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Stabilization and destabilization of fractional oscillators via a delayed feedback control

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ABSTRACT

This paper discusses the problem of stabilization and destabilization of fractional oscillators by use of a delayed feedback control. A mathematical part of the problem consists in stability analysis of appropriate fractional delay differential equations with the derivative order varying between 1 and 2. Derived stability criteria are efficient and easy to apply when stabilizing or destabilizing fractional oscillators in the standard as well as inverted form. As a by-product of our results, we explicitly describe critical values of a delay control parameter when stability property turns into instability and vice versa. Evaluations of these stability switches are possible also in the limit harmonic case which brings new insights into classical stability results on this topic.

1. Introduction

The model of a fractional oscillator originates from the equation of motion for the classical harmonic oscillator where the second-order derivative (appearing in the acceleration term) is replaced by a fractional-order derivative between 1 and 2. Existing analysis of this fractional model revealed some similarities compared to a damped harmonic oscillator. However, this damping property does not follow from friction (or other external sources) as in the classical harmonic case, but from the internal structure of the fractional oscillator itself. For more details, including the cause of this intrinsic damping, physical interpretations of fractional oscillators, and their response to some external forces, we refer to [1–4].

The decay of amplitudes of fractional oscillators is supported by another specific property, namely a weak oscillation. While in the classical subcritically damped case, an oscillation around the equilibrium state occurs, the fractional oscillator approaches the equilibrium from one side only (after a few overshoots) with oscillating velocity. Thus, from a certain moment, its behavior begins to resemble rather critical or supercritical damping whose decay rate is not exponential, but algebraic [5].

The classical damped harmonic oscillator has also its fractional extension. When studying the problem of the motion of a rigid plate in a Newtonian fluid, the usefulness of a fractional damping term compared to classical damping was justified theoretically as well as empirically in [6], and later analyzed in [7,8]. From the mathematical point of view, the first derivative term (representing a damping proportional to velocity) was replaced by terms with rational derivative orders (mostly 1/2 or 3/2). As it was shown in these and several following papers [9,10], this fractional model keeps

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the same stability properties compared to the classical one, but has a different asymptotics. In particular, the fractional damping term implies an algebraic decay of amplitudes to the equilibrium position.

Coming back to the fractional oscillator model, its fractional-order derivative guarantees asymptotic stability of the system (as recalled above, this is not true for the oscillator with fractional damping). On the other hand, considering the inverted oscillator (when the internal force linearly depending on the deflection is acting towards this deflection), the asymptotic stability property does not hold for any derivative order between 1 and 2.

These observations can be extended via adding a suitable feedback control. The most simple feedback control is that proportional to a current state of the system. However, it is easy to check that such a control cannot change stability properties of neither harmonic nor fractional oscillator (including their inverted forms). Moreover, in the practical implementation of feedback controls, it is very likely that time delays will occur. Therefore, it is important to understand the sensitivity of the control system with respect to a time delay in the feedback loop. Depending on the values of a delay, (asymptotic) stabilization or destabilization of many integer-order dynamical systems were reported. While general reasons for a feedback stabilization consist in bringing the unstable (or even chaotic) system into a stable position, the importance of a feedback destabilization appears in situations when we need to destabilize the stable, but in a certain sense undesirable (e.g. pathological) state [11–13].

A mathematical platform for such feedback (de)stabilization of fractional models is provided by fractional delay differential equations (FDDEs); for basics of the corresponding theory we refer, e.g. to [14–16]. Their stability analysis currently belongs among rapidly developing research topics due to its practical as well as theoretical importance. However, effective stability criteria for FDDEs are still rare (for some basic results we refer to [17–22]). From this viewpoint, fractional and harmonic oscillators (whose position is at the border between stability and instability) represent very suitable test models for stability analysis of their extended delayed feedback loops.

Following such outlines, we wish to discuss these problems: As a preliminary motivation, we consider the (undamped) harmonic oscillator controlled via a time-delayed feedback loop that is linearly depending on the state of the system (but not on its velocity). We wish to describe the structure of all delay and gain control parameters that enable either damping of the oscillator (more precisely, its asymptotic stabilization around the equilibrium), or obversely its destabilization. As the main problem, we consider the fractional oscillator (including its inverted form) with the same type of a control and put similar questions on critical values of the control parameters, including a derivative order.

The paper is organized as follows. Section 2 introduces a short survey of some existing results, related notions and methods. These methods involve, among others, techniques originating from root analysis of the appropriate characteristic equation. A detailed description of locations of characteristic roots including their basic properties is provided in Section 3. Section 4 presents stability criteria for studied linear FDDEs with the derivative order between 1 and 2. These results indicate that when considering the derivative order as a varying bifurcation parameter, the first-order derivative can be taken for the critical bifurcation value. More precisely, if the derivative order is changing between 0 and 1, the stability areas display similar topological properties. However, exceeding the value 1, quite new features accompanying stability investigations appear. Section 5 contains applications of these results to the problems of stabilization and destabilization of fractional oscillators (including their inverted forms) via delayed feedback controls. As a by-product of these investigations, we present appropriate results for the classical harmonic oscillator as well. The final section concludes the paper with a survey of the presented results and possible perspectives.

2. A brief mathematical background

The dynamics of a fractional oscillator is given by the fractional differential equation

$$D^{\alpha}y(t)+\omega^{\alpha}y(t)=0, \qquad t>0 \tag{2.1}$$

where $\alpha\in(1,2)$ and $\omega>0$ is a parameter corresponding to frequency [3]. The symbol $D^{\alpha}f(t)$ used here stands for the Caputo fractional derivative which is, for any positive real α and a given function f, introduced as a composition of the standard $\lceil\alpha\rceil$th order derivative ($\lceil\cdot\rceil$ means a ceiling function) and the fractional $(\lceil\alpha\rceil-\alpha)$th order integral

$$I^{\lceil\alpha\rceil-\alpha}f(t)=\int_0^t\frac{(t-\xi)^{\lceil\alpha\rceil-\alpha-1}}{\Gamma(\lceil\alpha\rceil-\alpha)}f(\xi)\mathrm{d}\xi\,, \qquad t>0,$$

i.e.

$$D^{\alpha}f(t)=I^{\lceil\alpha\rceil-\alpha}\frac{\mathrm{d}^{\lceil\alpha\rceil}f(t)}{\mathrm{d}t^{\lceil\alpha\rceil}}\,, \qquad t>0.$$

If we put $I^0f(t)\equiv f(t)$ and α is a positive integer, then the Caputo derivative coincides with the standard (integer-order) derivative (for more details on fractional operators we refer, e.g. to [23,24]). Thus, if particularly $\alpha=2$, then (2.1) becomes the model for the classical harmonic oscillator

$$y''(t)+\omega^2y(t)=0, \qquad t>0. \tag{2.2}$$

It is well-known [2] that while (2.2) is non-asymptotically stable, its fractional analogue (2.1) is asymptotically stable for any $\alpha\in(1,2)$. In the inverted case, both the classical model

$$y''(t)-\omega^2y(t)=0, \qquad t>0$$

as well as its fractional analogue

$$D^{\alpha}y(t) - \omega^{\alpha}y(t) = 0, \qquad t > 0$$

are unstable.

To change stability or instability properties of these models, we introduce the control term u into their right-hand sides. If we employ the basic feedback control $u(t) = Ky(t)$, where K is a real gain parameter, then it is known (and easy to verify) that its impact on stability properties of these models is limited. In particular, we are not able to achieve asymptotical stabilization of (2.2) for any real K. We show that this situation changes if we consider the delay feedback control of the form $u(t) = Ky(t-\tau)$ where, in addition to a gain parameter K, we employ also the real time lag $\tau > 0$ as the second control parameter.

From the mathematical point of view, we investigate stability properties of the FDDE

$$D^{\alpha}y(t) = ay(t) + by(t-\tau), \qquad t > 0 \tag{2.3}$$

where $\alpha \in (1, 2)$, $\tau > 0$ and a, b are real parameters. Stability properties of (2.3) are known in both the limit integer-order cases. It might be useful to recapitulate them [25,26].

