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FORMAL CONCEPT ANALYSIS WITH GRADED AFFIRMATIONS AND DENIALS

FORMÁLNÍ KONCEPTUÁLNÍ ANALÝZA
S POTVRZENÍMI A ZAPŘENÍMI VE STUPNÍCH

HABILITATION THESIS
HABILITAČNÍ PRÁCE

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Preface

Formal Concept Analysis (FCA) is a method of analysis of relational data which has proved to be useful in many areas of computer science. In its basic setting FCA is one-valued: it works only with affirmations that objects have attributes. If a user needs to express a denial of incidence, i.e. that an object does not have an attribute, he can easily achieve it using a logical negation. This is no longer the case for graded settings, where the affirmations and denials of incidences between objects and attributes are a matter of degrees. Management of graded affirmations is well elaborated in the literature because it represents a direct generalization of a one-valued character of FCA. In contrast, graded denials have received little attention. This habilitation thesis provides a thoroughly elaborated framework for handling data with graded denials and data with both graded denials and graded affirmations in FCA. A special attention is given to structures behind FCA in a graded setting.

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Contents

1 Introduction

The need to extract potentially useful information from an ever-growing amount of available data is generally recognized by both academia and business. The extracted information usually comes in the form of a reasonably small number of understandable patterns such as clusters, if-then rules (association rules, functional dependencies), etc. The process of such extraction is called Knowledge Discovery in Databases (KDD). Many KDD methods and techniques have been developed in the past few decades; one being Formal Concept Analysis (FCA) [29, 23]. Its core notion, formal concept, is a mathematical formalization of a traditional view of conceptual knowledge. As people naturally reason about reality in terms of concepts the patterns delivered by FCA are easy to understand and interpret.

Formal Concept Analysis is a method of knowledge representation, information management and data analysis invented by Rudolf Wille. Solid mathematical and computational foundations of FCA were developed in the 1980s. In the past two decades or so, FCA has enjoyed considerable interest in various communities. Many papers on applications of FCA in various domains have appeared, including those in premier journals and conferences. The method is based on a formalization of a certain philosophical view of conceptual knowledge which goes back to Port-Royal logic [1, 41].

Some of the most interesting applications of FCA are arguably in computer science. It has been applied in software engineering [61, 36, 62], web mining [26, 27], organization of web search results [25, 24], text mining and linguistics [37], analysis of medical and biological data [17, 40, 39], and crime data [51, 52].

The basic input data for FCA is a flat table, called a formal context, in which rows represent objects, columns represent attributes. Each entry of the table contains a cross if the corresponding object has the corresponding attribute, and is otherwise left blank (Fig. 1).

	a	b	c	d	e	f	g
1	×	×		×	×	×	×
2	×	×	×				×
3				×			×
4	×	×	×				×

Figure 1: Formal context with objects $1, 2, 3, 4$ and attributes $a, b, \ldots, g$.

The basic notion in FCA is that of a formal concept. A formal concept consists

	extent	intent
c_1	$\varnothing$	Y
c_2	$\{2, 4\}$	$\{a, b, c, g\}$
c_3	$\{1\}$	$\{a, b, d, e, f, g\}$
c_4	$\{1, 2, 4\}$	$\{a, b, g\}$
c_5	$\{1, 3\}$	$\{d, g\}$
c_6	X	$\{g\}$

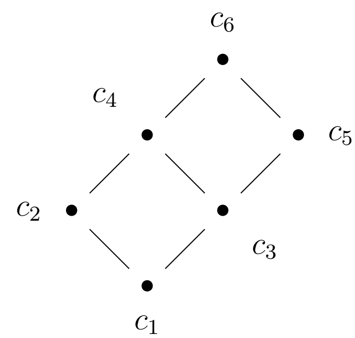


Figure 2: The formal concepts of the formal context in Fig. 1 and its concept lattice.

of two collections: *extent*—a collection of all objects sharing the same attributes, and *intent*—a collection of all the shared attributes.

FCA represents knowledge discovered in the input data in two ways. The first one is a *concept lattice*—a hierarchy of formal concepts present in the formal context (Fig. 2). The second one is *attribute implications*—if-then rules describing dependencies among attributes in the formal context.

FCA in its basic setting deals with one-valued data; i.e. presence of an element in a formal context, in a concept, or in an attribute implication represents an affirmation, while absence represents a lack of affirmation. In particular, each cross in the formal context is seen as an affirmation of the form

> "the object x has the attribute y".

