beta k+1+\alpha-\beta}(-A\,t^{\alpha-\beta}) \right) + \right. \\ &\left. + b_2 t^{\alpha k+1} E^{(k)}_{\alpha-\beta,\beta k+2}(-A\,t^{\alpha-\beta}) \right] \end{aligned}$$

While in the previous subsection there was the number of terms in the solution less than the number of initial conditions, here we observe the opposite effect.

5.1.5 The Mention about Green function

In subsections 5.1.2, 5.1.3 and 5.1.4 we studied only initial-value problems with homogeneous equations. If there is a nonzero function $f(t)$ on the right-hand side, it is sufficient to add the particular solution of this new equation with homogeneous initial conditions. This is a consequence of the superposition principle.

In subsection 5.1.1 we demonstrated this fact on the equation with one differential term. The particular solution was given by the convolution of $f(t)$ and any other function. It can be proven that in general the equation (5.2) with zero initial conditions has the particular solution of the form (5.18).

$$y(t) = \int_0^t G(t-\tau) f(\tau) \mathrm{d}\tau \tag{5.18}$$

The function $G(t-\tau)$ is called Green function and for nonlinear equations has more general form $G(t, \tau)$. Green function does not depend on the type of a fractional derivative because according to (3.16) the differences between them are combinations of initial values which are zero here.

For LFDE in the form (5.2) we may calculate Green function by using the Laplace transform similarly to subsection 5.1.1.

$$G(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^{\sigma_m} + \sum_{k=1}^{m-1} A_k s^{\sigma_k} + A_0}, s, t\right\} \tag{5.19}$$

The process of inverting is shown e.g. in [1] but we are not going to present it here.

5.2 The Method of the Reduction to a Volterra Integral Equation

This method follows directly from the proof of the existence and uniqueness theorem outlined in chapter 4. There we introduced the series which converges to the solution of an initial-value problem.

For the initial-value problem with the Riemann-Liouville derivative (4.4) appropriate sequence can be calculated in the following way (under suitable assumptions):

$$y_0(t) = \sum_{k=1}^{n} \frac{b_k}{\Gamma(\alpha - k + 1)}(t-a)^{\alpha-k} \tag{5.20}$$

$$y_j(t) = y_0(t) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau, y_{j-1}(\tau))\,\mathrm{d}\tau \tag{5.21}$$

where n is the number of initial conditions, $j = 1, 2, 3, \ldots$ and $f(t, y)$ is the right-hand side of the equation (4.4). Hence the solution is

$$y(t) = \lim_{j\to\infty} y_j(t). \tag{5.22}$$

This method can be easily (in some sense) applied to nonlinear equations too. We always obtain a recursion formula so in general there is a problem with the specification of a convergence interval.

Anyway, this method gives the solution in closed form for the linear two-term equation with constant coefficients and even for the equation:

$$\begin{aligned} \mathfrak{D}_a^{\alpha} y(t) - \lambda (t-a)^{\beta} y(t) &= 0 \\ \mathfrak{D}_a^{\alpha-k} y(a) &= b_k \end{aligned} \tag{5.23}$$

where b_k, λ are real constants, $k = 1, \ldots, m$ and $\beta > -\alpha$. Because we already know the solution of linear two-term equations with constant coefficients from subsection 5.1.1, we will solve the second problem now. Let us note without the proof that the problem (5.23) satisfies all necessary assumptions in order to this method.

Example 5.2.1. Solve the initial-value problem (5.23) with the Riemann-Liouville fractional derivative, $n = -[-\alpha]$.

Applying the formulas (5.20) and (5.21) we get the expressions

$$y_0(t) = \sum_{k=1}^{n} \frac{b_k}{\Gamma(\alpha - k + 1)}(t-a)^{\alpha-k},$$

$$y_j(t) = y_0(t) + \frac{\lambda}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} (\tau - a)^{\beta} y_{j-1}(\tau)\,\mathrm{d}\tau.$$

Let us compute terms $y_1(t)$, $y_2(t)$ and see what happens.

$$
\begin{aligned}
y_1(t) &= y_0(t) + \frac{\lambda}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \sum_{k=1}^{n} \frac{b_k}{\Gamma(\alpha-k+1)} (\tau-a)^{\alpha-\beta-k} \, \mathrm{d}\tau = \\
&= y_0(t) + \lambda \sum_{k=1}^{n} \frac{b_k}{\Gamma(\alpha-k+1)} \mathbf{D}_a^{-\alpha} \left[(t-a)^{\alpha+\beta-k}\right] = \\
&= y_0(t) + \lambda \sum_{k=1}^{n} \frac{b_k (t-a)^{2\alpha+\beta-k}}{\Gamma(\alpha-k+1)} \frac{\Gamma(\alpha+\beta-k+1)}{\Gamma(2\alpha+\beta-k+1)} \\
y_2(t) &= y_0(t) \lambda \mathbf{D}_a^{-\alpha} \left[(t-a)^{\beta} y_1(t)\right] = y_1(t) + \lambda^2 \mathbf{D}_a^{-\alpha} \left[(t-a)^{\beta} (y_1(t) - y_0(t))\right] = \\
&= y_1(t) + \lambda^2 \sum_{k=1}^{n} \frac{b_k (t-a)^{3\alpha+2\beta-k}}{\Gamma(\alpha-k+1)} \frac{\Gamma(\alpha+\beta-k+1)}{\Gamma(2\alpha+\beta-k+1)} \frac{\Gamma(2\alpha+2\beta-k+1)}{\Gamma(3\alpha+2\beta-k+1)}
\end{aligned}
$$

It can be proven that generally we have

$$
y_j(t) = \sum_{k=1}^{n} \frac{b_k (t-a)^{\alpha-k}}{\Gamma(\alpha-k+1)} \left[1 + \sum_{i=1}^{j} \left(\prod_{r=1}^{i} \frac{\Gamma[r(\alpha+\beta)-k+1]}{\Gamma[r(\alpha+\beta)+\alpha-k+1]} \right) \left(\lambda (t-a)^{\alpha+\beta} \right)^i \right].
$$

When we shift the index in the product and consider $m \to \infty$, we obtain the solution of this homogeneous equation containing the generalized Mittag-Leffler function (2.15):

$$
y_h(t) = \sum_{k=1}^{n} \frac{b_k (t-a)^{\alpha-k}}{\Gamma(\alpha-k+1)} E_{\alpha, 1+\frac{\beta}{\alpha}, 1+\frac{\beta-k}{\alpha}} \left(\lambda (t-a)^{\alpha+\beta} \right). \tag{5.24}
$$

Example 5.2.2. Solve the equation in the initial-value problem (5.23) with the Caputo fractional derivative and with initial conditions $y^{(k)}(a) = b_k$ for $k = 0, \ldots, n_c - 1$.

Here we solve the linear initial-value problem which we discussed generally for sequential derivative before. If we look at the Caputo derivative as its special case, chapter 4 implies the following procedure. Again by application of (5.21) with

$$
y_0(t) = \sum_{k=0}^{n_c-1} \frac{b_k}{k!} (t-a)^k
$$

and by the same steps as above we get the expression for the j^{th} term

$$
y_j(t) = \sum_{k=0}^{n_c-1} \frac{b_k}{k!} (t-a)^k \left[1 + \sum_{i=1}^{j} \left(\prod_{r=1}^{i} \frac{\Gamma[r(\alpha+\beta)-\alpha+k+1]}{\Gamma[r(\alpha+\beta)+k+1]} \right) \left(\lambda (t-a)^{\alpha+\beta} \right)^i \right].
$$

Then by using a limit and shift of the index we get the solution:

$$
y(t) = \sum_{k=0}^{n_c-1} \frac{b_k}{k!} (t-a)^k E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta+k}{\alpha}} \left(\lambda (t-a)^{\alpha+\beta} \right).
$$

We derived the solution of the homogeneous equation (5.23) with appropriate initial conditions in the Riemann-Liouville and the Caputo senses. It can be proven that the functions in the sums which form both solutions, are independent (for a proof see [2]). A nonhomogeneous version of this equation will be considered in section 5.3.

We saw that due to the linearity it is not difficult to obtain the formula for the j^{th} term of the series and then to pass to the limit. Generally the situation is not so simple.

5.3 The Power Series Method

In this section we will discuss the power series method of solving LFDEs with Riemann-Liouville derivatives. The main idea is contained in the formula:

$$\mathbf{D}_a^\alpha (t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}, \;\; \beta > \alpha > 0 \tag{5.25}$$

which implies that a fractional derivative does not change the quality of a power function. It can be generalized for series and even for functions defined by them. So it seems to be possible to find a solution of an equation in a form of a series, but there are two problems. First, we must be very careful about a convergence interval of series and second, it is not so easy to choose appropriate powers generating a series which would help us to solve the problem.

In some cases these points are more clear like in the following example.

Example 5.3.1. Solve the initial-value problem:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(f(t)(y(t)+1)\right) + \lambda \mathbf{D}_0^{\frac{1}{2}} y(t) &= 0 \\ y(0) &= A \end{aligned}$$

Due to the order of the fractional derivative let us suppose that the function $f(t)$ can be written in the form of series.

$$f(t) = \sum_{k=0}^{\infty} f_k\, t^{\frac{k}{2}}$$

It is reasonable to search the solution of the similar form:

$$y(t) = \sum_{k=0}^{\infty} y_k\, t^{\frac{k}{2}}.$$

If we substitute these two sums into the equation, we get after some computations

$$\sum_{k=1}^{\infty} \frac{k}{2}\left[\left(\sum_{j=0}^{k} y_{k-j} f_j\right) + f_k\right] t^{\frac{k-2}{2}} + \lambda \sum_{k=0}^{\infty} y_k \frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k}{2}+\frac{1}{2})} t^{\frac{k-1}{2}} = 0.$$

By the help of the initial condition we have directly $y_0 = A$. Other coefficients have to be determined by chain-solving the equation for the particular powers of t.

$$t^{\frac{k-2}{2}}: \quad \sum_{j=0}^{k} y_{k-j} f_j + f_k + \frac{2y_{k-1}\lambda}{k}\frac{\Gamma(\frac{k}{2}+\frac{1}{2})}{\Gamma(\frac{k}{2})} = 0, \quad k = 1, 2, \ldots$$

So we obtained the recursion formula in implicit form from which we can calculate the function $y(t)$ in successive steps.