Theorem 1. *(i) Let $\alpha = 1$. Then (2.3) is asymptotically stable if and only if either*

$$a \le b < -a \qquad \text{and} \qquad \tau \text{ is arbitrary},$$

or

$$|a| + b < 0 \qquad \text{and} \qquad \tau < \frac{\arccos(-a/b)}{(b^2 - a^2)^{1/2}}.$$

(ii) Let $\alpha = 2$ and $b > 0$. Then (2.3) is asymptotically stable if and only if $a < 0$ and there exists a non-negative even integer ℓ such that

$$2\ell\pi < \tau\sqrt{-a} < (2\ell + 1)\pi \tag{2.4}$$

and

$$\tau^2 b < \min(-(2\ell)^2\pi^2 - \tau^2 a, (2\ell + 1)^2\pi^2 + \tau^2 a). \tag{2.5}$$

Remark 1 (a). Both the parts (i) and (ii) were derived by use of root analysis of the corresponding characteristic quasi-polynomials

$$P_1(z) \equiv z - a - b\exp(-z\tau) \qquad \text{and} \qquad P_2(z) \equiv z^2 - a - b\exp(-z\tau), \tag{2.6}$$

respectively. More precisely, the well-known stability condition says that (2.3) with $\alpha = 1$ or $\alpha = 2$ is asymptotically stable if and only if all roots of $(2.6)_1$ or $(2.6)_2$, respectively, have negative real parts (see, e.g. [27]). A crucial problem, namely how to convert this theoretical condition into an efficient form, was solved by use of the D-partition method [26] (the case $\alpha = 1$) and the Pontryagin criterion for location of quasi-polynomial roots in the open left complex half-plane [25] (the case $\alpha = 2$).

(b) The case $b < 0$ was not explicitly discussed in [25]. However, using similar arguments as those used for $b > 0$, the extension of the part (ii) to $b < 0$ can be provided as well. We add that these results, even in a more efficient form, can be also obtained as particular (limit) cases of our next considerations.

The above stated stability conditions for both the integer-order cases are of a quite different nature as illustrated by Figs. 1 and 2 depicting appropriate stability regions in the (a, b)–plane.

Following both the integer–order cases, conditions for asymptotic stability of (2.3) originate from the requirement on location of all characteristic roots left to the imaginary axis. More precisely, when considering (2.3), the generalized characteristic quasi-polynomial takes the form

$$P_{\alpha}(z) = z^{\alpha} - a - b\exp(-z\tau)$$

and the corresponding theoretical stability condition can be stated as follows:

Lemma 1. *Let $\alpha > 0$, $\tau > 0$ and a, b be real numbers.*

(i) If all the roots of $P_{\alpha}(z)$ have negative real parts, then (2.3) is asymptotically stable.

(ii) If there exists a root of $P_{\alpha}(z)$ with a positive real part, then (2.3) is not stable.

The proof of this assertion originates from the Laplace transform method and can be found in several earlier papers for $\alpha \in (0, 1)$ (see, e.g. [21, Lemma 1]). Its extension to arbitrary positive real values α uses an analogous argumentation.

Recently, results on effective stability conditions for (2.3) with $\alpha \in (0, 1)$ have appeared in [17,18,21]. The stability area in the (a, b)-plane was described either via boundary parametric curves, or via implicit formulae involving the (unique)

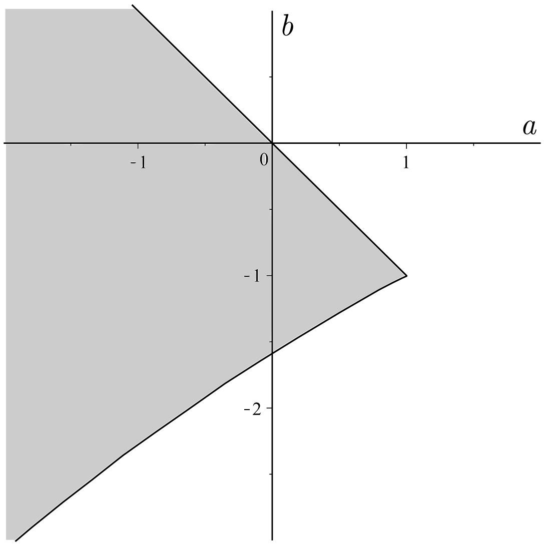

Fig. 1. Stability region of (2.3) for $\alpha = 1$ and $\tau = 1$.

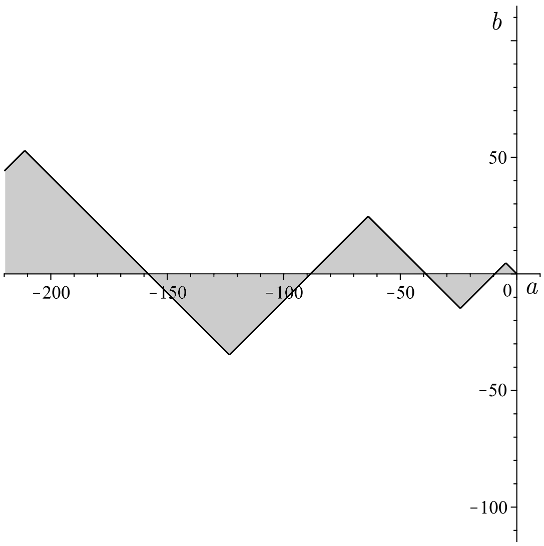

Fig. 2. Stability region of (2.3) for $\alpha = 2$ and $\tau = 1$.

switch between stability and instability with respect to increasing τ (this type of description is used in Theorem 1(i) for $\alpha = 1$). In both these descriptions, stability properties of (2.3) with $\alpha \in (0, 1)$ essentially share the same structure as in the case $\alpha = 1$. As Figs. 1 and 2 indicate, if $\alpha \in (1, 2)$, then the structure of stability properties of (2.3) become qualitatively different. From this point of view, $\alpha = 1$ can be viewed as an important bifurcation value with respect to a changing derivative order. This fact brings another reason why to study (2.3): we wish to clarify a mathematical background of this phenomenon and provide an interpolation between both the systems of stability conditions for (2.3) (recalled in Theorem 1) with α continuously varying between 1 and 2.

3. Distribution of characteristic roots

In this section, we analyze location of characteristic roots of $P_\alpha(z)$ with respect to the imaginary axis (and discuss also some related root properties). Thus we prepare a necessary mathematical apparatus for conversion of Lemma 1 into an effective from.

In addition to some elementary and well-known root properties of $P_\alpha(z)$ (such as complex conjugacy), the series of the following properties holds.

Proposition 1. *Let $\alpha \in (1,2)$, $\tau > 0$ and a, b be real numbers. Then $z = r\exp(\mathrm{i}\varphi)$ ($r \geq 0$, $\varphi \in (-\pi, \pi]$) is a root of $P_\alpha(z)$ with multiplicity greater than one if and only if either $z = 0$ or there exists an integer k such that $\alpha\varphi - \varphi + \tau r \sin(\varphi) = k\pi$ and $\tau r \sin(\alpha\varphi) + \alpha \sin(\alpha\varphi - \varphi) = 0$. Moreover, any root of $P_\alpha(z)$ has multiplicity at most three.*

Proof. Let z be a root of $P_\alpha(z)$ with multiplicity two. Then it also must hold

$$\alpha z^{\alpha-1} + b\tau \exp(-z\tau) = 0, \tag{3.1}$$

and we get $a = z^\alpha + \frac{\alpha}{\tau} z^{\alpha-1}$, $b = -\frac{\alpha}{\tau} z^{\alpha-1}\exp(z\tau)$. Since a, b are real numbers, the first part of the assertion follows by elaborating these two expressions for zero imaginary parts with respect to $z = r\exp(\mathrm{i}\varphi)$ ($r \geq 0$, $\varphi \in (-\pi, \pi]$).

Further, a triple root of $P_\alpha(z)$ satisfies, in addition to (3.1), also $\alpha(\alpha-1)z^{\alpha-2} - b\tau^2\exp(-z\tau) = 0$. Such a system admits a unique solution $z = (1-\alpha)/\tau$ occurring if $a = (1-\alpha)^{\alpha-1}/\tau^\alpha$, $b = -\alpha(1-\alpha)^{\alpha-1}\exp(1-\alpha)/\tau^\alpha$ (provided these values are feasible). Similarly we can show that there are no roots of multiplicity greater than three. □

Proposition 2. *Let $\alpha \in (1,2)$, $\tau > 0$, $a < 0$ and b be real numbers. Then there exists $\delta = \delta(\alpha, a) > 0$ such that all the roots z of $P_\alpha(z)$ satisfy $|\arg(z)| > \pi/2$ whenever $|b| < \delta$.*

Proof. Let $z = r\exp(\mathrm{i}\varphi)$ where $r \geq 0$ and $|\varphi| < \pi/2$. We substitute it into $P_\alpha(z) = 0$ and rewrite the real and imaginary parts as

$$r^\alpha \cos(\alpha\varphi) + |a| - b\exp(-r\tau\cos(\varphi))\cos(r\tau\sin(\varphi)) = 0,$$
$$r^\alpha \sin(\alpha\varphi) + b\exp(-r\tau\cos(\varphi))\sin(r\tau\sin(\varphi)) = 0,$$

which implies

$$(r^\alpha - |a|)^2 + 2r^\alpha|a|(1+\cos(\alpha\varphi)) = b^2\exp(-2r\tau\cos(\varphi)). \tag{3.2}$$