An absence of such affirmation does not generally mean that the object x does not have the attribute y. The Port-Royal logic additionally works another object-attribute incidence—denial of the form

> "the object x does not have the attribute y".

When a denial needs to be processed by FCA, one can easily introduce a negative attribute, for example 'not y', and add the affirmation

> "the object x has the attribute 'not y'".

This way of managing denials in FCA can be found in [48, 49, 56, 57, 58, 60, 59]. We see that denials are easily handled in the basic setting of FCA with one-valued data, however this is no longer the case for graded data.

In everyday life we use concepts which are not sharply bounded (e.g. 'great dancer' or 'middle aged man'). In terms of FCA, objects and attributes need not be divided sharply by a formal concept into those to which the formal concept applies and those to which it does not. That is to say, a formal concept applies to different objects to different, possibly intermediate, degrees. For example, the concept 'middle aged man' may apply to a 45-year old person to degree 1, to a 55-year old person to degree 0.5, and to a 65-year old person to degree 0.2. There are several ways to generalize FCA by which we are able to process such indeterminacy or uncertainty [8, 9, 54, 47, 38, 22] (see also [53] and references therein). Many of them are based on Zadeh's theory of fuzzy sets [68].

In this work, we stick with the graded setting introduced independently by Belohlavek and Pollandt [8, 9, 54] where the formal context contains truth degrees taken from a particular structure of truth degrees. Truth degree a in entry $\langle x, y\rangle$ represents an affirmation that

the object x has the attribute y *at least* to degree a.

Denials are then statements of the form:

the object x has the attribute y *at most* to degree b.

Unlike in the basic setting, here we cannot simply substitute denials by affirmations of negative attributes. The reason is that the law of double negation does not generally hold true in the graded setting. Consequently, applying negation leads to degradation of the input data.

Two main kinds of concept-forming operators, antitone (or standard) and isotone (of attribute/object-oriented), were studied [9, 30, 54, 55], compared [13, 15] and even covered under a unifying framework [10, 50]. The antitone concept-forming operators handle object-attribute incidences as affirmations, and concepts are based on sharing attributes (at least in some degree). The isotone concept-forming operators handle incidences of objects and attributes as denials, and concepts are based on the absence of the same attributes (having them at most in some degree).

The graded affirmations in FCA have been thoroughly studied in the literature while the study of graded denials is the main content of this thesis.

Contributions This thesis consists of eight commented selected papers whose unifying scheme is managing graded denials in FCA. They start with extensive studies of isotone concept-forming operators in FCA for graded data and lead to a general framework for FCA that handles both graded affirmations and graded denials.

The list of the papers follows.[1] The bracketed numbers correspond to the reference numbers in the bibliography.

[43] Jan Konecny. Isotone fuzzy Galois connections with hedges. *Information Sciences*, 181(10):1804–1817, 2011.

[16] Radim Belohlavek and Jan Konecny. A calculus for containment of fuzzy attributes. *Soft Computing*, pages 1–12, 2017.

[15] Radim Belohlavek and Jan Konecny. Concept lattices of isotone vs. antitone Galois connections in graded setting: Mutual reducibility revisited. *Information Sciences*, 199:133–137, 2012.

[3] Eduard Bartl and Jan Konecny. L-concept analysis with positive and negative attributes. *Information Sciences*, 360:96–111, 2016.

[4] Eduard Bartl and Jan Konecny. Rough fuzzy concept analysis. *Fundamenta Informaticae*, 156(2):141–168, 2017.

[44] Jan Konecny and Michal Krupka. Block relations in formal fuzzy concept analysis. *International Journal of Approximate Reasoning*, 73:27–55, 2016.

[45] Jan Konecny and Michal Krupka. Complete relations on fuzzy complete lattices. *Fuzzy Sets and Systems*, 320:64–80, 2017.

[46] Jan Konecny and Manuel Ojeda-Aciego. On homogeneous L-bonds and heterogeneous L-bonds. *International Journal of General Systems*, 45(2):160–186, 2016.

The thesis is structured as follows. Section 2 provides unified preliminaries to all the enclosed papers. It represents a brief introduction to FCA in the graded setting, [8, 9, 54]. Section 3 then contains the papers, each preceded by a short summary of its content.

[1] Whenever author's contribution is of interest we declare that all authors contributed equally.