There are some special cases where we can express the k^{th} term of the sequence or even to find the closed-form solution. One of those examples is considered in [1]:

$$y(0) = 0, \quad f(t) = 1 - \sqrt{t}, \quad \lambda = \frac{2}{\sqrt{\pi}}.$$

If we perform all calculations, we get the solution in closed form:

$$y(t) = \sqrt{t}.$$

In the previous example we chose the powers of t as multiples of $\frac{1}{2}$. There are two reasons to do so - the order of the fractional derivative (see formula (5.25)), and the product of $f(t)$ and $y(t)$ has to stay a multiple of $\frac{1}{2}$. A convergence interval is not generally determined because we do not know all coefficients y_k, however in the mentioned special case it is the positive real half-axis.

This method may be in principle used even for nonlinear equations, if we manage to find suitable powers of t.

At the end let us note, that if the initial condition is given for $\mathbf{D}_0^{\alpha-1}y(t)\big|_{t=0}$, we should expect a solution in the form

$$y(t) = t^{\alpha-1}\sum_{k=0}^{\infty} y_k t^{Xk} \tag{5.26}$$

where X is a suitable number. In more complicated situations we may suppose the solution as a linear combination of the functions (5.26) with various values of α and X.

The Compositional Method

In [2] there is introduced so-called compositional method which essentially starts from the power series method. We prepare some series in advance and then we calculate its parameters by substituting into the solved equation. Of course we have to estimate the solution in a right form which is the main disadvantage of the method.

Anyway, in some cases this method may be useful as we will see now. The following formula holds under the assumptions $\alpha > 0$, $m > 0$, $l > m - 1 - \frac{1}{\alpha}$ and $\alpha(jm + l) \notin \mathbb{Z}^-$ for $j \in \mathbb{N}_0$.

$$\begin{aligned}
&\mathbf{D}_a^{\alpha}\left[A(t-a)^{\alpha(l-m+1)}E_{\alpha,m,l}\left(\lambda(t-a)^{\alpha m}\right)\right] = \\
&\quad = A\frac{\Gamma(\alpha(l-m+1)+1)}{\Gamma(\alpha(l-m)+1)}(t-a)^{\alpha(l-m)} + A\lambda(t-a)^{\alpha l}E_{\alpha,m,l}\left(\lambda(t-a)^{\alpha m}\right).
\end{aligned} \tag{5.27}$$

Example 5.3.2. Let us consider the initial-value problem:

$$\begin{aligned}
\mathbf{D}_a^{\alpha}y(t) &= \lambda(t-a)^{\beta}y(t) + \sum_{r=1}^{p} f_r(t-a)^{\mu_r} \\
\mathbf{D}_a^{\alpha-k}y(a) &= b_k
\end{aligned} \tag{5.28}$$

with f_r, b_k, α, β real constants. The equation has non-constant coefficients but is still linear so the superposition principle holds. The solution satisfying the homogeneous equation and the original initial values was derived in the previous section by using the method of a Volterra integral. Hence, it is sufficient to find the solution of this problem with zero initial conditions.

If we expect the solution as the product of the generalized Mittag-Leffler function and the power function, we can compare its derivative (5.27) and the right-hand side of the equation. This procedure leads to the equations:

$$\begin{aligned}
\mu_r &= \alpha(l-m) \\
\beta &= \alpha(m-1) \\
f_r &= A\frac{\Gamma(\alpha(l-m+1)+1)}{\Gamma(\alpha(l-m)+1)}
\end{aligned}$$

for all r. It is clear that the constants m, l in (5.27) are $m = 1 + \frac{\beta}{\alpha}$ and $l = 1 + \frac{\mu_r + \beta}{\alpha}$, thus the particular solution is the sum of these functions for all r:

$$y_p(t) = \sum_{r=1}^{p} \frac{\Gamma(\mu_r + 1) f_r}{\Gamma(\mu_r + \alpha + 1)} (t-a)^{\alpha + \mu_r} E_{\alpha, 1+\frac{\beta}{\alpha}, 1+\frac{\beta+\mu_r}{\alpha}} \left(\lambda (t-a)^{\alpha+\beta}\right). \tag{5.29}$$

Due to the superposition principle we may write the solution of the problem (5.25) as the sum

$$y(t) = y_h(t) + y_p(t)$$

where $y_h(t)$ is from (5.24) and $y_p(t)$ was just computed.

Now we know how to solve another type of linear initial-value problems (5.28), so-called two-term LFDEs with quasi-polynomial coefficients and a quasi-polynomial free term.

Other mutations of the compositional method work with Bessel type functions in an analogous way, but we will not consider them here.

5.4 The Method of the Transformation to ODE

Some initial-value problems may be solved by the transformation to an ordinary differential equation. In principal there is a condition w.r.t. the order of derivatives - if there is only one derivative in the equation, its order has to be a rational number. If there are more differential terms, all their orders have to be rational numbers and moreover we should manage to get only integer-order terms by combining the used fractional orders. Other problems may occur during the computation because it is difficult to apply a fractional derivative on terms containing an unknown function.

We will demonstrate this method on examples with one differential term. For better understanding the method's spirit, let us first introduce the following simple example with the Riemann-Liouville semiderivative.

Example 5.4.1. Solve the initial-value problem

$$\mathbf{D}_0^{\frac{1}{2}} y(t) = y(t)$$
$$\mathbf{D}_0^{-\frac{1}{2}} y(t)\big|_{t=0} = b.$$

We see that if we apply again the semiderivative on the entire equation, we get the ordinary differential equation of the first order. During computation it is necessary to remember the formula for the composition of fractional derivatives (3.24).

$$y'(t) - \mathbf{D}_0^{-\frac{1}{2}} y(t)\big|_{t=0} \frac{t^{-\frac{3}{2}}}{\Gamma(-\frac{1}{2})} = \mathbf{D}_0^{\frac{1}{2}} y(t)$$

Now we use the initial condition and even the original equation and after some calculations we obtain the nonhomogeneous linear ODE with constant coefficients.

$$y'(t) - y(t) = -\frac{b}{2\sqrt{\pi}} t^{-\frac{3}{2}}$$

First we can calculate the general solution of the appropriate homogeneous equation which is $y_h(t) = C\,\mathrm{e}^t$, where C is a constant. We know from the theory of linear ODEs that the solution of the original nonhomogeneous equation can be found in the form $y(t) = C(t)\,\mathrm{e}^t$. The function $C(t)$ can be determined by substituting back into ODE.

$$\begin{aligned} C'(t)\,\mathrm{e}^t + \lambda^2 C(t)\,\mathrm{e}^t - C(t)\,\mathrm{e}^t &= -\frac{b}{2\sqrt{\pi}} t^{-\frac{3}{2}} \\ C(t) = -\frac{b}{2\sqrt{\pi}} \int_0^t \tau^{-\frac{3}{2}} \mathrm{e}^{-\tau}\,\mathrm{d}\tau & \end{aligned}$$

This integral diverges at the first glance, but we may identify so-called incomplete Gamma function there, hence we may write $C(t)$ in the form (for more details see [10])

$$C(t) = b\left(\operatorname{erf}\left(\sqrt{t}\right) + \frac{\mathrm{e}^{-t}}{\sqrt{\pi t}}\right).$$

Thus the solution of the ODE is

$$y(t) = C\,\mathrm{e}^t + b \operatorname{erf}\left(\sqrt{t}\right)\mathrm{e}^t + \frac{b}{\sqrt{\pi t}}.$$

The last thing that remains, isto determine the unknown constant C. The only condition we did not use yet, is the original FDE. We are not going to compute the semiderivative of $y(t)$ here but only introduce the result.

$$\mathbf{D}_0^{\frac{1}{2}} y(t) = C\left(\frac{1}{\sqrt{\pi t}} + \operatorname{erf}\left(\sqrt{t}\right)\mathrm{e}^t\right) + \frac{b}{\sqrt{\pi}}\mathrm{e}^t$$

Let us substitute it and find the constant C.

$$C\left(\frac{1}{\sqrt{\pi t}} + \operatorname{erf}\left(\sqrt{t}\right)\mathrm{e}^t\right) + b\,\mathrm{e}^t = C\,\mathrm{e}^t + b\operatorname{erf}\left(\sqrt{t}\right)\mathrm{e}^t + \frac{b}{\sqrt{\pi t}}$$

It is clear that $C = b$ and then the solution of the initial-value problem is

$$y(t) = b\left(\mathrm{e}^t + \operatorname{erf}\left(\sqrt{t}\right)\mathrm{e}^t + \frac{1}{\sqrt{\pi t}}\right).$$

It is easy to check that we would get the same result by using the Laplace transform method.

The method of the transformation to ODE is more general and it could be used even for linear FDEs with nonconstant coefficients but the problem is the big technical complication. Let us show this on the following example.

Example 5.4.2. Solve the initial-value problem:

$$\begin{gathered} \mathbf{D}_0^{\frac{1}{3}} y(t) - \lambda t\, y(t) = f(t) \\ \mathbf{D}_0^{\frac{1}{3}-1} y(t)\big|_{t=0} = 0, \end{gathered}$$

where the function $f(t)$ satisfies the condition $\mathbf{D}_0^{\frac{1}{3}-1} f(t)\big|_{t=0} = 0$.

Before we start the computations, it is necessary to derive the formula for a fractional derivative of the product of the function $f(t)$ with a polynomial. It is a particular case of the Leibniz rule which we did not introduce before (see [1], [10]).