Since $r \geq 0$ and $|\varphi| < \pi/2$, the left-hand side of (3.2) has a minimum equal either to a^2 or $a^2\sin^2(\alpha\varphi)$ for $|\alpha\varphi| \leq \pi/2$ or $|\alpha\varphi| > \pi/2$, respectively. The right-hand side of (3.2) reaches its maximum for $r = 0$ and its value is b^2. Thus, (3.2) has no solution for any b such that $|b| < \delta = |a|\sin(\alpha\pi/2)$ which implies the assertion. □

Lemma 2. *Let $\alpha \in (1,2)$, $\tau > 0$ and a, b be real numbers. Then all functions $z = z(a,b)$ defined implicitly by $P_\alpha(z) = 0$ are continuous whenever $z(a,b)$ is not zero.*

Proof. Denote $z = r\exp(\mathrm{i}\varphi)$ and let F and G be real and imaginary parts of P_α, respectively. For the sake of clarity, we will use the notation $P_\alpha(a,b;z)$ to point out the dependence on parameters a, b. Then we can expand $P_\alpha(a,b;z) = 0$ as

$$F(a,b;r,\varphi) \equiv r^\alpha\cos(\alpha\varphi) - a - b\exp(-r\tau\cos(\varphi))\cos(r\tau\sin(\varphi)) = 0\,,$$
$$G(a,b;r,\varphi) \equiv r^\alpha\sin(\alpha\varphi) + b\exp(-r\tau\cos(\varphi))\sin(r\tau\sin(\varphi)) = 0\,.$$

The implicit function theorem states that if the fixed values $\hat{a}, \hat{b}, \hat{r}, \hat{\varphi}$ satisfy $F(\hat{a},\hat{b};\hat{r},\hat{\varphi}) = 0$, $G(\hat{a},\hat{b};\hat{r},\hat{\varphi}) = 0$ and $F_r(\hat{a},\hat{b};\hat{r},\hat{\varphi})G_\varphi(\hat{a},\hat{b};\hat{r},\hat{\varphi}) - F_\varphi(\hat{a},\hat{b};\hat{r},\hat{\varphi})G_r(\hat{a},\hat{b};\hat{r},\hat{\varphi}) \neq 0$, then there exists an open set $\mathcal{O} \in \mathbb{R}^2$ containing $(\hat{a},\hat{b})$ such that there exist unique continuously differentiable functions r, φ such that $r(\hat{a},\hat{b}) = \hat{r}$, $\varphi(\hat{a},\hat{b}) = \hat{\varphi}$ and $F(a,b;r(a,b),\varphi(a,b)) = 0$, $G(a,b;r(a,b),\varphi(a,b)) = 0$ for all $(a,b) \in \mathcal{O}$.

In our case, $F_rG_\varphi - F_\varphi G_r$ equals

$$r\cdot\left[\left(\alpha r^{\alpha-1} - \tau|b|\exp(-r\tau\cos(\varphi))\right)^2 + 2\alpha r^{\alpha-1}\tau|b|\exp(-r\tau\cos(\varphi))\,(1+\mathrm{sgn}(b)\cos(r\tau\sin\varphi + \alpha\varphi - \varphi))\right]. \tag{3.3}$$

Since z (then also r) is nonzero, (3.3) can be equal to zero provided the term in the square bracket is zero. Assume that this does occur. Since both the terms in the square brackets are nonnegative, we obtain

$$|b| = \frac{\alpha}{\tau}r^{\alpha-1}\exp(r\tau\cos(\varphi)) \qquad \text{and} \qquad r\tau\sin(\varphi) + \alpha\varphi - \varphi = \left(2k + \frac{1+\mathrm{sgn}(b)}{2}\right)\pi \quad \text{for a suitable } k \in \mathbb{Z}.$$

After some technical rearrangements, we can see that these conditions are identical to those for multiple roots described in Proposition 1.

Thus we get that if $\hat{a}, \hat{b}$ are fixed and $\hat{z}$ is a nonzero root of $P_\alpha(\hat{a},\hat{b};z)$, i.e. $\hat{z}^\alpha - \hat{a} - \hat{b}\exp(-\hat{z}\tau) = 0$, then there exists a complex-valued function $z(a,b)$ such that $z(\hat{a},\hat{b}) = \hat{z}$ and $P_\alpha(a,b;z(a,b)) = 0$. By a continuous extension of the corresponding open set $\mathcal{O}$, we obtain that this function is unique and continuous for all points $(a,b,z(a,b))$ provided $z(a,b)$ is not a multiple root of $P_\alpha(a,b;z)$.

$P_\alpha(a,b;z)$ has countably many nonzero roots. It defines countably many branches of $z(a,b)$ which do not intersect each other unless a, b imply a multiple root z of $P_\alpha(a,b;z)$; in this case, the corresponding branches cross each other. Since all the functions involved in $P_\alpha(a,b;z)$ are continuous at these points, we conclude that the branches $z(a,b)$ do not lose their continuity property there. □

Remark 2. Lemma 2 admits that the zero root occurring for $a+b=0$ might not depend continuously on a, b. Indeed, for values of α not admitting negative real roots of $P_\alpha(z)$, we can observe that the zero root "disappears" when crossing the line $a+b=0$.

Now we proceed to analysis of conditions ensuring that all the roots of $P_\alpha(z)$ have negative real parts. Doing it, we employ the D-partition method. This method, when applied to $P_\alpha(z)$, originates from analytical descriptions of curves in the (a,b)-plane such that $P_\alpha(z)$ admits either zero or purely imaginary roots just when its coefficients a, b are lying on these curves. Such descriptions are independent of values of the parameter α and appeared in several earlier papers [18,21]. It holds that $P_\alpha(z)$ has the zero root if and only if $a+b=0$, and $P_\alpha(z)$ has a purely imaginary root $z=\pm \mathrm{i}s/\tau$ $(s>0)$ if and only if $s \neq j\pi$ for any integer j and

$$a=\frac{s^\alpha \sin(s+\alpha\pi/2)}{\tau^\alpha \sin(s)} \qquad \text{and} \qquad b=-\frac{s^\alpha \sin(\alpha\pi/2)}{\tau^\alpha \sin(s)}. \tag{3.4}$$

From the geometrical point of view, $P_\alpha(z)$ admits the zero or purely imaginary roots if and only if the couple (a,b) is lying either on the line $a+b=0$, or on some of the curves Γ_j $(j=0,1\dots)$ given in the parametric forms

$$\Gamma_j: \begin{aligned} a_j(s) &= \frac{s^\alpha \sin(s+\alpha\pi/2)}{\tau^\alpha \sin(s)}, \\ b_j(s) &= -\frac{s^\alpha \sin(\alpha\pi/2)}{\tau^\alpha \sin(s)}, \end{aligned} \qquad s \in (j\pi, j\pi+\pi). \tag{3.5}$$

This system of curves plays a crucial role in stability investigations of (2.3). Properties of these curves significantly differ with respect to $\alpha \in (0,1]$ or $\alpha \in (1,2)$. In the sequel, we explore the latter case in a more detail.

Lemma 3. *Let $\alpha \in (1,2)$, $\tau > 0$ be real numbers and let Γ_j $(j=0,1\dots)$ be the curves defined by (3.5). Then it holds:*

(i) The line $a+b=0$ is tangent to the curve Γ_0 at the origin, and the line

$$p_0^-: b = a - \left(\frac{\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right)$$

is the asymptote to Γ_0 as $s \to \pi^-$. Moreover, $b_0(s) < 0$, $b_0(s) < a_0(s) - (\pi/\tau)^\alpha \cos(\alpha\pi/2)$ and $b_0(s) < -a_0(s)$ for all $s \in (0,\pi)$.

(ii) If j is a positive odd integer, then Γ_j has asymptotes p_j^+ (as $s \to j\pi^+$) and p_j^- (as $s \to (j+1)\pi^-$) given by

$$p_j^+: b = a - \left(\frac{j\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) \qquad \text{and} \qquad p_j^-: b = -a + \left(\frac{j\pi+\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right).$$

Moreover, $b_j(s) > 0$, $b_j(s) > a_j(s) - (j\pi/\tau)^\alpha \cos(\alpha\pi/2)$ and $b_j(s) > -a_j(s) + ((j\pi+\pi)/\tau)^\alpha \cos(\alpha\pi/2)$ for all $s \in (j\pi, j\pi+\pi)$.

(iii) If j is a positive even integer, then Γ_j has asymptotes p_j^+ (as $s \to j\pi^+$) and p_j^- (as $s \to (j+1)\pi^-$) given by

$$p_j^+: b = -a + \left(\frac{j\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) \qquad \text{and} \qquad p_j^-: b = a - \left(\frac{j\pi+\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right).$$

Moreover, $b_j(s) < 0$, $b_j(s) < -a_j(s) + (j\pi/\tau)^\alpha \cos(\alpha\pi/2)$ and $b_j(s) < a_j(s) - ((j\pi+\pi)/\tau)^\alpha \cos(\alpha\pi/2)$ for all $s \in (j\pi, j\pi+\pi)$.