2 Formal Concept Analysis for Graded Data

We introduce basic notions on complete residuated lattices, fuzzy sets and fuzzy relations and then we turn to FCA for graded data. The content of this section is not to be considered a contribution of this thesis. The only exception is the semantics of graded denials assigned to attribute-oriented concept-forming operators.

2.1 Complete Residuated Lattices

We use complete residuated lattices as basic structures of truth degrees. The truth degrees taken from these structures are used to express the strength of affirmations and denials in formal contexts and in both outputs of formal concept analysis.

A complete residuated lattice [8, 34, 64] is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that

- $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, i.e. a partially ordered set in which arbitrary infima and suprema exist (the partial order of $\mathbf{L}$ is denoted by $\leqslant$);
- $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation which is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$;
- $\otimes$ and $\rightarrow$ satisfy adjointness, i.e. $a \otimes b \leqslant c$ iff $a \leqslant b \rightarrow c$.

Elements of L are called truth degrees. Operations $\otimes$ (multiplication) and $\rightarrow$ (residuum) play the role of truth functions of "fuzzy conjunction" and "fuzzy implication." 0 and 1 denote the least and greatest elements. Throughout this work, $\mathbf{L}$ denotes an arbitrary complete residuated lattice.

Common examples of complete residuated lattices include those defined on the unit interval (i.e. $L = [0, 1]$), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding residuum $\rightarrow$ given by $a \rightarrow b = \max\{c \mid a \otimes c \leqslant b\}$. The three most important pairs of adjoint operations on the unit interval are

- Łukasiewicz

$$a \otimes b = \max(a + b - 1, 0),$$
$$a \rightarrow b = \min(1 - a + b, 1),$$

- Gödel

$$a \otimes b = \min(a, b),$$
$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leqslant b, \\ b & \text{otherwise,} \end{cases}$$

- Goguen (product)

$$a \otimes b = a \cdot b,$$

$$a \to b = \begin{cases} 1 & \text{if } a \leqslant b, \\ \frac{b}{a} & \text{otherwise.} \end{cases}$$

Instead of a unit interval we can also consider a finite chain, e.g.

$$L = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}.$$

All operations on this chain are then defined analogously, see [8].

2.2 Truth-Stressing and Truth-Depressing Hedges

We endow the complete residuated lattices with additional unary operations— truth-stressing and truth-depressing hedges. These operations will serve as parameters for semantics of concept-forming operators as well as for semantics of attribute implications.

Truth-stressing hedges were studied from the point of fuzzy logic as logical connectives 'very true', see [35]. Our approach is close to that in [35]. A *truth-stressing hedge* is a mapping $^* : L \to L$ satisfying

$$1^* = 1, \quad a^* \leqslant a, \quad a \leqslant b \text{ implies } a^* \leqslant b^*, \quad a^{**} = a^* \tag{1}$$

for each $a, b \in L$.

On every complete residuated lattice $\mathbf{L}$, there are two important truth-stressing hedges:

(i) identity, i.e. $a^* = a$ $(a \in L)$;

(ii) globalization, i.e.

$$a^* = \begin{cases} 1, & \text{if } a = 1, \\ 0, & \text{otherwise.} \end{cases}$$

A *truth-depressing hedge* is a mapping $^\square : L \to L$ such that following conditions are satisfied

$$0^\square = 0, \quad a \leqslant a^\square, \quad a \leqslant b \text{ implies } a^\square \leqslant b^\square, \quad a^{\square\square} = a^\square \tag{2}$$

for each $a, b \in L$.

A truth-depressing hedge is a truth function of logical connective 'slightly true', see [63]. In [63] a stricter definition of the truth-depressing hedge with a connection to truth-stressing hedges is given. For our purposes, it is enough to assume conditions (2).

On every complete residuated lattice $\mathbf{L}$, there are two important truth-depressing hedges:

(i) identity, i.e. $a^{\square} = a$ $(a \in L)$;

(ii) antiglobalization, i.e.

$$a^{\square} = \begin{cases} 0, & \text{if } a = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Let $\bullet : L \to L$ be a truth-stressing hedge or truth-depressing hedge. By fix $(\bullet)$ we denote a set of truth degrees $a \in L$ with $a = a^{\bullet}$; that is

$$\text{fix}\,(\bullet) = \{a \in L \mid a = a^{\bullet}\}.$$

2.3 L-sets and L-relations

In the basic setting, a formal concept is given by two sets—an extent which contains objects covered by the concept, and an intent which contains attributes covered by the concept. In the graded setting, the presence of objects and attributes in extents and intents is a matter of degree. We model the extents and intents using $\mathbf{L}$-sets. Similarly, incidences between objects and attributes in the input context are a matter of degree and we model them using $\mathbf{L}$-relations.