First let us derive this rule for a fractional integral ($r \in \mathbb{N}$, $\alpha > 0$).

$$\begin{aligned}
\mathbf{D}_0^{-\alpha}[t^r f(t)] &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau^r f(\tau)\,\mathrm{d}\tau = \left| \begin{array}{c} \tau^r = (t-(t-\tau))^r = \\ = \sum_{k=0}^r \frac{(-1)^k r!}{k!(r-k)!} t^{r-k} (t-\tau)^k \end{array} \right| = \\
&= \sum_{k=0}^r (-1)^k \frac{r!}{(r-k)!} t^{r-k} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{k!} (t-\tau)^{\alpha+k-1} f(\tau)\,\mathrm{d}\tau = \\
&= \sum_{k=0}^r (-1)^k \frac{(\alpha+k-1)\cdots\alpha}{k!} \cdot \frac{\mathrm{d}^k}{\mathrm{d}t^k} t^r \cdot \mathbf{D}_0^{-\alpha-k} f(t) = \\
&= \sum_{k=0}^r \binom{-\alpha}{k} \frac{\mathrm{d}^k}{\mathrm{d}t^k} t^r \cdot \mathbf{D}_0^{-\alpha-k} f(t)
\end{aligned}$$

With this result and a well-known formula for the classical derivative of a product of two functions we can derive the relation even for α-derivative, $n = [\alpha] + 1$, $\alpha > 0$.

$$\begin{aligned}
\mathbf{D}_0^{\alpha}[t^r f(t)] &= \frac{\mathrm{d}^n}{\mathrm{d}t^n} \mathbf{D}_0^{-(n-\alpha)}[t^r f(t)] = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \sum_{k=0}^r \binom{-n+\alpha}{k} \cdot \frac{\mathrm{d}^k}{\mathrm{d}t^k} t^r \cdot \mathbf{D}_0^{-n+\alpha-k} f(t) = \\
&= \sum_{k=0}^r \binom{-n+\alpha}{k} \sum_{j=0}^n \binom{n}{j} \frac{\mathrm{d}^{k+j}}{\mathrm{d}t^{k+j}} t^r \cdot \mathbf{D}_0^{\alpha-k-j} f(t) = \left| \begin{array}{c} k+j=s \\ \text{all terms for} \\ s>r \text{ disappear} \end{array} \right| = \\
&= \sum_{s=0}^r \frac{\mathrm{d}^s}{\mathrm{d}t^s} t^r \cdot \mathbf{D}_0^{\alpha-s} f(t) \sum_{j=0}^s \binom{-n+\alpha}{s-j} \binom{n}{j}
\end{aligned}$$

It can be proven that the following formula holds

$$\sum_{j=0}^s \binom{-n+\alpha}{s-j} \binom{n}{j} = \binom{\alpha}{s},$$

thus we can write the result for $\alpha \in \mathbb{R}$ in the common form

$$\mathbf{D}_0^{\alpha}[t^r f(t)] = \sum_{k=0}^r \binom{\alpha}{k} \frac{\mathrm{d}^k}{\mathrm{d}t^k} t^r \cdot \mathbf{D}_0^{\alpha-k} f(t). \tag{5.30}$$

Now we start solving our initial-value problem. Let us mention that due to the zero initial value there will be no additional terms from the composition formulas (3.24) and (3.23). Obviously we have to apply the operator $\mathbf{D}_0^{\frac{1}{3}}$ three times to get the integer-order derivative. The first application gives after using the formula (5.30):

$$\mathbf{D}_0^{\frac{2}{3}} y(t) - \lambda t\, \mathbf{D}_0^{\frac{1}{3}} y(t) - \frac{\lambda}{3} \mathbf{D}_0^{\frac{1}{3}-1} y(t) = \mathbf{D}_0^{\frac{1}{3}} f(t).$$

Due to the zero initial condition we may rewrite $\mathbf{D}_0^{\frac{1}{3}-1} y(t) = \mathbf{D}_0^{-1} \mathbf{D}_0^{\frac{1}{3}} y(t)$ and then by substituting from the original equation we get all terms $\mathbf{D}_0^{\frac{1}{3}} y(t)$ out.

$$\mathbf{D}_0^{\frac{2}{3}} y(t) - \lambda^2 t^2 y(t) - \frac{\lambda^2}{3} \mathbf{D}_0^{-1}[t\, y(t)] = \mathbf{D}_0^{\frac{1}{3}} f(t) + \lambda t\, f(t) + \frac{\lambda}{3} \mathbf{D}_0^{-1} f(t)$$

According to the formula (3.22) also $\mathbf{D}_0^{\frac{1}{3}}\mathbf{D}_0^{-1}y(t) = \mathbf{D}_0^{-1}\mathbf{D}_0^{\frac{1}{3}}y(t)$ holds (even for $f(t)$), hence another application of $\mathbf{D}_0^{\frac{1}{3}}$ gives

$$\mathbf{D}_0^1 y(t) - \lambda^2\,\mathbf{D}_0^{\frac{1}{3}}[t^2 y(t)] - \frac{\lambda^2}{3}\,\mathbf{D}_0^{-1}\mathbf{D}_0^{\frac{1}{3}}[t\,y(t)] = \mathbf{D}_0^{\frac{2}{3}}f(t) + \lambda\,\mathbf{D}_0^{\frac{1}{3}}[t\,f(t)] + \frac{\lambda}{3}\,\mathbf{D}_0^{-\frac{2}{3}}f(t)$$

We arrange the composed terms separately:

$$\begin{aligned} \mathbf{D}_0^{\frac{1}{3}}[t^2 y(t)] &= t^2\,\mathbf{D}_0^{\frac{1}{3}}y(t) + \frac{2}{3}t\,\mathbf{D}_0^{-1}\mathbf{D}_0^{\frac{1}{3}}y(t) + \frac{2}{9}\,\mathbf{D}_0^{-2}\mathbf{D}_0^{\frac{1}{3}}y(t) \\ \mathbf{D}_0^{\frac{1}{3}-1}[t\,y(t)] &= t\,\mathbf{D}_0^{-1}\mathbf{D}_0^{\frac{1}{3}}y(t) - \frac{2}{3}\,\mathbf{D}_0^{-2}\mathbf{D}_0^{\frac{1}{3}}y(t) \end{aligned}$$

By combining the same terms the left-hand side of the equation gets the form:

$$\mathbf{D}_0^1 y(t) - \lambda^2 t^2\,\mathbf{D}_0^{\frac{1}{3}}y(t) - \lambda^2 t\,\mathbf{D}_0^{-1}\mathbf{D}_0^{\frac{1}{3}}y(t) + \frac{8\lambda^2}{9}\,\mathbf{D}_0^{-2}\mathbf{D}_0^{\frac{1}{3}}y(t).$$

Now we again replace $\mathbf{D}_0^{\frac{1}{3}}y(t)$ by the expression from the original equation and then we apply the formula (5.30) on the terms with the integer-order integrals. Finally we obtain the equation which contains only classical integrals and the first derivative of the unknown function $y(t)$:

$$\begin{aligned} &\mathbf{D}_0^1 y(t) - \lambda^3 t^3 y(t) - \lambda^3 t^2\,\mathbf{D}_0^{-1}y(t) + \frac{17\lambda^3}{9}t\,\mathbf{D}_0^{-2}y(t) - \frac{16\lambda^3}{9}\,\mathbf{D}_0^{-3}y(t) = \\ &\quad = \mathbf{D}_0^{\frac{2}{3}}f(t) + \lambda t\,\mathbf{D}_0^{\frac{1}{3}}f(t) + \frac{2\lambda}{3}\,\mathbf{D}_0^{-\frac{2}{3}}f(t) + \lambda^2 t^2 f(t) + \lambda^2 t\,\mathbf{D}_0^{-1}f(t) - \frac{8\lambda^2}{9}\,\mathbf{D}_0^{-2}f(t) \end{aligned}$$

We managed to convert the nonhomogeneous LFDE with a polynomial coefficient to the nonhomogeneous linear ordinary integral-differential equation with polynomial coefficients. This procedure was quite tough and the resulting equation is too complicated for analytical solution. For special functions $f(t)$ this method could be useful for numerical solving if a fractional calculus toolbox is not available.

So we saw that this method primarily does not work with series but with functions written in closed forms. It is the only method which really needs the ability to calculate differintegrals of various functions (right-hand side) and in more general cases even the Leibniz rule and the rule for the differintegration of a compositional function. Moreover, for equations with nonconstant coefficients this method even complicates the situation. These are the reasons why this method has only theoretical significance and is not used very often.

Chapter 6

Applications of Fractional Calculus

In the last two decades fractional calculus starts to intervene significantly in engineering, physics, economics etc. It is due to new possibilities which fractional calculus brings into the modelling of various problems. The main points are the global character of differintegrals and their linearity.

In this chapter we will introduce four applications. The first one, the problem of the tautochrone, is one of the basic examples where fractional calculus was used, and it demonstrates its utility for solving some types of integral equations. Next we will study the hydrogeology application known as the fractional advection-dispersion equation which shows the fundamental connection between fractional derivatives and the stable distributions. In the third section we will discuss the fractional oscillator and its properties. There will be also pointed out the link to the stable distributions. In the last section we will consider the problem of fractional viscoelasticity.

6.1 The Historical Example: The Tautochrone Problem

This well-known example was for the first time studied by Abel in the early 19^{th} century. It was one of the basic problems where the framework of the fractional calculus was used although it is not essentially necessary (see [10]).

The problem is the following: to find a curve in the (x, y)-plane such that the time required for a particle to slide down the curve to its lowest point is independent of its initial placement on the curve; suppose the homogeneous gravity field and no friction. Let us fix the lowest point of a curve at the origin and the position of a curve in the positive quadrant of the plane, denoting by (x, y) the initial point and (x^*, y^*) any point intermediate between $(0, 0)$ and (x, y).

According to the energy conservation law we may write

$$\frac{m}{2}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 = mg(y - y^*),$$

where σ is the length along the curve measured from the origin, m the mass of the particle, g the gravitational acceleration. Considering $\frac{\mathrm{d}\sigma}{\mathrm{d}t} < 0$ and $\sigma = \sigma(y^*(t))$, we rewrite the formula in the form

$$\sigma' \frac{\mathrm{d}y^*}{\mathrm{d}t} = -\sqrt{2g(y - y^*)},$$

which we integrate from $y^* = y$ to $y^* = 0$ and from $t = 0$ to $t = T$. After some calculations we get the integral equation:

$$\int_0^y \frac{\sigma'(y^*)}{\sqrt{y - y^*}} \, \mathrm{d}y^* = \sqrt{2g}T.$$

Here one can easily recognize the Caputo differintegral and write

$${}^{\mathrm{C}}\mathrm{D}_0^{\frac{1}{2}} \sigma(y) = \frac{\sqrt{2g}}{\Gamma(\frac{1}{2})} T.$$

Let us note that T is the time of descent, so it is a constant. By applying the $\frac{1}{2}$-integral to both sides of the equation and by using the formulas for the composition of the Caputo differintegrals (3.26) and for the fractional integral of the constant (3.34), we get the relation between the length along the curve and the initial position in y direction.

$$\sigma(y) = \frac{\Gamma(1)\sqrt{2g}T}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})} y^{\frac{1}{2}} = \frac{2\sqrt{2g}T}{\pi} y^{\frac{1}{2}}$$

The formula describing coordinates of points generating the curve can be written by the help of the relation:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}y} = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2},$$

which after the substitution of $\sigma(y)$ gives

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \sqrt{\frac{2gT^2}{\pi^2 y} - 1}.$$

It can be shown that the solution of this equation is so-called tautochrone, i.e. one arch of the cycloid which arises by rolling of the circle along the green line in figure 6.1. The parametric equations of the tautochrone are

$$x = \frac{A}{2}[u + \sin(u)]$$
$$y = \frac{A}{2}[1 - \cos(u)],$$

where $A = \frac{2gT^2}{\pi^2}$. In particular for $T = \frac{\pi}{\sqrt{2g}}$, i.e. for $A = 1$, the tautochrone is drawn in figure 6.1 by the red color.