Proof. Verifications of all the three parts are of a technical nature. We derive the part (i), the remaining parts can be proven analogously.

Let $j=0$. Then $\lim_{s\to 0^+}(a_0(s), b_0(s)) = (0,0)$ due to $\alpha > 1$, and the form of the tangent line $a+b=0$ easily follows from the property $\lim_{s\to 0^+} b_0(s)/a_0(s) = -1$. Similarly, the value of slope k and b-intercept q of the asymptote p_0^- to Γ_0 (as $s \to \pi^-$) follow from the standard formulae

$$k = \lim_{s\to\pi^-} \frac{b_0(s)}{a_0(s)} = 1 \qquad \text{and} \qquad q = \lim_{s\to\pi^-} (b_0(s) - a_0(s)) = -\left(\frac{\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right).$$

Further, the property $b_0(s) < 0$ is evident. The second inequality appearing in the part (i) is equivalent to

$$\frac{s^\alpha}{\tau^\alpha \sin(s)}\left(\sin\left(\frac{\alpha\pi}{2}\right) + \sin(s)\cos\left(\frac{\alpha\pi}{2}\right) + \cos(s)\sin\left(\frac{\alpha\pi}{2}\right)\right) > \left(\frac{\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right),$$

and also to

$$s^\alpha\left(1 + \tan\left(\frac{\alpha\pi}{2}\right)\frac{1+\cos(s)}{\sin(s)}\right) < \pi^\alpha$$

due to $s \in (0,\pi)$ and $\alpha \in (1,2)$. The validity of the last inequality is easy to verify.

Similar calculations also imply $b_0(s) < -a_0(s)$ for all $s \in (0,\pi)$. □

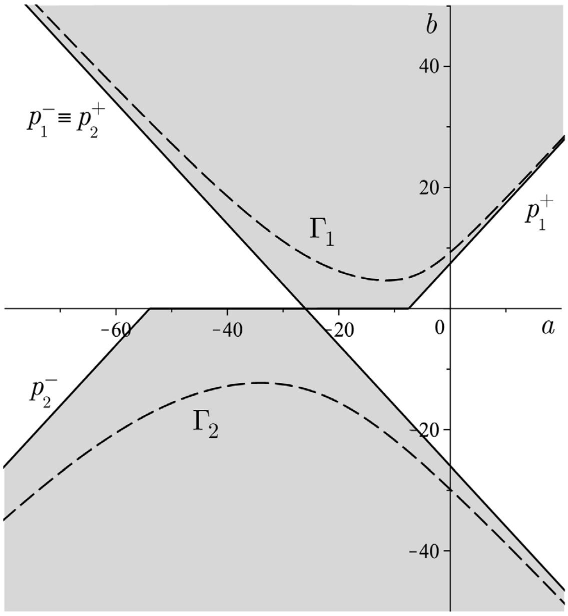

Fig. 3. The common asymptote to Γ_1 and Γ_2 and the corresponding trapezoids for $\alpha = 1.8$ and $\tau = 1$.

Remark 3. Lemma 3 implies that each of the curves Γ_j $(j = 0, 1 \dots)$ is contained in an infinite trapezoid bounded by the a-axis and two straight lines (namely its asymptotes). These asymptotes intersect the a-axis at the values $a = ((j\pi + \pi)/\tau)^\alpha \cos(\alpha\pi/2)$. Also, we can see that each pair Γ_j, Γ_{j+1} shares a common asymptote. The situation is depicted in Fig. 3.

Lemma 4. *Let $\alpha \in (1, 2)$, $\tau > 0$ be real numbers, and let Γ_j $(j = 0, 1 \dots)$ be the curves defined by (3.5). Further, let $X_{m,n} = (a_{m,n}, b_{m,n})$ be intersections of Γ_m and Γ_n (if they exist). Then it holds:*
(i) The intersection $(a_{m,n}, b_{m,n})$ exists (and it is unique) if and only if m, n have the same parity.
(ii) $a_{m,m+2k} < 0$ for all $k \in \mathbb{Z}$ such that $k > -m/2$.
(iii) $a_{m,m+2k} > a_{m,m+2(k+1)}$ for all $k \in \mathbb{Z}$ such that $k > -m/2$.
(iv) $a_{m,m+2k} > a_{m+2\ell,m+2k+2\ell}$ for all $k \in \mathbb{Z}$ such that $k > -m/2$ and $\ell = 1, 2 \dots$.

Proof. (i) By Lemma 3, the sign of functions $b_j(s)$ from (3.5) depends on parity of j. Hence, to find the intersections between Γ_m and Γ_n, we put $n = m + 2k$ for a suitable $k \in \mathbb{Z}$ and search for a couple $(s_{m,m+2k}, s_{m+2k,m})$ such that $s_{m,m+2k} \in (m\pi, m\pi + \pi)$, $s_{m+2k,m} \in ((m + 2k)\pi, (m + 2k + 1)\pi)$ and

$$a_m(s_{m,m+2k}) = a_{m+2k}(s_{m+2k,m}), \qquad b_m(s_{m,m+2k}) = b_{m+2k}(s_{m+2k,m}).$$

Substituting (3.5) into these relations one gets

$$\frac{(s_{m,m+2k})^\alpha \sin(s_{m,m+2k} + \alpha\pi/2)}{\sin(s_{m,m+2k})} = \frac{(s_{m+2k,m})^\alpha \sin(s_{m+2k,m} + \alpha\pi/2)}{\sin(s_{m+2k,m})} \quad \text{and} \quad \frac{(s_{m,m+2k})^\alpha}{\sin(s_{m,m+2k})} = \frac{(s_{m+2k,m})^\alpha}{\sin(s_{m+2k,m})} \tag{3.6}$$

which leads to $\sin(s_{m,m+2k} + \alpha\pi/2) = \sin(s_{m+2k,m} + \alpha\pi/2)$, hence

$$s_{m+2k,m} = (2m + 2k + 3 - \alpha)\pi - s_{m,m+2k}.$$

Let $s_{m,m+2k} = m\pi + \xi$ for a suitable $\xi \in (0, \pi)$. Then using (3.6), intersections between Γ_m and Γ_{m+2k} can be expressed via roots of the function

$$g_{m,m+2k}(\xi) = \frac{\sin(\alpha\pi + \xi)}{\sin(\xi)} - \left(\frac{(2m + 2k + 3 - \alpha)\pi}{m\pi + \xi} - 1\right)^\alpha. \tag{3.7}$$

Since $\lim_{\xi\to 0^+} g_{m,m+2k}(\xi) = \infty$, $\lim_{\xi\to\pi^-} g_{m,m+2k}(\xi) = -\infty$ and $g'_{m,m+2k}(\xi) < 0$ for all $\xi \in (0, \pi)$, the root $\xi_{m,m+2k}$ of $g_{m,m+2k}$ exists and it is determined uniquely.

(ii) Let s_j be a root of $a_j(s)$, $s \in (j\pi, j\pi + \pi)$. Obviously, this root is determined uniquely as $s_j = (j + 1 - \alpha/2)\pi$ for all $j = 0, 1, \dots$. Moreover, $a_j(s) > 0$ for $s < s_j$ and $a_j(s) < 0$ for $s > s_j$. We put $\xi_0 = s_j - j\pi = (1 - \alpha/2)\pi$. The property (ii) now follows from $g_{m,m+2k}(\xi_0) > 0$ for any m, k.

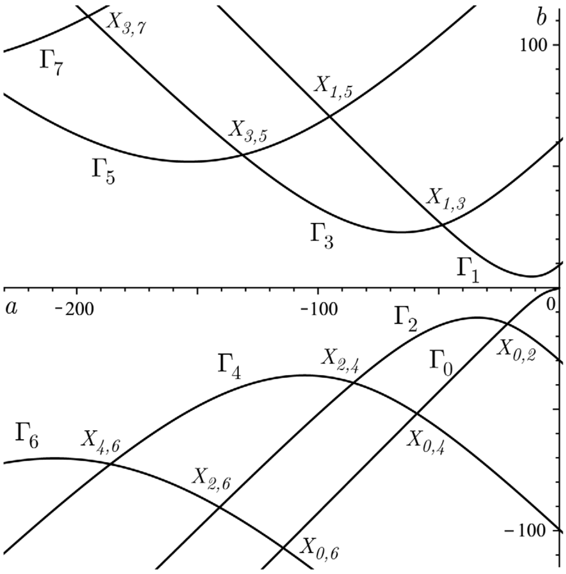

Fig. 4. Some intersections $X_{m,n} = (a_{m,n}, b_{m,n})$ for $\alpha = 1.8$, $\tau = 1$ and $m, n \in \{0, 1, 2, 3, 4, 5, 6, 7\}$.