An $\mathbf{L}$-*set* [32, 31] A in a universe set X is a mapping assigning to each $x \in X$ some truth degree $A(x) \in L$. The set of all $\mathbf{L}$-sets in a universe X is denoted L^X. An $\mathbf{L}$-set $A \in L^X$ is also denoted $\{{}^{A(x)}/x \mid x \in X\}$. If for all $y \in X$ distinct from $x_1, x_2, \ldots, x_n$ we have $A(y) = 0$, we also write

$$\{{}^{A(x_1)}/x_1, {}^{A(x_2)}/x_1, \ldots, {}^{A(x_n)}/x_n\}.$$

If there is exactly one $x \in X$ s.t. $A(x) > 0$ (i.e. $A = \{{}^{A(x)}/x\}$) we call A a singleton.

The operations with $\mathbf{L}$-sets are defined componentwise. For instance, for $a \in L$ and $A \in L^X$ we define $\mathbf{L}$-sets $a \to A$ and $a \otimes A$ in X by $(a \to A)(x) = a \to A(x)$ and $(a \otimes A)(x) = a \otimes A(x)$ for all $x \in X$ respectively. The intersection of $\mathbf{L}$-sets $A, B \in L^X$ is an $\mathbf{L}$-set $A \cap B$ in X such that $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in X$. Similarly, this is utilized for the union of $\mathbf{L}$-sets.