We have seen that the knowledge of the rules of fractional calculus is very useful for solving this type of integral equations.

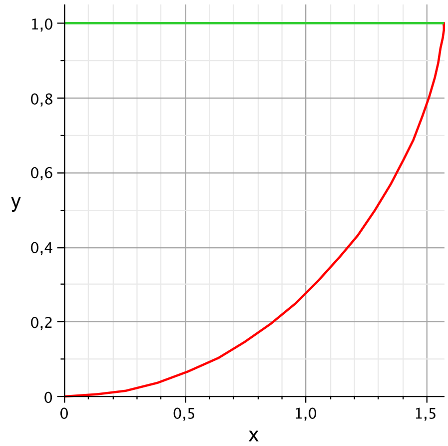


Figure 6.1: The Tautochrona for $A = \frac{2gT^2}{\pi^2} = 1$.

6.2 Fractional Advection-Dispersion Equation

The classical advection-dispersion equation describes the propagation of a tracer (small particles, fluid) in an infinite tube where the main fluid flows. Its basic one-dimensional form is

$$\frac{\partial c(x,t)}{\partial t} = -v\frac{\partial c(x,t)}{\partial x} + D\frac{\partial^2 c(x,t)}{\partial x^2}, \tag{6.1}$$

where $c(x,t)$ is the concentration of the tracer, v the velocity of the main flow (advection term) and D the diffusion coefficient (dispersion term). The well-known fundamental solution (i.e. for the initial condition $c(x,0) = \delta(x)$) has the Gaussian profile with the mean value moving with the speed v and with variance increasing in proportion to the square root of the time t. During the derivation of this equation we suppose that Fick's first law holds:

$$J = -D\frac{\partial c(x,t)}{\partial x},$$

so the diffusion flux J is proportional to minus the gradient of the concentration $c(x,t)$.

Of course there occur situations when Fick's first law cannot be used. For example if the tracer is being trapped into small regions (e.g. vortices) where is spending a random time - this case can be described by a model with the time γ-derivative on the left-hand side of the equation (6.1) with $\gamma \in (0,1\rangle$. In this thesis we will study the problem with the combination of the spatial left and right α-derivatives instead of the second derivative on the right-hand side of (6.1) with $\alpha \in (1,2\rangle$ and $\beta \in \langle -1,1\rangle$.

$$\frac{\partial c(x,t)}{\partial t} = -v\frac{\partial c(x,t)}{\partial x} + \frac{D}{2}\left[(1+\beta)\mathbf{D}^{\alpha}_{-\infty,x}c(x,t) + (1-\beta)_{-}\mathbf{D}^{\alpha}_{\infty,x}c(x,t)\right] \tag{6.2}$$

The bottom index x denotes the variable according to which we differentiate, β describes different probabilities of particle's step to the left and right direction (same probabilities occur for $\beta = 0$). We need here both left and right derivatives since we consider influences of all particles.

This model fits for heterogeneous structures with locations where particles go for a long distance by a large speed, so very fast particles are more probable. Such situations occur in porous media and in case of tracer's propagation in an aquifer in hydrogeology (see [5]).

The derivation of the equation (6.2) requires many results from probability theory so we are not going to present it here, but we will give some good reasons for using this equation at the end on the basis of the obtained solution. The traditional derivation can be found e.g. in [5] or [6], the eulerian approach is indicated in [4].

6.2.1 The Solution

Similarly to the case of the classical advection-dispersion equation, we start with the Fourier transform of the equation (6.2) using the relations (3.32) and (3.33).

$$\frac{\partial \hat{c}(k,t)}{\partial t} = \left[-\mathrm{i}vk + \frac{D}{2}(1+\beta)(\mathrm{i}k)^\alpha + \frac{D}{2}(1-\beta)(-\mathrm{i}k)^\alpha\right]\hat{c}(k,t)$$

We get an ODE w.r.t. the Fourier image $\hat{c}(k,t)$. If we assume the initial condition $c(x,0) = c_0(x)$, this equation has the following solution.

$$\hat{c}(k,t) = \hat{c}_0(k)\,\exp\left[\left(-\mathrm{i}vk + \frac{D}{2}(1+\beta)(\mathrm{i}k)^\alpha + \frac{D}{2}(1-\beta)(-\mathrm{i}k)^\alpha\right)t\right]$$

Now by using the formula

$$(\pm\mathrm{i}k)^\alpha = |k|^\alpha \exp\left(\pm\mathrm{i}\frac{\pi\alpha}{2}\mathrm{sgn}(k)\right) = |k|^\alpha\left[\cos\left(\frac{\pi\alpha}{2}\right) \pm \mathrm{i}\,\mathrm{sgn}(k)\sin\left(\frac{\pi\alpha}{2}\right)\right]$$

we get after some computations the better expression for the solution.

$$\hat{c}(k,t) = \hat{c}_0(k)\,\exp\left[-\mathrm{i}vkt - D\left|\cos\left(\frac{\pi\alpha}{2}\right)\right||k|^\alpha t\left(1+\mathrm{i}\,\beta\mathrm{sgn}(k)\mathrm{tg}\left(\frac{\pi\alpha}{2}\right)\right)\right]$$

According to (2.28) we can see directly that the solution is given by the convolution of the initial condition with so-called fundamental solution (analogously to the classical theory). The problem is that there does not exist an appropriate closed-form inverse, so the solution is given just by its Fourier transform. Anyway, if we substitute $k \to -k$, we can find out that we have the same formula like for the characteristic function of so-called α-stable density or density for Lévy skew α-stable distribution which we are going to introduce now. Hence, the fundamental solution coincides with α-stable density and we may write:

$$c(x,t) = c_0(x) * f_\alpha\left(x;\beta,(Dt)^{\frac{1}{\alpha}}\left|\cos\left(\frac{\pi\alpha}{2}\right)\right|^{\frac{1}{\alpha}}, vt\right). \tag{6.3}$$

6.2.2 Lévy Skew α-stable Distributions

Lévy skew α-stable distributions (simply stable distributions) form a family of continuous four-parameter probability distributions which includes even the classical Gaussian distribution. A stable distribution X is defined by its characteristic function (see e.g. [5])

$$\psi(k) = E\left(\mathrm{e}^{\mathrm{i}kX}\right) = \exp\left[\mathrm{i}\mu k - |ck|^\alpha\left(1-\mathrm{i}\beta\mathrm{sgn}(k)\Phi\right)\right], \tag{6.4}$$

$$\Phi = \begin{cases} \mathrm{tg}\left(\frac{\pi\alpha}{2}\right), & \alpha \neq 1 \\ -\frac{2}{\pi}\log|k|, & \alpha = 1 \end{cases}$$

where $\mu \in \mathbb{R}$ is for $\alpha > 1$ a mean value, $c \geq 0$ a scale (analogy to variance), $\beta \in \langle -1, 1 \rangle$ an analogy of skewness and the index $\alpha \in (0, 2\rangle$ is called a stability parameter. Apparently we can express stable distribution density by the inverse Fourier transform of this characteristic function by substituting $k \to -k$:

$$f_\alpha(x; \beta, c, \mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(-k)\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}k. \tag{6.5}$$

It can be proven that for $|x| \to \infty$ stable distributions follow power-law

$$f_\alpha \sim \frac{1}{|x|^{\alpha+1}}.$$

This property is called a power-law tail distribution or simply heavy tails and for $\alpha < 2$ it clearly implies the infinite variance of the appropriate random variable.

For better illustration of the stable density's shape, look at figure 6.2 for zero skewness and at figure 6.3 for non-zero skewness. We did not plot any curves with $\alpha \leq 1$ because we do not need them in this thesis. We see that the main difference between the Gaussian distribution and other symmetric stable distributions is just in heavy tails.

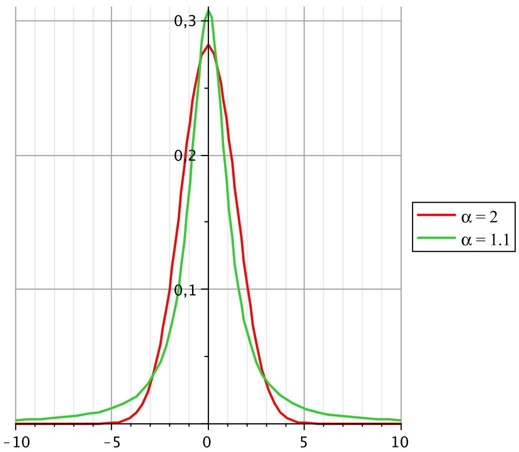


Figure 6.2: Stable distributions with $\beta = 0$, $c = 1$, $\mu = 0$.

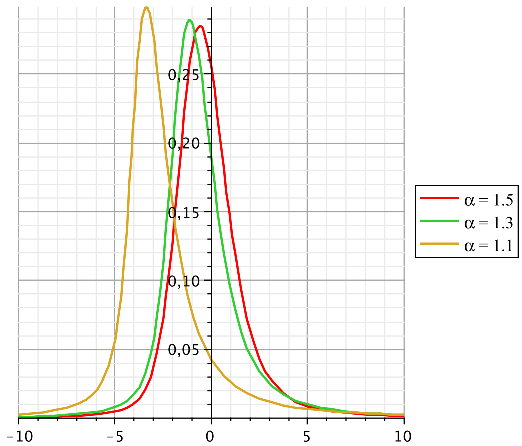


Figure 6.3: Stable distributions with $\beta = 1$, $c = 1$, $\mu = 0$.