(iii) Since $g_{m,m+2k}(\xi) < g_{m,m+2(k+1)}(\xi)$ for all $\xi \in (0, \pi)$, the roots of these functions satisfy $\xi_{m,m+2k} > \xi_{m,m+2(k+1)}$. Moreover, $a_j(s)$ is monotonically decreasing whenever $a_j(s) < 0$, which implies $a_{m,m+2k} > a_{m,m+2(k+1)}$.

(iv) The statement is a consequence of the property (iii). Indeed, $a_{m,m+2k} > a_{m,m+2k+2\ell} = a_{m+2k+2\ell,m} > a_{m+2k+2\ell,m+2\ell} = a_{m+2\ell,m+2k+2\ell}$. □

Remark 4. For the sake of lucidity, the location and ordering of intersections $X_{m,n} = (a_{m,n}, b_{m,n})$ between Γ_m and Γ_n is depicted in Fig. 4.

Lemma 5. *Let $\alpha \in (1, 2)$, $\tau > 0$, $a < 0$ and b be real numbers, and let a couple $(a, b) \in \Gamma_j$ for a unique j. If $|b|$ increases, then a new root of $P_\alpha(z)$ with a positive real part appears.*

Proof. Let $z = z(a, b)$ be the corresponding purely imaginary root of $P_\alpha(z)$. By Lemma 2, there exists a neighborhood of (a, b) such that $z(a, b)$ is a continuous function on this neighborhood. Moreover, we can calculate

$$\frac{\partial z(a, b)}{\partial b} = \frac{\exp(-z\tau)}{\alpha z^{\alpha-1} + b\tau \exp(-z\tau)} = \frac{z^\alpha - a}{b(\alpha z^{\alpha-1} + \tau z^\alpha - \tau a)},$$

which is the well-defined expression because the root $z = z(a, b)$ is simple. Substituting $z = \mathrm{i}s$ $(s > 0)$ and evaluating the real part we get

$$\mathcal{R}\left(\frac{\partial z(a, b)}{\partial b}\right)\Big|_{z=\mathrm{i}s} = \frac{1}{b}\frac{x_1x_3 + x_2x_4}{x_3^2 + x_4^2}$$

where $x_1 = s^\alpha \cos(\alpha\pi/2) - a$, $x_2 = s^\alpha \sin(\alpha\pi/2)$, $x_3 = \alpha s^{\alpha-1}\cos((\alpha-1)\pi/2) + \tau s^\alpha \cos(\alpha\pi/2) - \tau a$ and $x_4 = \alpha s^{\alpha-1}\sin((\alpha-1)\pi/2) + \tau s^\alpha \sin(\alpha\pi/2)$. If $a < 0$ then

$$x_1x_3 + x_2x_4 = \tau(s^\alpha + a)^2 - 2as^\alpha\tau(1 + \cos(\alpha\pi/2)) - \alpha a s^{\alpha-1}\sin(\alpha\pi/2) > 0.$$

Since analogously we can dispose with the complex conjugate case $z = -\mathrm{i}s$ $(s > 0)$, unifying both the cases we get

$$\operatorname{sgn}\left(\frac{\partial \mathcal{R}(z(a, b))}{\partial b}\Big|_{z=\pm\mathrm{i}s}\right) = \operatorname{sgn}(b)$$

which proves the assertion. □

4. Effective stability criteria for (2.3) with $\alpha \in (1, 2)$

Lemma 1 supported by the results of Section 3 shows that the line $a + b = 0$ and the curves Γ_j given by (3.5) will play a crucial role in the formulation of efficient stability conditions for (2.3). Before we state these conditions, some auxiliary notation might be useful.

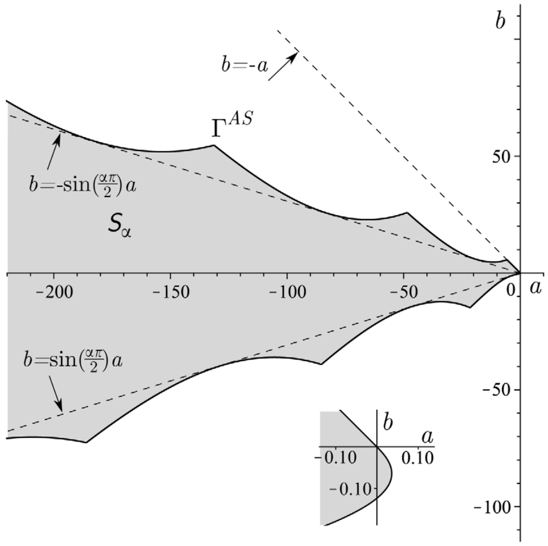

Fig. 5. Stability boundary Γ^{AS} and stability region $\mathcal{S}_\alpha$ of (2.3) for $\alpha = 1.8$ and $\tau = 1$.

Let P be the line segment

$$a = -s, \quad b = s, \qquad s \in (0, T)$$

where $T = (3\pi - \alpha\pi)^\alpha/(2\tau^\alpha|\cos(\alpha\pi/2)|)$, and let $\tilde{\Gamma}_j$ ($j = 0, 1, \dots$) be the parts of Γ_j with the endpoints $X_{j,j-2}$ and $X_{j,j+2}$ given by its intersections with the neighboring curves Γ_{j-2} and Γ_{j+2} (see Fig. 4). If $j = 0$ or $j = 1$, then the curves Γ_{-2} and Γ_{-1} are not defined; in such a case, we introduce the corresponding endpoints as $(0, 0)$ and $(-T, T)$, respectively (for analytical descriptions of all the other endpoints we refer to Lemma 4 and its proof). Further, we put

$$\Gamma^{AS} = \bigcup_{j=0}^{\infty} \tilde{\Gamma}_j \cup P$$

and denote by $\mathcal{S}_\alpha$ the open area in the (a, b)-plane containing the negative part of a-axis and bounded by Γ^{AS}. Also, we set $\mathcal{U}_\alpha = \mathbb{R}^2 \setminus \mathrm{cl}(\mathcal{S}_\alpha)$ where $\mathrm{cl}(\cdot)$ means the closure of a given set. The area $\mathcal{S}_\alpha$ (including some angular bounds of the curve Γ^{AS} and a detail of the situation near the origin) is depicted in Figs. 5 and 6.

Then an effective reformulation of Lemma 1 is provided by

Theorem 2. *Let $\alpha \in (1, 2)$, $\tau > 0$ and a, b be real numbers.*
(i) If $(a, b) \in \mathcal{S}_\alpha$, then (2.3) is asymptotically stable.
(ii) If $(a, b) \in \mathcal{U}_\alpha$, then (2.3) is not stable.

Proof. By Lemma 1, it is enough to show that if $(a, b) \in \mathcal{S}_\alpha$, then all the roots of $P_\alpha(z)$ have negative real parts, and if $(a, b) \in \mathcal{U}_\alpha$, then there exists a root of $P_\alpha(z)$ with a positive real part.

Proposition 2 implies that there exists a neighborhood $\mathcal{O}$ of the negative a-axis where all the roots of $P_\alpha(z)$ have negative real parts. In addition, all the roots of $P_\alpha(z)$ depend continuously on its parameters a, b due to Lemma 2. Hence, the neighborhood $\mathcal{O}$ can be expanded until it is bounded by the appropriate parts of the line $a + b = 0$ and the curves Γ_j ($j = 0, 1 \dots$). The properties described in Lemmas 3 and 4 (supported by some simple calculations) yield that such an expansion of $\mathcal{O}$ is bounded just by the curve Γ^{AS}. Finally, by Lemma 5, when crossing Γ^{AS} through any curve $\tilde{\Gamma}_j$ (while expanding the neighborhood $\mathcal{O}$), a root of $P_\alpha(z)$ with a positive real part appears. To complete the proof, it is enough to employ an obvious property claiming that $P_\alpha(z)$ admits a root with the positive real part whenever $a + b > 0$. □

Remark 5. Geometry of the stability boundary Γ^{AS} can be described as follows: If $a < 0$ and $b > 0$, then this curve is formed by a part of the line $a + b = 0$ connecting the origin and the curve Γ_1. Then Γ^{AS} follows Γ_1 until it is crossed by Γ_3, etc. If $b < 0$, then Γ^{AS} follows the curve Γ_0 (that crosses the b-axis at $b = -(\pi/\tau - \alpha\pi/(2\tau))^\alpha$) until it is crossed by Γ_2, then follows appropriate part of Γ_2 until reaching Γ_4, etc.