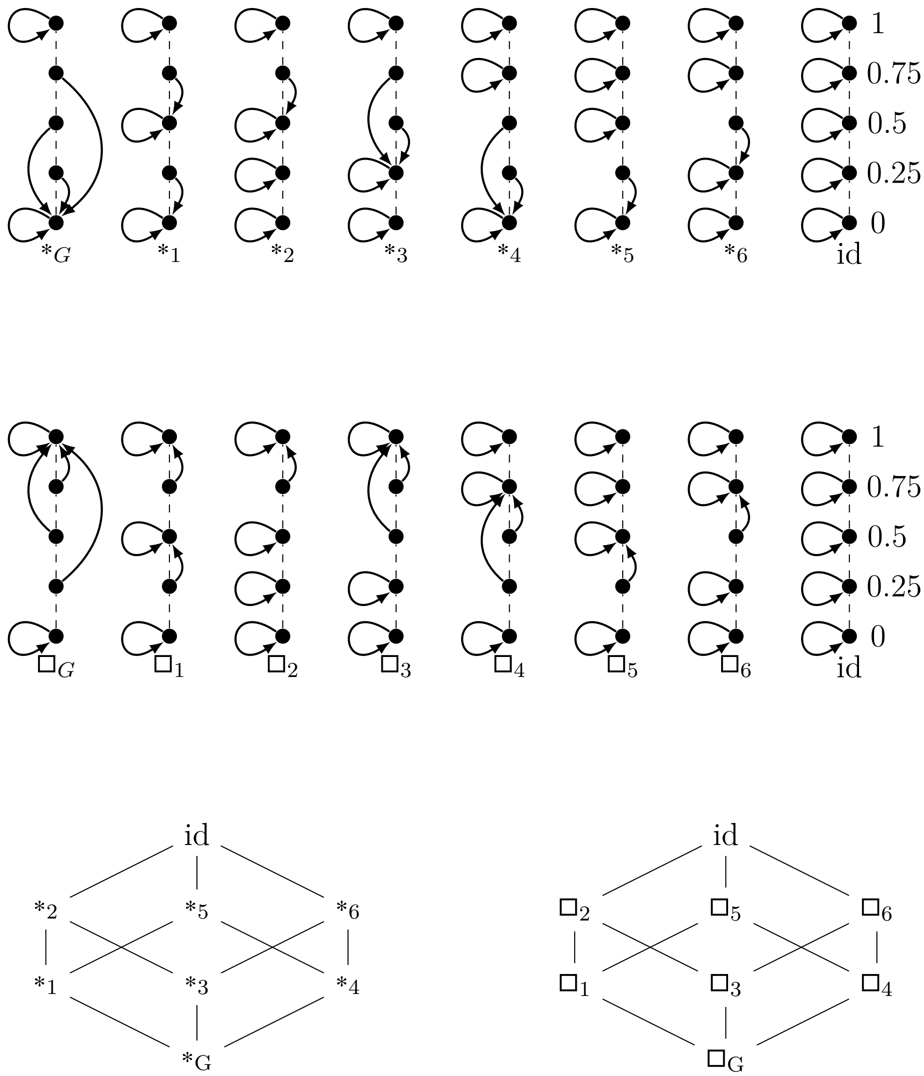


Figure 3: Truth-stressing hedges (top) and truth-depressing hedges (middle) on a five element chain and their ordering w.r.t. $\text{fix}(\cdot) \subseteq \text{fix}(\cdot)$ (bottom).

Additionally, for $a \in L$ and an **L**-set $B \in L^X$ we define left a-multiplication $a \otimes B$ left a-shift $a \rightarrow B$ and a-complement $B \rightarrow a$ respectively by

$$(a \otimes B)(x) = a \otimes B(x)$$
$$(a \rightarrow B)(x) = a \rightarrow B(x)$$
$$(B \rightarrow a)(x) = B(x) \rightarrow a$$

for all $x \in X$.

Intersection and union of two **L**-sets can be generalized to any number of **L**-sets and even to **L**-sets of **L**-sets. For an **L**-set $U : L^X \rightarrow L$, the intersection $\bigcap U$ and union $\bigcup U$ of U are **L**-sets in X, defined by

$$\bigcap U(x) = \bigwedge_{A \in L^X} U(A) \rightarrow A(x), \tag{3}$$
$$\bigcup U(x) = \bigvee_{A \in L^X} U(A) \otimes A(x), \tag{4}$$

for any $x \in X$.

An **L**-set $A \in L^X$ is called crisp if $A(x) \in \{0, 1\}$ for each $x \in X$. Crisp **L**-sets can be identified with ordinary sets. For a crisp set A, we also write $x \in A$ for $A(x) = 1$ and $x \notin A$ for $A(x) = 0$.

For $A, B \in L^X$ we define the *degree of inclusion of A in B* by

$$S(A, B) = \bigwedge_{x \in X} (A(x) \rightarrow B(x)). \tag{5}$$

The degree of inclusion generalizes the classical inclusion relation. Described verbally, $S(A, B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A, B) = 1$. As a consequence, we have $A \subseteq B$ iff $A(x) \leqslant B(x)$ for each $x \in X$. Further, we set

$$A \approx^X B = S(A, B) \wedge S(B, A). \tag{6}$$

The value $A \approx^X B$ is interpreted as the *degree to which the sets A and B are similar.*

A *binary* **L***-relation* (binary fuzzy relation) between X and Y can be thought of as an **L**-set in the universe $X \times Y$. That is, a binary **L**-relation $I \in L^{X \times Y}$ between a set X and a set Y is a mapping assigning to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$ (a degree to which x and y are related by I). In the case $X = Y$ we call such **L**-relation also an **L**-relation on X.

A binary **L**-relation R on a set X is called *reflexive* if $R(x,x) = 1$ for any $x \in X$, *symmetric* if $R(x,y) = R(y,x)$ for any $x, y \in X$, and *transitive* if $R(x,y) \otimes R(y,z) \leqslant R(x,z)$ for any $x, y, z \in X$. R is called an **L**-*tolerance*, if it is reflexive and symmetric, **L**-*equivalence* if it is reflexive, symmetric and transitive. If R is an **L**-equivalence such that for any $x, y \in X$ from $R(x,y) = 1$ it follows $x = y$, then R is called an **L**-*equality* on X. **L**-equalities are often denoted by $\approx$. The similarity $\approx^X$ of **L**-sets (6) is an **L**-equality on L^X.