The importance of the Gaussian distribution for physical applications is due to the central limit theorem which states that the sum of a number of random variables with finite variances tends to the Gaussian distribution as the number of variables grows. In other words, it does not matter how an event is distributed, if it has a finite variance, the distribution of a large number of these events is approximately the Gaussian.

With stable distributions we may generalize this theorem by removing the assumption of a finite variance. The sum of a number of random variables with power-law tail distributions tends to a stable distribution as the number of variables grows. That is the point which indicates the suitability of using fractional differential equation.

6.2.3 The Profile of the Fundamental Solution

Now we will consider the fundamental solution of the equation (6.2) which corresponds to the initial condition $c_0(x) = \delta(x)$ in (6.3) and coincides with the α-stable density.

The stable distributions include the Gaussian as the special case $\alpha = 2$, so the formula (6.3) describes even the solution of the classical advection-dispersion equation. We also considered only $\alpha \in (1, 2\rangle$ in the solution. We saw, that the mean value does not depend on α, so in the symmetric case the top of the profile moves exactly in the same way like for the Gaussian. On the contrary the scale depends on α thus for a smaller α it increases faster with time, but this effect is balanced by the cosinus term which magnifies the scale for greater α in a short-time horizon. The result is plotted in figures 6.4 and 6.5.

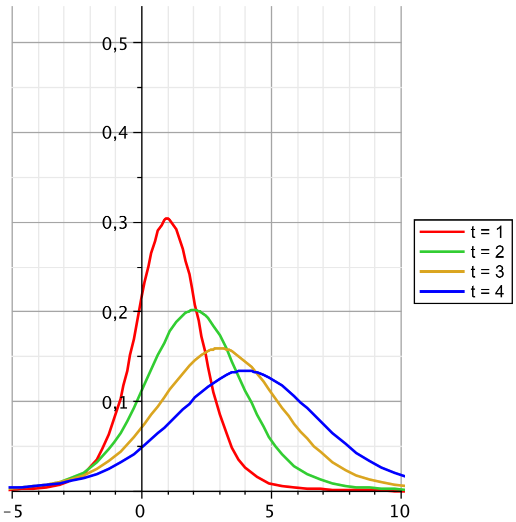


Figure 6.4: The fundamental solution with parameters $\alpha = 1.7$, $v = 1$, $D = 1$ and $\beta = 0$ during time.

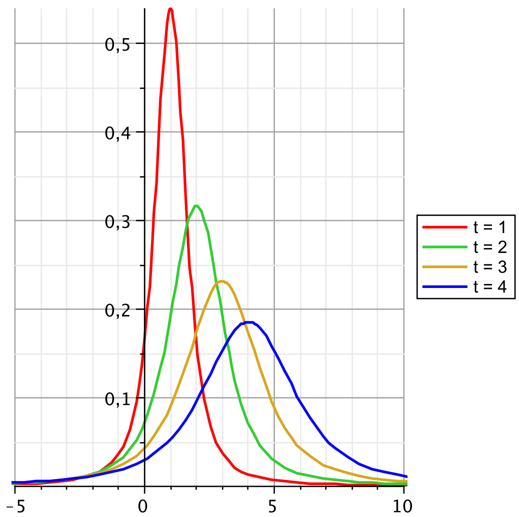


Figure 6.5: The fundamental solution with parameters $\alpha = 1.3$, $v = 1$, $D = 1$ and $\beta = 0$ during time.

The influence of the skewness analogy β is linked by figures 6.6 and 6.7. We see that the positive parameter β causes a very slow motion of the solution curve's maximum, the negative value of β has the opposite effect.

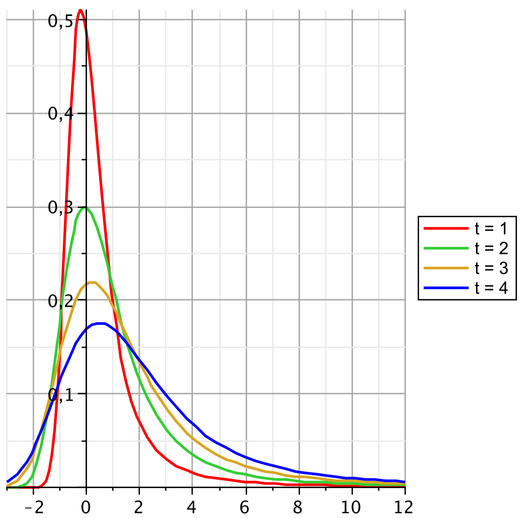


Figure 6.6: The fundamental solution with parameters $\alpha = 1.3$, $v = 1$, $D = 1$ and $\beta = 1$ during time.

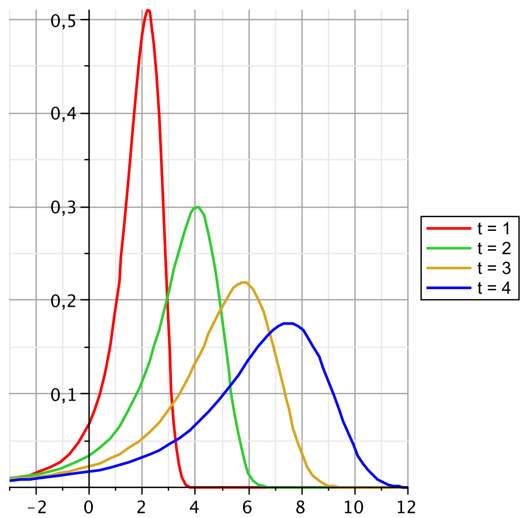


Figure 6.7: The fundamental solution with parameters $\alpha = 1.3$, $v = 1$, $D = 1$ and $\beta = -1$ during time.

6.2.4 The Reasons for Using Fractional Derivative

Instead of the question why to use fractional derivative, we should ask why to prefer the classical derivative. The integer-order derivative is a local operator, hence it is unsuitable in the situation where heavy tails (infinite variance) are expected, because the influence of larger neighbourhood cannot be neglected anymore. On the contrary, a fractional derivative is global so it manages to consider these effects and moreover it provides the possibility of different probabilities for the motion forward and backward.

The result in the form of a stable distribution is the confirmation of the good choice of the model, because it represents the limit case for any distribution with heavy tails. Therefore the situation is completely analogous to the classical advection-dispersion equation which in addition is included in our solution as the special case.

6.3 Fractional Oscillator

The harmonic oscillator is the fundamental model used in classical mechanics intervening even more general physical, chemical and engineering applications. The generalization to the language of fractional derivatives, which means replacing the second derivative by a fractional-order one in the equation of classical oscillator, is still in progress and properties of the fractional oscillator are investigated for various types of differintegrals.

Stanislavsky in [7] interprets the fractional oscillator as an ensemble average of ordinary harmonic oscillators governed by a stochastic time arrow. Used oscillators differ a little from each other in a frequency so that the interference causes a behaviour similar to the classical oscillator with damping. Let us note that the essence of the dissipation is completely different in these two cases. Stanislavsky does not introduce explicitly a type of the fractional operator because he develops the model via properties of normal modes, but the derived solution leads us to some more general kind of the sequential fractional derivative than the Riemann-Liouville and the Caputo ones. The interesting point of this article is the link to stable distributions - the normal modes are given by weighted average w.r.t. the probability density of a stable distribution. Nevertheless we will not follow this approach here because it would need tools from complex analysis which have not been introduced in this thesis.

Many authors (e.g. [8]) consider the equation of fractional oscillator in the form

$$\mathbf{D}^{\alpha}_{-\infty}x(t) + \omega^{\alpha}x(t) = f_0\omega^{\alpha-2}g(t), \tag{6.6}$$

where $\alpha \in (1, 2\rangle$ is the order of the derivative, ω is the vibration eigenfrequency, $x(t)$ the displacement from the equilibrium, f_0 is a constant and $g(t)$ denotes so-called source function. We use the Riemann-Liouville derivative with the lower bound tending to minus infinity in spite of the one with zero lower bound. Before starting our studies we assume all quantities to be constant (mostly zero), the beginning is determined by the non-zero value of the source function as we will see later.

We will study the example of two bodies linked by a spring with dissipation, which shows us an interesting behaviour of the system caused by using the fractional derivative. The model is described by the system of two equations (compare with (6.6)):

$$\begin{aligned}
\mathbf{D}^{\alpha}_{-\infty}x_1(t) + \omega^{\alpha}[x_1(t) - x_2(t) + \xi_0] &= f_0\omega^{\alpha-1}\delta(t) \\
\mathbf{D}^{\alpha}_{-\infty}x_2(t) + \omega^{\alpha}[x_2(t) - x_1(t) - \xi_0] &= 0.
\end{aligned}$$

We see that the Delta function $\delta(t)$ plays the role of a source function, so in fact the process starts at time $t = 0$. The reason why we do not set it as the lower bound, is hidden in the initial conditions. Now we completely avoid the initial conditions which are hardly physical interpreted for the Riemann-Liouville derivative except the zero ones. The meaning of the other used quantities is better illustrated by figure 6.8. Now we introduce

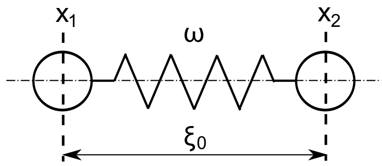


Figure 6.8: The scheme of the example with used quantities.

two new variables η, ξ. The first one $\eta = \frac{1}{2}(x_2 + x_1)$ represents the position of the center of gravity, the second one $\xi = x_2 - x_1$ is the distance between the bodies' centers. By subtracting and adding the previous equations we get two independent equations:

$$\begin{aligned} \mathbf{D}^{\alpha}_{-\infty}\eta(t) &= \frac{1}{2} f_0 \omega^{\alpha-1}\delta(t) \\ \mathbf{D}^{\alpha}_{-\infty}\xi(t) + 2\omega^{\alpha}[\xi(t) - \xi_0] &= -f_0\omega^{\alpha-1}\delta(t). \end{aligned}$$

The first equation can be solved directly by applying $\mathbf{D}^{-\alpha}_{-\infty}$ to both sides of the equation and considering all derivatives of η in minus infinity to be zero.

$$\eta(t) = \frac{f_0}{2\Gamma(\alpha)}(\omega t)^{\alpha-1} \tag{6.7}$$

The FDE for ξ is more difficult due to the ξ_0 term. We know that ξ_0 is the initial distance of the bodies therefore a non-zero constant for $t < 0$. It is reasonable to shift the coordinates: $\rho(t) = \xi(t) - \xi_0$. If we substitute this relation into the fractional derivative, we have to calculate the derivative of the constant function. According to the formula for a differintegral of the power function (3.36) we get again the power function but for lower bound tending to minus infinity it vanishes. That is the point where we appreciate the infinite lower bound because in this way we obtain the simple equation

$$\mathbf{D}^{\alpha}_{-\infty}\rho(t) + 2\omega^{\alpha}\rho(t) = -f_0\omega^{\alpha-1}\delta(t).$$

Now we have to use the Fourier transform but the procedure is analogous to the use of the Laplace transform so we will not calculate it here. The result is then given for $t \geq 0$ by (5.10) and for $t < 0$ we define it as zero. After shifting back we have the solution for the distance between the bodies' centers:

$$\xi(t) = \xi_0 - f_0(\omega t)^{\alpha-1}E_{\alpha,\alpha}(-2(\omega t)^{\alpha}). \tag{6.8}$$

The plot of the function $\xi(t)$ is presented in figure 6.9. The case $\alpha = 2$ corresponds to harmonic oscillations, for $\alpha < 2$ a dissipation occurs. There we see the behaviour qualitatively similar to the classical damped oscillator.