Theorem 2 yields a geometric description of the stability area $\mathcal{S}_\alpha$ of (2.3) in the (a, b)-plane. In the sequel, we provide an alternative stability criterion for the case $a < 0$ that better agrees with the form of the conditions of Theorem 1 (the

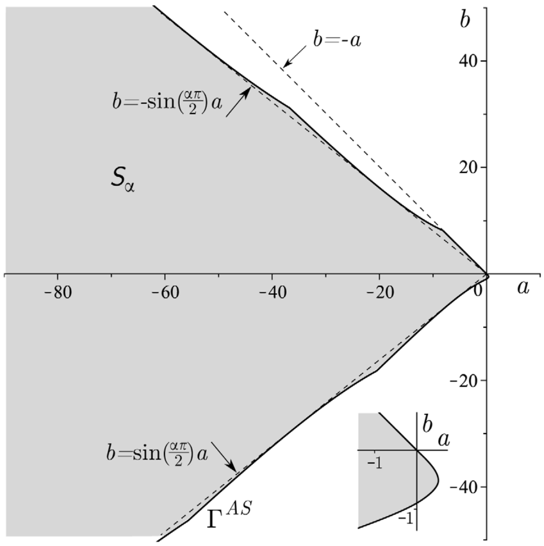

Fig. 6. Stability boundary Γ^{AS} and stability region $\mathcal{S}_\alpha$ of (2.3) for $\alpha = 1.4$ and $\tau = 1$.

case $a > 0$ is not considered here because the corresponding stability conditions are quite straightforward as illustrated in Figs. 5 and 6). On this account, for any $\alpha \in (1,2)$, $\tau > 0$, $a < 0$ and b being real numbers, we introduce the symbols

$$\tau_\ell^+ = \frac{\ell\pi + \frac{(2-\alpha)\pi}{2} + \arcsin\left(\left|\frac{a}{b}\right|\sin(\frac{\alpha\pi}{2})\right)}{\left(a\cos(\frac{\alpha\pi}{2}) + \sqrt{b^2 - a^2\sin^2(\frac{\alpha\pi}{2})}\right)^{1/\alpha}} \quad \text{and} \quad \tau_\ell^- = \frac{\ell\pi + \pi + \frac{(2-\alpha)\pi}{2} - \arcsin\left(\left|\frac{a}{b}\right|\sin(\frac{\alpha\pi}{2})\right)}{\left(a\cos(\frac{\alpha\pi}{2}) - \sqrt{b^2 - a^2\sin^2(\frac{\alpha\pi}{2})}\right)^{1/\alpha}},$$

$\ell = 0, 1, 2 \dots$. Then it holds

Theorem 3. *Let $\alpha \in (1,2)$, $\tau > 0$, $a < 0$ and b be real numbers.*

(i) If $-\sin(\alpha\pi/2) < b/a < \sin(\alpha\pi/2)$, then (2.3) is asymptotically stable.

(ii) If $b/a > \sin(\alpha\pi/2)$, then there exists an integer $N_1 \geq 0$ such that (2.3) is asymptotically stable for any $\tau \in (\tau_{2k-2}^-, \tau_{2k}^+)$, and it is not stable for any $\tau \in (\tau_{2k}^+, \tau_{2k+2}^-)$ where $k = 0, \dots, N_1$ (here we set $\tau_{-2}^- = 0$, $\tau_{2N_1+2}^- = \infty$).

(iii) If $-1 < b/a < -\sin(\alpha\pi/2)$, then there exists an integer $N_2 \geq 0$ such that (2.3) is asymptotically stable for any $\tau \in (\tau_{2k-1}^-, \tau_{2k+1}^+)$, and it is not stable for any $\tau \in (\tau_{2k+1}^+, \tau_{2k+3}^-)$ where $k = 0, \dots, N_2$ (here we set $\tau_{-1}^- = 0$, $\tau_{2N_2+3}^- = \infty$).

(iv) If $b/a < -1$, then (2.3) is not stable.

Proof. By (3.4),

$$\frac{b}{a} = -\frac{\sin(\alpha\pi/2)}{\sin(s + \alpha\pi/2)}. \tag{4.1}$$

This immediately implies the property (i) because all the couples (a, b) lying on Γ^{AS} have to satisfy $|b/a| \geq \sin(\alpha\pi/2)$.

The proofs of the parts (ii), (iii) are technical. The formulae for τ_ℓ^+, τ_ℓ^- can be derived via expressing s from (4.1) in the form

$$s = (-1)^{\ell+1}\arcsin\left(\frac{a}{b}\sin\left(\frac{\alpha\pi}{2}\right)\right) + \ell\pi - \frac{\alpha\pi}{2},$$

using its proper sign analysis, substitution into (3.4) and some straightforward rearrangements. The existence of the values N_1, N_2 follows from the property of the intersections $X_{j,j+2} = (a_{j,j+2}, b_{j,j+2})$ between the curves Γ_j, Γ_{j+2} discussed in Lemma 4. Indeed, the ratio

$$\left|\frac{b_{j,j+2}}{a_{j,j+2}}\right| = \frac{\sin(\alpha\pi/2)}{|\sin(s_{j,j+2} + \alpha\pi/2)|}$$

is monotonically decreasing and tending to $\sin(\alpha\pi/2)$ as $j \to \infty$ because the sequence $\{s_{j,j+2} - j\pi\}_{j=0}^\infty$ is decreasing and its limit equals $(3-\alpha)\pi/2$ due to (3.7) (see the proof of Lemma 4).

The property (iv) is a direct consequence of the fact that the line $a + b = 0$ forms a part of Γ^{AS}. □

Remark 6. It might be interesting to discuss the conclusions of Theorem 2 with $\alpha \to 2^-$ and compare them with the stability conditions of Theorem 1 (ii) derived for $\alpha = 2$. To emphasize dependence on the derivative order α, we denote the curves from (3.5) as $\Gamma_{\alpha,j}$ and their asymptotes (described in Lemma 3) as $p^+_{\alpha,j}$, $p^-_{\alpha,j}$ throughout this remark.

We describe the calculation with j being odd. First, we determine the distance between the curve $\Gamma_{\alpha,j}$ and its asymptotes $p^+_{\alpha,j}$, $p^-_{\alpha,j}$ in the sense of the supremum metric. On this account, we introduce the symbol $d_I(\Gamma_{\alpha,j}, p)$ as the difference between b–coordinates of $\Gamma_{\alpha,j}$ and a line p maximized over all $a \in I$. Further, we denote $a^+_j = (j\pi)^\alpha/\tau^\alpha \cos(\alpha\pi/2)$, $a^-_j = (j\pi+\pi)^\alpha/\tau^\alpha \cos(\alpha\pi/2)$, and let $s^+_j \in (j\pi, j\pi+\pi)$ and $s^-_j \in (j\pi, j\pi+\pi)$ be such that $a(s^+_j) = a^+_j$ and $a(s^-_j) = a^-_j$, respectively. Using this notation, we wish to show that $d_{[a^+_j,\infty)}(\Gamma_{\alpha,j}, p^+_{\alpha,j}) \to 0$ as $\alpha \to 2^-$. Indeed, it holds

$$
\begin{aligned}
d_{[a^+_j,\infty)}(\Gamma_{\alpha,j}, p^+_{\alpha,j}) &= \sup_{t\in(j\pi,s^+_j]} \frac{s^\alpha}{\tau^\alpha \sin(s)}\left(\sin\left(t+\frac{\alpha\pi}{2}\right)+\sin\left(\frac{\alpha\pi}{2}\right)\right) - \left(\frac{j\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right)\\
&= \sup_{t\in(j\pi,s^+_j]} \frac{s^\alpha-(j\pi)^\alpha}{\tau^\alpha}\left|\cos\left(\frac{\alpha\pi}{2}\right)\right| - \frac{s^\alpha}{\tau^\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\frac{1+\cos(s)}{\sin(s)}\\
&= \frac{(s^+_j)^\alpha-(j\pi)^\alpha}{\tau^\alpha}\left|\cos\left(\frac{\alpha\pi}{2}\right)\right| - \frac{(s^+_j)^\alpha}{\tau^\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\frac{1+\cos(s^+_j)}{\sin(s^+_j)}
\end{aligned}
$$

due to the monotony property of the maximized expression on the interval $(j\pi, s^+_j]$. Moreover, by use of the identity $((s^+_j)^\alpha - (j\pi)^\alpha)|\cos(\alpha\pi/2)|/\cos(s^+_j) = (s^+_j)^\alpha \sin(\alpha\pi/2)/\sin(s^+_j)$, we obtain

$$
d_{[a^+_j,\infty)}(\Gamma_{\alpha,j}, p^+_{\alpha,j}) = \frac{((s^+_j)^\alpha-(j\pi)^\alpha)\left|\cos\left(\frac{\alpha\pi}{2}\right)\right|}{\tau^\alpha|\cos(s^+_j)|}.
$$

Since $s^+_j \to j\pi$ as $\alpha \to 2^-$, we actually get $d_{[a^+_j,\infty)}(\Gamma_{\alpha,j}, p^+_{\alpha,j}) \to 0$ as $\alpha \to 2^-$. Similarly, we can show that $d_{(-\infty,a^-_j]}(\Gamma_{\alpha,j}, p^-_{\alpha,j}) \to 0$ as $\alpha \to 2^-$ and, if we denote by p_* the a-axis, then also $d_{[a^+_j,a^-_j]}(\Gamma_{\alpha,j}, p_*) \to 0$ as $\alpha \to 2^-$. A similar line of arguments leads to analogous results also for j being even.