Let $\sim$ be an **L**-equivalence on X. We say that an **L**-set A in X is *compatible with* $\sim$ (or *extensional w.r.t.* $\sim$, if for any $x, x' \in X$ it holds

$$A(x) \otimes (x \sim x') \leqslant A(x'). \tag{7}$$

A binary **L**-relation R on X is *compatible with* $\sim$, if for each $x, x', y, y' \in X$,

$$R(x,y) \otimes (x \sim x') \otimes (y \sim y') \leqslant R(x', y'). \tag{8}$$

Composition Operators We use three composition operators, $\circ$, $\triangleleft$, and $\triangleright$, and consider the corresponding compositions $I = A \circ B$, $I = A \triangleleft B$, and $I = A \triangleright B$ (for $I \in L^{X \times Y}, A \in L^{X \times F}, B \in L^{F \times Y}$). In the compositions, $I(x,y)$ is interpreted as the degree to which the object x has the attribute y; $A(x,f)$ as the degree to which the factor f applies to the object x; $B(f,y)$ as the degree to which the attribute y is a manifestation (one of possibly several manifestations) of the factor f. The composition operators are defined by

$$(A \circ B)(x,y) = \bigvee_{f \in F} A(x,f) \otimes B(f,y), \tag{9}$$

$$(A \triangleleft B)(x,y) = \bigwedge_{f \in F} A(x,f) \to B(f,y), \tag{10}$$

$$(A \triangleright B)(x,y) = \bigwedge_{f \in F} B(f,y) \to A(x,z). \tag{11}$$

Note that these operators were extensively studied by Bandler and Kohout, see e.g. [42]. They have natural verbal descriptions. For instance, $(A \circ B)(x,y)$ is the truth degree of the proposition "there is factor f such that f applies to object x and attribute y is a manifestation of f"; $(A \triangleleft B)(x,y)$ is the truth degree of "for every factor f, if f applies to object x then attribute y is a manifestation of f". Note also that for $L = \{0,1\}$, $A \circ B$ coincides with the well-known composition of binary relations.

Theorem 1 ([42, 8], associativity and distributivity of composition operators)**.** *We have*

$$R \circ (S \circ T) = (R \circ S) \circ T, \tag{12}$$
$$R \triangleleft (S \triangleright T) = (R \triangleleft S) \triangleright T, \tag{13}$$
$$R \triangleleft (S \triangleleft T) = (R \circ S) \triangleleft T, \tag{14}$$
$$R \triangleright (S \circ T) = (R \triangleright S) \triangleright T. \tag{15}$$

Furthermore, we have that

$$(\bigcup_i R_i) \circ S = \bigcup_i (R_i \circ S), \quad \text{and} \quad R \circ (\bigcup_i S_i) = \bigcup_i (R \circ S_i), \tag{16}$$
$$(\bigcap_i R_i) \triangleright S = \bigcap_i (R_i \triangleright S), \quad \text{and} \quad R \triangleright (\bigcup_i S_i) = \bigcap_i (R \triangleright S_i), \tag{17}$$
$$(\bigcup_i R_i) \triangleleft S = \bigcap_i (R_i \triangleleft S), \quad \text{and} \quad R \triangleleft (\bigcap_i S_i) = \bigcap_i (R \triangleleft S_i). \tag{18}$$

Remark 1. *In [10] it is shown that* $\circ, \triangleright$, *and* $\triangleleft$ *can be considered to be the same composition as it can be covered by a general framework. We do not use the general framework in this thesis because most results contained here use specific properties of compositions defined by* (9),(10), *and* (11).

2.4 L-Galois Connections, L-closures and L-interiors

Now we introduce the fundamental mappings behind FCA in the graded setting, specifically antitone and isotone **L**-Galois connections and **L**-closure and **L**-interior operators.

An *antitone* **L**-*Galois connection* [5] between the sets X and Y is a pair $\langle f, g \rangle$ of mappings $f : L^X \to L^Y$, $g : L^Y \to L^X$, satisfying

$$S(A_1, A_2) \leqslant S(f(A_2), f(A_1)) \qquad S(B_1, B_2) \leqslant S(g(A_2), g(B_1)) \tag{19}$$
$$A \subseteq g(f(A)) \qquad B \subseteq f(g(B)) \tag{20}$$

for every $A, A_1, A_2 \in L^X, A, A_1, A_2 \in L^Y$.

An *isotone* **L**-*Galois connection* [30] between the sets X and Y is a pair $\langle {}^{\cap}, {}^{\cup} \rangle$ of mappings ${}^{\cap} : L^X \to L^Y$, ${}^{\cup} : L^Y \to L^X$, satisfying

$$S(A_1, A_2) \leqslant S(f(A_1), f(A_2)) \qquad S(B_1, B_2) \leqslant S(g(A_1), g(B_2)) \tag{21}$$
$$A \subseteq g(f(A)) \qquad B \supseteq f(g(B)) \tag{22}$$

for every $A, A_1, A_2 \in L^X, A, A_1, A_2 \in L^Y$.