The situation is much more interesting for the evolution of the center of gravity drawn in figure 6.10. Again, the case $\alpha = 2$ describes the undamped case, so the entire system moves uniformly straightly to infinity as the consequence of the initial impulse. Unusual behaviour occurs for damping ($\alpha < 2$) because then the center of gravity tends to infinity with power law, whereas in the classical damped case there is always a finite constant stationary position. The research on this field still continues and applications are searched.

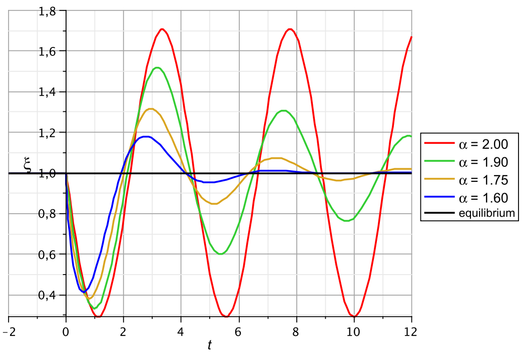


Figure 6.9: The time evolution of the distance between the bodies' centers for various α with $\omega = 1$, $f_0 = 1$ and $\xi_0 = 1$.

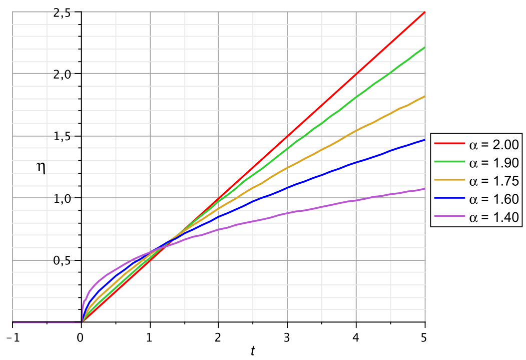


Figure 6.10: The time evolution of the center of gravity position for various values of α with $\omega = 1$ and $f_0 = 1$.

6.4 Viscoelasticity

Viscoelasticity is a scientific discipline describing the behaviour of materials and it belongs to the fields of the most extensive applications of differintegrals. The approach introduced here is not based on theoretical derivations (nevertheless this is possible), but it starts just from the similarity of mathematical models supported by experimental results. The fractional models describe very well e.g. polymers. For more information about viscoelasticity see [9] or [1].

There are two main quantities in viscoelasticity, a stress $\sigma(t)$ and a strain $\epsilon(t)$. Various models differ from each other in the way how to relate them. In the first subsection we will discuss the classical models of viscoelasticity, i.e. both extreme cases: the ideal solid and the ideal fluid, and then some viscoelastic models composed of them. In the second subsection we will introduce the fractional model and fractional generalizations of some models mentioned before.

For system's response examination usually the impulse response function is used (i.e. the excitation is the Dirac delta function) because it provides easy calculation of responses to all other excitations. But in viscoelasticity the step response function is more telling and realistic, thus we will use as the excitation the Heaviside step function $H(t)$ defined by (6.9).

$$H(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0 \end{cases} \tag{6.9}$$

We will investigate responses of the stress $\sigma(t)$ to the strain $\epsilon(t) = H(t)$ and vice-versa, i.e. responses of the strain $\epsilon(t)$ to the stress $\sigma(t) = H(t)$. Appropriate response functions are called the relaxation modulus $G(t)$ and the stress creep compliance $J(t)$. The calculation of $G(t)$ and $J(t)$ will be performed by the help of the Laplace transform. According to the definition of $H(t)$ we will suppose in this section all quantities to be equal to zero for $t < 0$, thus we will write only expressions for $t \geq 0$.

6.4.1 Classical Models

There are two fundamental models used in viscoelasticity. The first one is the ideal solid which describes the relation between the stress and the strain by Hooke's law (6.10) where E is so-called modulus of elasticity. Its relaxation modulus and stress creep compliance can be calculated easily and they are drawn in figures 6.11 and 6.12 respectively. The main qualitative problem of this simple model is the absence of the experimentally observed stress relaxation for a constant strain.

$$\sigma(t) = E\,\epsilon(t) \tag{6.10}$$

The second model is the ideal fluid described by Newton's law (6.11) where η is so-called viscosity. Its response functions are again plotted in figures 6.11 and 6.12, their formulas are obvious. We may observe the immediate total stress relaxation for the step strain there and infinite growing of the strain for the step stress. Both these properties should be improved.

$$\sigma(t) = \eta\,\frac{\mathrm{d}\epsilon(t)}{\mathrm{d}t} \tag{6.11}$$

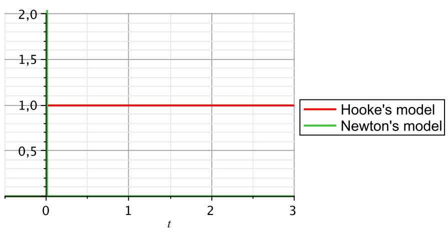


Figure 6.11: The relaxation moduli for ideal solid ($E = 1$) and ideal fluid ($\eta = 1$).

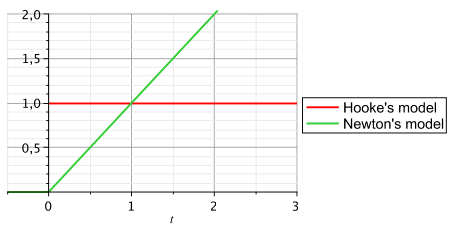
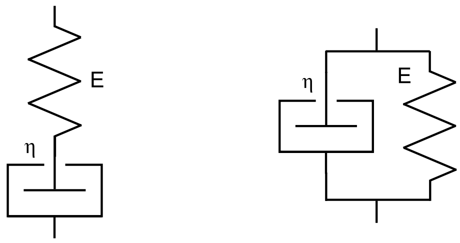


Figure 6.12: The stress creep compliances for ideal solid ($E = 1$) and ideal fluid ($\eta = 1$).

In nature does not exist any ideal solids and any ideal fluids either. Real materials have properties somewhere between those two limit cases. In the theory of viscoelasticity we usually work with a schematic representation of those two mentioned models. Hooke's element is symbolized by a spring whereas Newton's element by a dashpot. This representation provides an easy way how to get the models with intermediate properties by the help of Hooke's and Newton's laws. The simplest combinations are the serial connection

(a) Hooke's element (b) Newton's element

Figure 6.13: The schematic symbols of the basic elements.



(a) Maxwell's element (b) Voigt's element

Figure 6.14: The simplest combinations of the basic models.

(so-called Maxwell's model) and the parallel connection (so-called Voigt's model).

Maxwell's model is characterized by the equation (6.12), so there is the absolute strain term missing.

$$\frac{1}{\eta}\,\sigma(t) + \frac{1}{E}\,\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} = \frac{\mathrm{d}\epsilon(t)}{\mathrm{d}t} \tag{6.12}$$

Its step response functions $G_M(t)$, $J_M(t)$ are given by the following relations, their graphs are depicted in figures 6.16 and 6.17. We see that the stress relaxation process is more realistic, but there is still the infinite growing of the strain for the step stress.

$$G_M(t) = E\,\mathrm{e}^{-\frac{E}{\eta}t}$$
$$J_M(t) = \frac{1}{E}\,H(t) + \frac{1}{\eta}\,t$$

The parallel combination of the ideal elements gives us Voigt's model which is described by the formula (6.13).

$$\sigma(t) = E\,\epsilon(t) + \eta\,\frac{\mathrm{d}\epsilon(t)}{\mathrm{d}t} \tag{6.13}$$

The step response functions $G_V(t)$, $J_V(t)$ are drawn in figures 6.16 and 6.17 by the gold color. The stress creep compliance seems to be quite realistic, but there is the stress relaxation missing.

$$G_V(t) = E\,H(t) + \eta\,\delta(t)$$
$$J_V(t) = \frac{1}{E}\left(1 - \mathrm{e}^{-\frac{E}{\eta}t}\right)$$

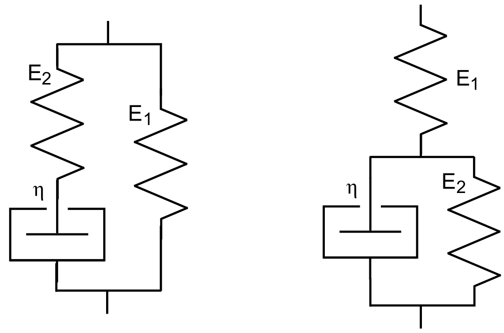
All models we presented until now differ from experimental observations at some points, hence we have to continue by generalization. The next steps are Zener's and Kelvin's models schematically represented in figure 6.15.