Note that the asymptotes $p^+_{\alpha,j}$, $p^-_{\alpha,j}$ themselves tend to the lines $b = -a - (j\pi)^2/\tau^2$ and $b = a + (j\pi)^2/\tau^2$ as $\alpha \to 2^-$. Hence, one can observe that the stability area described in Theorem 2 actually approaches (when $\alpha \to 2^-$) the system of triangles bounded by these lines and the negative a-axis (see Fig. 2). Thus the stability conditions of Theorem 2 are converting to those of Theorem 1 (ii) when $\alpha \to 2^-$.

This fact provides another interesting consequence. If we make this limit transition also in Theorem 3 (serving as an alternative description of $\mathcal{S}_\alpha$ using τ as a driving parameter), then we can easily deduce that (2.3) with $\alpha = 2$ is asymptotically stable if and only if $0 < |b| < -a$ and

$$
\frac{\ell\pi}{\sqrt{-a-|b|}} < \tau < \frac{(\ell+1)\pi}{\sqrt{-a+|b|}} \tag{4.2}
$$

where ℓ is a non-negative integer that is even for $b > 0$ and odd for $b < 0$. This form of conditions seems to be more effective compared to that of Theorem 1 (ii), especially with respect to explicit evaluations of stability switches for a varying delay parameter.

5. Stabilization and destabilization of harmonic and fractional oscillator

Now we apply conclusions of the previous section to the problem stated in the introductory part. First, we consider the classical harmonic oscillator and show how it can be asymptotically stabilized or destabilized via a feedback control depending only on a position of the controlled object. Although these results can be partially derived from Theorem 1 (ii), we deduce them more straightforwardly from (4.2). Then we study the linear fractional oscillator (which is, in the uncontrolled case, asymptotically stable for any derivative order between 1 and 2) and describe conditions for its destabilization. Also, we consider both these types of oscillators in their inverted forms and discuss conditions for their stabilizations.

5.1. Delayed feedback control of a harmonic oscillator

We consider a controlled harmonic oscillator in the form

$$
y''(t) + \omega^2 y(t) = u(t), \qquad t > 0, \tag{5.1}
$$

$\omega > 0$ is a real number representing frequency. Based on an elementary analysis, the uncontrolled linear oscillator (when u is identically zero) is non-asymptotically stable and cannot be asymptotically stabilized via a feedback control of the

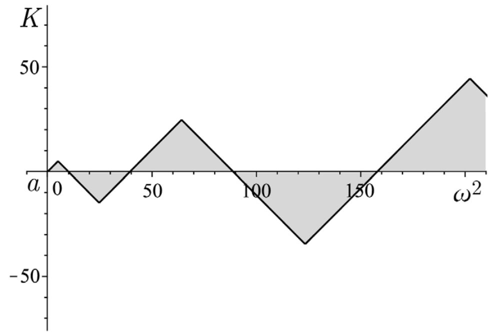

Fig. 7. The stability region for harmonic oscillator (5.1) with control (5.2) depicted in the (ω^2, K)-plane for $\tau = 1$.

form $u(t) = Ky(t)$ for any real gain parameter K (obviously, it can be destabilized whenever $K \geq \omega^2$). When implementing a time lag into the controller, i.e. when

$$u(t) = Ky(t-\tau), \tag{5.2}$$

the situation becomes different. There exists a nonempty set of couples (K, τ) such that (5.1)–(5.2) is either asymptotically stable or unstable. To describe these sets, we can set $a = -\omega^2$ and $b = K$ in (2.4), (2.5) and obtain their analytical descriptions explicitly depending on the gain parameter K. More precisely, (5.1)–(5.2) is asymptotically stable if and only if there exists a non-negative integer ℓ such that

$$2\ell\pi/\tau < \omega < (2\ell+1)\pi/\tau \qquad \text{and} \qquad |K| < \min\left(\omega^2 - (\ell\pi/\tau)^2, ((\ell+1)\pi/\tau)^2 - \omega^2\right)$$

where ℓ is even or odd for K positive or negative, respectively (see Fig. 7 for the corresponding stabilization region). If we need to evaluate an explicit dependence of stability of (5.1)–(5.2) on a time lag parameter, then (4.2) provides a more suitable platform. Indeed, (4.2) with $a = -\omega^2$ and $b = K$ immediately implies that if the gain K is fixed such that $0 < |K| < \omega^2$, then (5.1)–(5.2) is asymptotically stable if and only if

$$\frac{\ell\pi}{\sqrt{\omega^2 - |K|}} < \tau < \frac{(\ell+1)\pi}{\sqrt{\omega^2 + |K|}} \tag{5.3}$$

where ℓ is a non-negative integer that is even for $K > 0$ and odd for $K < 0$. In addition, we can notice that the inequality

$$\frac{\ell\pi}{\sqrt{\omega^2 - |K|}} < \frac{(\ell+1)\pi}{\sqrt{\omega^2 + |K|}}$$

can actually occur provided ℓ is less than the value

$$\ell^* = \frac{\omega^2 - |K| + \sqrt{\omega^4 - K^2}}{2|K|}$$

that enables to determine the number of such stability switches.

Quite analogously, as a counterpart of the condition (5.3), we can formulate the conditions for destabilization of the harmonic oscillator via the control (5.2).

Finally, it may be useful to describe this asymptotic stabilization or destabilization area in the domain (K, τ) of both the control parameters. Appropriate boundary curves $\tau_\ell = \tau_\ell(K)$ can be obtained directly from (5.3) in the form

$$\tau_\ell = \frac{\ell\pi}{\sqrt{\omega^2 + (-1)^{\ell+1}K}}, \qquad -\omega^2 < K < \omega^2, \quad \ell = 0, 1, \dots$$

and they are depicted in Fig. 8.

5.2. Delayed feedback control of an inverted linear oscillator

We consider the controlled inverted linear oscillator

$$y''(t) - \omega^2 y(t) = u(t), \qquad t > 0 \tag{5.4}$$

that is for $u = 0$ unstable. Obviously, using the control $u(t) = Ky(t)$, it can be stabilized (only non-asymptotically) provided $K < -\omega^2$ (in fact, thus we convert (5.4) into the standard harmonic case). If we employ the control (5.2), then our previous analysis easily shows that asymptotic stabilization of (5.4) is not possible for any couple (K, τ).

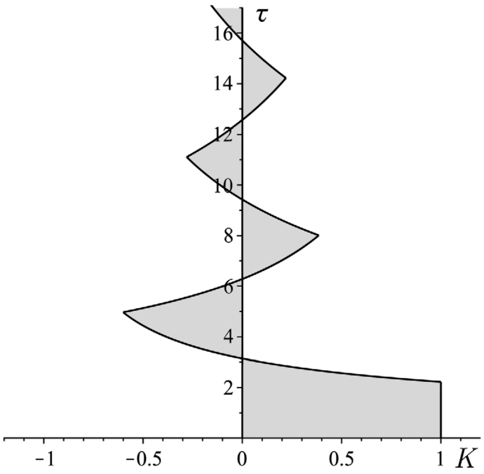

Fig. 8. The stability region for harmonic oscillator (5.1) with control (5.2) depicted in the (K, τ)-plane for $\omega = 1$.

5.3. Delayed feedback control of a fractional oscillator

Now we consider the controlled fractional oscillator

$$D^\alpha y(t) + \omega^\alpha y(t) = u(t), \qquad t > 0 \tag{5.5}$$

where $\alpha \in (1, 2)$ and $\omega > 0$ are real numbers. It is well-known [2] that this fractional model considered without a control term is asymptotically stable for any $\alpha \in (1, 2)$. Thus, using our results from the previous section, we can discuss its possible destabilization via (5.2).