Kelvin's model uses the equation (6.14) with three parameters.

$$\frac{E_1 + E_2}{\eta}\,\sigma(t) + \frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} = \frac{E_1 E_2}{\eta}\,\epsilon(t) + E_1\,\frac{\mathrm{d}\epsilon(t)}{\mathrm{d}t} \tag{6.14}$$

The step response functions can be found in figures 6.16 and 6.17. We can recognize some kind of stress relaxation and even the strain does not grow to infinity anymore.

$$G_K(t) = \frac{E_1^2}{E_1 + E_2}\,\mathrm{e}^{-\frac{E_1+E_2}{\eta}t} + \frac{E_1 E_2}{E_1 + E_2}$$
$$J_K(t) = \frac{E_1 + E_2}{E_1 E_2} - \frac{1}{E_2}\,\mathrm{e}^{-\frac{E_2}{\eta}t}$$



(a) Zener's element (b) Kelvin's element

Figure 6.15: The advanced combinations of the basic models.

Zener's model is described by equation (6.15) which is similar to Kelvin's model.

$$\frac{E_2}{\eta}\sigma(t) + \frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} = \frac{E_1 E_2}{\eta}\epsilon(t) + (E_1 + E_2)\frac{\mathrm{d}\epsilon(t)}{\mathrm{d}t} \tag{6.15}$$

The step responses in figures 6.16 and 6.17 are quite similar to the previous case too.

$$G_Z(t) = E_1 + E_2\,\mathrm{e}^{-\frac{E_2}{\eta}t}$$
$$J_Z(t) = \frac{1}{E_1} - \frac{E_2}{E_1(E_1+E_2)}\,\mathrm{e}^{-\frac{E_1 E_2}{\eta(E_1+E_2)}t}$$

Kelvin's and Zener's models qualitatively agree with experiments but problems occur in quantitative description. Both models expect a finite value of the stress at time $t = 0$ in spite of the jump change of the strain, none of them can achieve the total stress relaxation. Both models also predict a discontinuity of the stress creep compliance after applying the step excitation stress.

We would get more realistic results by adding another basic elements but this causes an increasing order of the equation. There is a more elegant way - fractional variants of the introduced models.

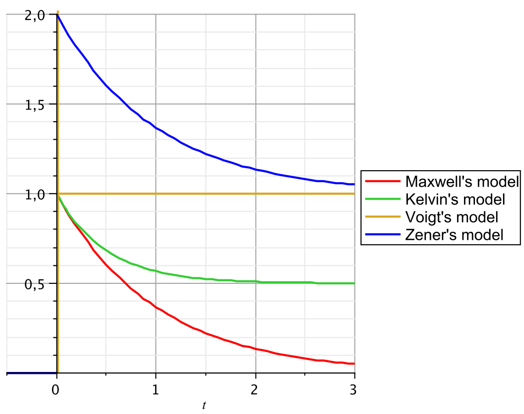


Figure 6.16: The relaxation moduli for classical combined models (all constants E, η, E_1, E_2 are equal to 1).

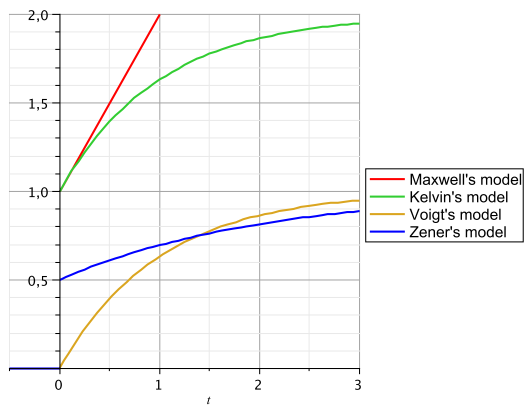


Figure 6.17: The stress creep compliances for classical combined models (all constants E, η, E_1, E_2 are equal to 1).

6.4.2 Fractional-order Models

Hooke's law can be considered as a proportionality of the stress to the 0-derivative of the strain, Newton's law describes a proportionality of the stress to the 1-derivative of the strain. Hence, it seems to be reasonable to introduce a more general model, a proportionality of the stress to the α-derivative of the strain where $\alpha \in \langle 0, 1\rangle$.

For including the entire history of the material we should use a differintegral with the lower bound minus infinity and then it also does not matter whether we work with the Riemann-Liouville or with the Caputo differintegral. Due to the assumption of zero values of all quantities for $t < 0$, the Riemann-Liouville differintegral with the lower bound $a = 0$ gives the same results like the previous ones and we are going to use it here (there is a better mathematical background for the operator $\mathbf{D}_0^\alpha$ in this thesis).

The fractional generalization of Hooke's and Newton's models is called Blair's model and is described by the formula (6.16). We introduce the new constant $\tau = \frac{\eta}{E}$ which guarantees a perfect coincidence with the ideal solid for $\alpha = 0$ and with the ideal fluid for $\alpha = 1$.

$$\sigma(t) = E\,\tau^\alpha\,\mathbf{D}_0^\alpha \epsilon(t) \tag{6.16}$$

The relaxation moduli and the stress creep compliances are plotted for various values of α in figures 6.18 and 6.19 respectively, their calculation can be performed via the Laplace transform.

$$G_B(t) = \frac{E\,\tau^\alpha}{\Gamma(1-\alpha)}\,t^{-\alpha}$$
$$J_B(t) = \frac{1}{E\,\tau^\alpha\Gamma(1+\alpha)}\,t^{\alpha}$$

Even this simple fractional model gives qualitatively realistic response functions: there is the stress relaxation, the infinite value of $G_B(t)$ for $t = 0$, the slower growing of the strain for the step stress and the continuity of $J_B(t)$. Moreover, a similar power-law stress relaxation was observed in real materials and it gives us a good reason for further improving of the fractional model.

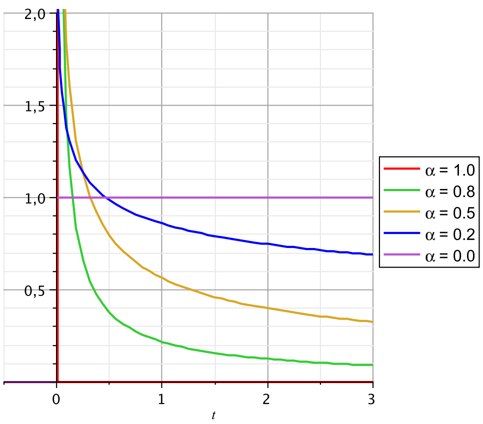


Figure 6.18: The relaxation moduli for Blair's model for various α ($E = 1$, $\tau = 1$).

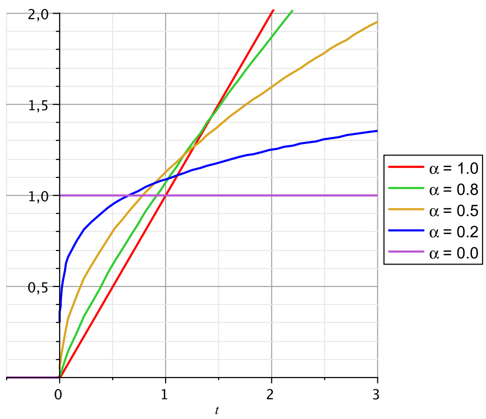


Figure 6.19: The stress creep compliances for Blair's model for α ($E = 1$, $\tau = 1$).

One may ask whether Blair's model is equivalent to a combination of Hooke's and Newton's elements. Such a representation is given in figure 6.20 and we see that we would need the infinite number of basic elements. That is why we use a special symbol for Blair's element drawn also in figure 6.20. The derivation of this representation is not trivial and we are not going to introduce it here. The required order α is given by the suitable combination of values E_i and η_i, in particular for all $E_i = E$ and $\eta_i = \eta$ we get $\alpha = \frac{1}{2}$.

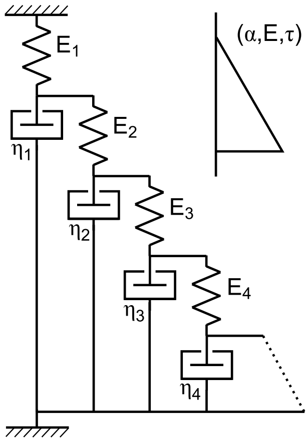


Figure 6.20: Blair's element represented by an infinite structure of the classical basic elements; the schematic symbol of Blair's element.

Introducing the fractional element provides a lot of possibilities for generalizations of models described in the previous subsection. Now we will consider Voigt's and Maxwell's models where one or both basic elements are substitute by Blair's element.

First we will study the generalized Voigt's model where Newton's element (dashpot) is replaced by Blair's. Clearly we get the equation (6.17) for this model with one fractional-order term, a_0, b_0 and b_1 are constants.

$$a_0\, \sigma(t) = b_0\, \epsilon(t) + b_1\, \mathbf{D}_0^{\alpha} \epsilon(t) \tag{6.17}$$

Its step response functions are more complicated, the stress creep compliance $J_{GV}(t)$ is given by a function of the Mittag-Leffler type. The comparison of this model with the classical Voigt's model is depicted in figures 6.22 and 6.23, we can see a more realistic relaxation modulus $G_{GV}(t)$ with infinite value at $t = 0$.

$$\begin{aligned} G_{GV}(t) &= \frac{b_0}{a_0}\, H(t) + \frac{b_1}{a_0 \Gamma(1-\alpha)}\, t^{-\alpha} \\ J_{GV}(t) &= \frac{a_0}{b_1}\, t^{\alpha}\, E_{\alpha,\alpha+1}\left(-\frac{b_0}{b_1} t^{\alpha}\right) \end{aligned}$$

Now we investigate a more complicated situation, the generalized Maxwell's model with two fractional-order terms. Here we will see that it is not possible to choose orders of differintegrals arbitrarily, so let us look at this example properly.

The schematic representation is in figure 6.21 where we assume without loss of generality that $\alpha > \beta$. In the serial connection the stress $\sigma(t)$ has the same value on both

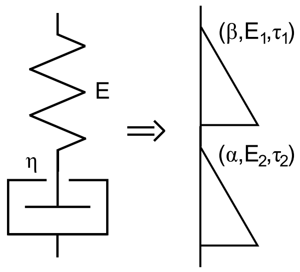


Figure 6.21: The schematical representation of the generalized Maxwell's model.

elements so we can write the equations of this system in the following form.

$$\sigma(t) = E_1\,\tau_1^\alpha\,\mathbf{D}_0^\alpha \epsilon_1(t)$$
$$\sigma(t) = E_2\,\tau_2^\beta\,\mathbf{D}_0^\beta \epsilon_2(t)$$

Then we apply the operator $\mathbf{D}_0^{\alpha-\beta}$ to the second equation (remember $\alpha > \beta$) and according to the composition rule (3.24) we get the followig equivalent system.

$$\sigma(t) = E_1\,\tau_1^\alpha\,\mathbf{D}_0^\alpha \epsilon_1(t)$$
$$\mathbf{D}_0^{\alpha-\beta}\sigma(t) = E_2\,\tau_2^\beta\,\mathbf{D}_0^\alpha \epsilon_2(t) + E_2\,\tau_2^\beta\,\mathbf{D}_0^{\beta-1}\epsilon_2(t)\Big|_{t=0}\frac{t^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}$$

We know that $\beta < 1$ so the last term of the second equation is the fractional integral and if we suppose $\epsilon_2(t)$ to be bounded in a neighborhood of the point $t = 0$ (which is reasonable from a physical point of view), this term disappears. Because the total strain of the system is $\epsilon(t) = \epsilon_1(t) + \epsilon_2(t)$, the sum of these two equations gives the formula for this model (a_0, a_1 and b_1 are constants depending on $E_1, E_2, \tau_1, \tau_2, \alpha$ and β).

$$a_0\,\sigma(t) + a_1\,\mathbf{D}_0^{\alpha-\beta}\sigma(t) = b_1\,\mathbf{D}_0^\alpha \epsilon(t) \tag{6.18}$$

The correct derivation of the equation gives a condition for the orders of the differintegrals because the order connected with $\sigma(t)$ has to be less than the order incident to $\epsilon(t)$.

The step responses $G_{GM}(t)$, $J_{GM}(t)$ are given by the help of the Mittag-Leffler function and power functions, the graphs are drawn in figures 6.22 and 6.23. Again we have the realistic relaxation modulus with the infinite value in $t = 0$ and the continuous stress creep compliance. The differences between this model and the generalized Voigt's model are the infinite growing of the strain for the step stress and types of used functions (power functions are substituted by the Mittag-Leffler function and vice-versa).

$$G_{GM}(t) = \frac{b_1}{a_1}\,t^{-\beta}\,E_{\alpha-\beta,1-\beta}\left(-\frac{a_0}{a_1}t^{\alpha-\beta}\right)$$
$$J_{GM}(t) = \frac{a_0}{b_1\Gamma(1+\alpha)}\,t^\alpha + \frac{a_1}{b_1\Gamma(1+\beta)}\,t^\beta$$

As we saw, even such simple fractional models behave more realistically than some advanced classical models described in the previous subsection. Podlubný in [1] mentions that the four-parameter model

$$\sigma(t) + a_1\,\mathbf{D}_0^\alpha \sigma(t) = b_0\,\epsilon(t) + b_1\,\mathbf{D}_0^\alpha \epsilon(t)$$

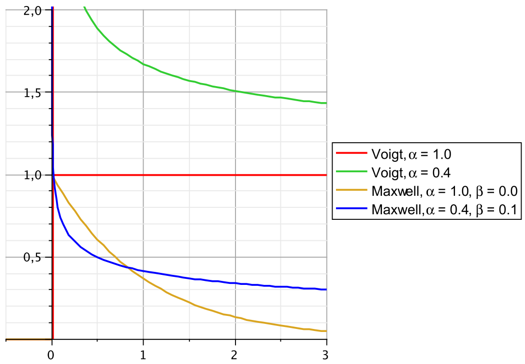


Figure 6.22: The relaxation moduli for the generalized Voigt's and Maxwell's models (a_0, a_1, b_0, b_1 equal 1).

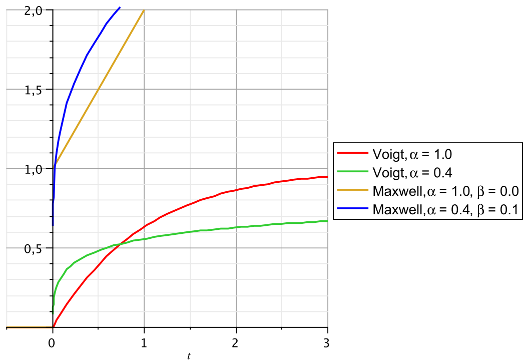


Figure 6.23: The stress creep compliances for the generalized Voigt's and Maxwell's models (a_0, a_1, b_0, b_1 equal 1).

gives satisfactory descriptions of most real materials.

If we would continue in generalization, we would get models which are extremely successful in describing viscoelastic properties of many polymeric materials (interconnected and branched structures, gels and filled polymers), especially their relaxation behaviour. For more experimental results see e.g. [9].

Chapter 7

Conclusions

This thesis may be divided into three main parts: the framework of the fractional calculus (chapters 2-3), theory of linear fractional differential equations (chapters 4-5) and applications of the fractional calculus (chapter 6).

At the beginning we recalled some techniques, classes of functions and basic integral transforms which are necessary for further investigation of the fractional calculus rules. These facts are in more rigorous way treated in [1], [2], [3] and [14].

Then we introduced some standard approaches to the definition of differintegrals, namely the Riemann-Liouville and the Caputo approach and the sequential fractional derivative, and studied their basic properties. In particular, we proved the linearity of differintegrals, derived the composition rules for Riemann-Liouville differintegrals (and gave the recipe for Caputo differintegrals), established conditions for the equivalence of these approaches and investigated their continuity w.r.t. the order of derivation. Next we derived the Laplace and Fourier transforms of differintegrals. For other properties we refer to [1], [2] or [10].

We gave some examples of differintegrals for important functions like the power function and functions of the Mittag-Leffler type. Finally we considered differintegrals of a discontinuous function where some of their effects were demonstrated.

Next we studied linear fractional differential equations (LFDEs). First of all we introduced the existence and uniqueness theorem for general linear fractional Cauchy problems with arbitrary differintegral (see [1]). Then we presented some methods of solving like the Laplace transform method, the reduction to a Volterra integral equation, the power series method and the transformation to an ODE. These and many other methods can be found e.g. in [1] or [2].

We can say that the structure of the solution of LFDEs is very similar to the one for linear ODEs including the theory of Green functions. On the other hand the theory of nonlinear FDEs is still uncompleted.

In the application part we investigated four problems where fractional calculus is often used, namely the tautochrone problem (see [10]), space-fractional advection-dispersion equation (see [5], [6]), fractional oscillator (see [7], [8]) and the problem of viscoelasticity (see [9], [1]).

The space-fractional advection-dispersion equation is used e.g. in hydrogeology for modelling spreading particles in a heterogeneous medium in one dimension. The fundamental solution is given in the form of the stable distribution density which provides the important link between fractional derivatives and the generalized central limit theorem.

Viscoelasticity is a discipline describing the behaviour of materials. We mentioned some typical models generalizing the standard ones via the replacement of the classical terms by the fractional ones. We saw that those models have more realistic behaviour and according to the literature, fractional models describe the viscoelastic properties of many polymeric materials very well.

Briefly, fractional calculus gives to the modelling of complex structures "another dimension" - that is why the most important application fields are rheology (viscoelasticity) and control theory. Moreover, the connection between fractional derivatives and stable distributions automatically means a good applicability of fractional models on heavy-tailed phenomena (fluid flow, processes in self-similar and porous structures, economics).

The perspectives of the research could be fractional kinetics of hamiltonian chaotic systems, a fractional regression model describing the memory effects of complex processes and also further improvement and applications to the models mentioned above, e.g. in biology, medicine or economics. A discrete version of fractional calculus, fractional difference calculus, could provide interesting possibilities in modelling discrete-time systems.

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List of Symbols

$\mathbb{N}$	the set of natural numbers
$\mathbb{N}_0$	the set of natural numbers with zero included
$\mathbb{Z}$	the set of integer numbers
$\mathbb{Z}^-$	the set of negative integer numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathfrak{Re}(z)$	a real part of a complex number z
$\delta(z)$	the Dirac delta function of a real variable z
$H(z)$	the Heaviside step function of a real variable z
$[z]$	the floor function of a real variable z
$\Gamma(z)$	the Gamma function of a variable z
$B(z,w)$	the Beta function of a variables z, w
$E_{\alpha,\beta}(z)$	the two-parameter Mittag-Leffler function of a variable z
$E_{\alpha,m,l}(z)$	the three-parameter Mittag-Leffler function of a variable z
$E_{\alpha,\beta}^{(m)}(z)$	m^{th} the derivative of the two-parameter Mittag-Leffler function with respect to variable z
${}_p\Psi_q\left[\begin{array}{c\|c} (a_l,\alpha_l) & \\ (b_j,\beta_j) & z \end{array}\right]$	the general Wright function of a variable z
$f_\alpha(x;,\beta,c,\mu)$	the stable distribution density
$\mathcal{L}\{f(t);t;s\}$	the Laplace transform of the function $f(t)$, independent variable of image is s
$\mathcal{L}^{-1}\{F(s);s;t\}$	the inverse Laplace transform of the function $F(s)$, the independent variable of image is t
$\mathcal{F}\{f(x);x;k\}$	the Fourier transform of the function $f(x)$, independent variable of image is k
$\mathcal{F}^{-1}\{\hat{f}(k);k;x\}$	the inverse Fourier transform of function $\hat{f}(k)$, independent variable of image is x
$\frac{\mathrm{d}^n}{\mathrm{d}t^n}f(t)$	n^{th} derivative of function $f(t)$

$\frac{\partial}{\partial t}f(t,\tau)$	partial derivative of function $f(t,\tau)$ with respect to t
$f^{(n)}(t)$	n^{th} derivative of function $f(t)$
$f(t) * g(t)$	the convolution of functions $f(t)$ and $g(t)$
I_a^α	α-integratio with lower bound a
$\mathbf{D}_a^\alpha$	the Riemann-Liouville differintegration of order α, the lower bound a
${}^{\mathrm{C}}\mathrm{D}_a^\alpha$	the Caputo differintegration of order α, the lower bound a
$\mathcal{D}_a^\alpha$	the sequential fractional derivative of order α, the lower bound a
$\mathfrak{D}_a^\alpha$	a general differintegration operator of order α, the lower bound a
${}_{-}\mathbf{D}_b^\alpha$	the right Riemann-Liouville differintegration of order α, the upper bound b
$L_1(a,b)$	the space of functions which have integrable their absolute value on interval (a,b) in Lebesgue sense