By Theorem 3, if $|K| < \omega^\alpha \sin(\alpha\pi/2)$, then (5.5) remains asymptotically stable for any non-negative value of τ, hence its destabilization is possible only when $|K| \geq \omega^\alpha \sin(\alpha\pi/2)$. If, particularly, $K > \omega^\alpha$, then (5.5) is destabilized via (5.2) unconditionally, i.e. for any $\tau > 0$. For other values of K, the stability behavior depends on τ. If $\omega^\alpha \sin(\alpha\pi/2) < K < \omega^\alpha$, then (5.5) becomes destabilized via (5.2) provided

$$K > \frac{s_{\omega,\ell}^\alpha \sin(\alpha\pi/2)}{\tau^\alpha \sin(s_{\omega,\ell})} \qquad \text{for a non-negative odd integer } \ell, \tag{5.6}$$

$s_{\omega,\ell}$ being determined uniquely by $\omega^\alpha = -s_{\omega,\ell}^\alpha \sin(s_{\omega,\ell} + \alpha\pi/2)/(\tau^\alpha \sin(s_{\omega,\ell}))$, $s_{\omega,\ell} \in (\ell\pi, \ell\pi + \pi)$. If $K < -\omega^\alpha \sin(\alpha\pi/2)$, then (5.5) becomes destabilized via (5.2) provided

$$K < \frac{-s_{\omega,\ell}^\alpha \sin(\alpha\pi/2)}{\tau^\alpha \sin(s_{\omega,\ell})} \qquad \text{for a non-negative even integer } \ell, \tag{5.7}$$

$s_{\omega,\ell}$ being introduced as above. The corresponding stability region in the (ω^α, K)-plane is depicted in Fig. 9.

The use of criteria (5.6) and (5.7) requires a numerical solving of nonlinear equations which negatively affects their practical usability. Therefore, we describe the destabilization boundary in the (K, τ)–plane since it can provide explicit formulae. Indeed, using Theorem 3 (parts (ii) and (iii)), we can evaluate stability switches with respect to a time lag. First, we determine the number ℓ^* of stability switches via roots $\xi_{j,j+2}$ of $g_{j,j+2}(\xi) = 0$ (see (3.7)) for j odd (if $K > 0$) and even (if $K < 0$). The value ℓ^* is given uniquely by

$$|\sin(\xi_{\ell^*,\ell^*+2} + \alpha\pi/2)| \geq \frac{\omega^\alpha}{|K|}\sin(\alpha\pi/2) \quad \text{and} \quad |\sin(\xi_{\ell^*-2,\ell^*} + \alpha\pi/2)| < \frac{\omega^\alpha}{|K|}\sin(\alpha\pi/2).$$

Then (5.5) is destabilized via (5.2) when

$$\tau_\ell^+ < \tau < \tau_\ell^-, \qquad \ell < \ell^*,$$
$$\tau > \tau_{\ell^*}^+$$

where ℓ is odd (if $K > 0$) or even ($K < 0$), and τ_ℓ^+, τ_ℓ^- are defined by Theorem 3. Similarly as in the harmonic case $\alpha = 2$, formulae for τ_ℓ^+, τ_ℓ^- can be interpreted as parts of the stability boundary curve $\tau = \tau(K)$ in the (K, τ)-plane (see Fig. 10).

5.4. Delayed feedback control of an inverted fractional oscillator

Finally, we consider the controlled inverted fractional oscillator

$$D^\alpha y(t) - \omega^\alpha y(t) = u(t), \qquad t > 0 \tag{5.8}$$

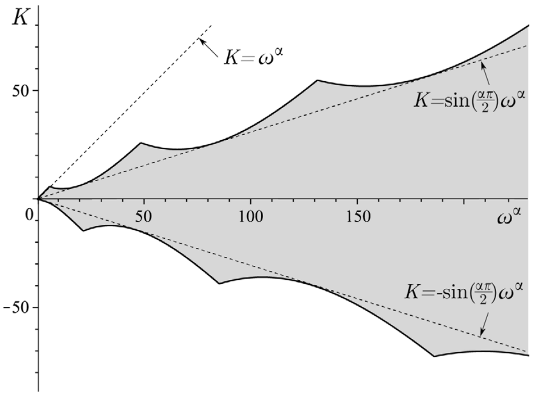

Fig. 9. The stability region for fractional oscillator (5.5) with control (5.2) depicted in the (ω^{α}, K)-plane for the values $\alpha = 1.8$ and $\tau = 1$.

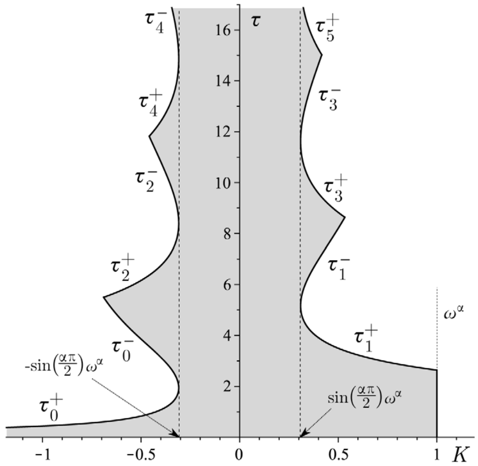

Fig. 10. The stability region for fractional oscillator (5.5) with control (5.2) depicted in the (K, τ)-plane for the values $\alpha = 1.8$ and $\omega = 1$.

that is unstable if $u = 0$. Introducing the control term (5.2) one can stabilize this system as demonstrated by the stability region in Fig. 11 (note that stabilization is not possible for $\alpha = 2$). Unlike the classical fractional oscillator case, there are no stability switches. The boundary curve for the asymptotic stability region in the (K, τ)-plane can be described as

$$\tau(K) = \frac{\frac{(2-\alpha)\pi}{2} + \arcsin\left(\frac{\omega^{\alpha}}{K}\sin(\frac{\alpha\pi}{2})\right)}{\left(\omega^{\alpha}\cos(\frac{\alpha\pi}{2}) + \sqrt{K^2 - \omega^{2\alpha}\sin^2(\frac{\alpha\pi}{2})}\right)^{1/\alpha}}, \qquad K \le -\omega^{\alpha}.$$

Asymptotic stabilization of (5.8) via (5.2) then occurs whenever $\tau < \tau(K)$ as depicted in Fig. 12.

6. Concluding remarks

We have discussed the problems connected with (asymptotic) stabilization and destabilization of fractional oscillators via a delayed feedback loop. The mathematical background of the problem consisted in analysis of appropriate FDDEs. Stability properties of these equations with the derivative order between 0 and 1 were described previously. Our analysis showed that the topological structure of stability regions (considered in the parameter space) significantly changes when the derivative order exceeds the value 1. In particular, a repeated occurrence of finitely many stability switches with respect to a changing delay parameter is a product of this analysis.

The derived stability criteria are easy to apply and result into effective conditions on control parameters ensuring required asymptotic stabilization or destabilization of the studied models. In the limit case, when the fractional oscillator becomes the harmonic one, stability switches can be evaluated fully explicitly. Thus, our stability criteria offer a new computational view on the existing results also in this classical case.

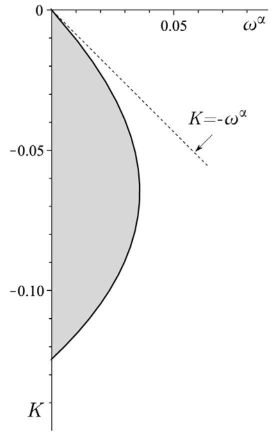

Fig. 11. The stability region for inverted fractional oscillator (5.8) with control (5.2) depicted in the (ω^α, K)-plane for the values $\alpha = 1.8$ and $\tau = 1$.

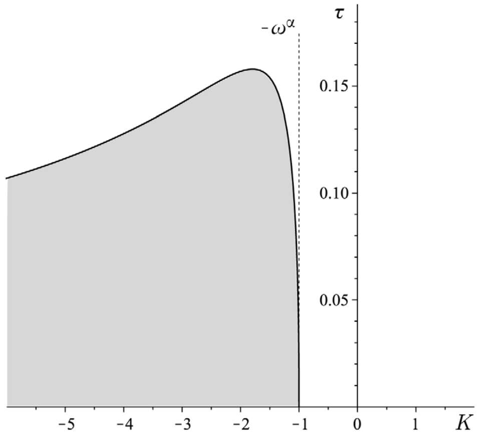

Fig. 12. The stability region for inverted fractional oscillator (5.8) with control (5.2) depicted in the (K, τ)-plane for the values $\alpha = 1.8$ and $\omega = 1$.

This research can be continued towards more advanced types of (nonlinear) fractional oscillators and controls of their stability or oscillatory properties. It seems to be unavoidable that the development of techniques for analysis of appropriate (nonlinear) FDDEs will require new approaches.

CRediT authorship contribution statement

Jan Čermák: Conceptualization, Methodology, Calculations, Investigation, Validation, Writing – review & editing. **Tomáš Kisela:** Conceptualization, Visualisation, Methodology, Calculations, Investigation, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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