The following theorem summarizes properties of both antitone and isotone Galois connections.

Theorem 2 ([5, 30]). *An antitone* $\mathbf{L}$*-Galois connection* $\langle f, a\rangle$ *satisfies the following properties:*

(i) $A_1 \subseteq A_2$ *implies* $f(A_2) \subseteq f(A_1)$ *and* $B_1 \subseteq B_2$ *implies* $g(B_2) \subseteq g(B_1)$

(ii) $S(A, g(B)) = S(B, f(A))$

(iii) $f\left(\bigcup_{i\in I} A_i\right) = \bigcap_{i\in I} f(A_i)$ *and* $g\left(\bigcup_{i\in I} B_i\right)^{\downarrow} = \bigcap_{i\in I} g(B_i)$

(iv) $f(g(f(A))) = f(A)$ *and* $g(f(g(B))) = g(B)$

for each $A, A_i \in L^X, B, B_i \in L^Y$.

An isotone $\mathbf{L}$*-Galois connection* $\langle f, g\rangle$ *satisfies the following properties:*

(i) $A_1 \subseteq A_2$ *implies* $f(A_1) \subseteq f(A_2)$ *and* $B_1 \subseteq B_2$ *implies* $g(B_1) \subseteq g(B_2)$

(ii) $S(A, g(B)) = S(f(A), B)$

(iii) $f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i)$ *and* $g\left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I} g(B_i)$

(iv) $f(g(f(A))) = f(A)$ *and* $g(f(g(B))) = g(B)$

for each $A, A_i \in L^X, B, B_i \in L^Y$.

Definition 1. [11, 6] A system of $\mathbf{L}$-sets $V \subseteq L^X$ is called an $\mathbf{L}$-*interior system* if

- V is closed under $\otimes$-multiplication, i.e. for every $a \in L$ and $A \in V$ we have that $a \otimes A \in V$;
- V is closed under union, i.e. for $A_j \in V$ $(j \in J)$ we have that $\bigcup_{j\in J} A_j \in V$.

$V \subseteq L^X$ is called an $\mathbf{L}$-*closure system* if

- V is closed under $\rightarrow$-shifts, i.e. for every $a \in L$ and $A \in V$ we have that $a \rightarrow A \in V$;
- V is closed under intersection, i.e. for $A_j \in V$ $(j \in J)$ we have that $\bigcap_{j\in J} A_j \in V$.

Theorem 3. *If* $\langle f, g\rangle$ *an antitone* $\mathbf{L}$*-Galois connection between sets* X *and* Y*, then the composition* $f \circ g$ *is an* $\mathbf{L}$*-closure system on* X *and the composition* $g \circ f$ *in an* $\mathbf{L}$*-closure system on* Y*.*

If $\langle f, g\rangle$ *an isotone* $\mathbf{L}$*-Galois connection between sets* X *and* Y*, then the composition* $f \circ g$ *is an* $\mathbf{L}$*-closure system on* X *and the composition* $g \circ f$ *in an* $\mathbf{L}$*-interior system on* Y*.*

2.5 L-ordered Sets

The set of all formal concepts in the graded setting with particular **L**-order forms a structure called **L**-ordered set. This structure is described in this section.

An **L**-*order* on a set U with an **L**-equality $\approx$ is a binary **L**-relation $\preceq$ on U which is compatible with $\approx$, reflexive, transitive and satisfies $(u \preceq v) \wedge (v \preceq u) \leqslant u \approx v$ for any $u, v \in U$ (antisymmetry). The tuple $\mathbf{U} = \langle\langle U, \approx\rangle, \preceq\rangle$ is called an **L**-ordered set [8, 9]. An immediate consequence of the definition is that for any $u, v \in U$ it holds

$$u \approx v = (u \preceq v) \wedge (v \preceq u). \tag{23}$$

If $\mathbf{U} = \langle\langle U, \approx\rangle, \preceq\rangle$ is an **L**-ordered set then the tuple $\langle U, {}^1\!\preceq\rangle$, where ${}^1\!\preceq$ is the 1-cut of $\preceq$, is a (partially) ordered set. We sometimes write $\leqslant$ instead of ${}^1\!\preceq$ and use the symbols $\wedge$, $\bigwedge$ resp. $\vee$, $\bigvee$ for denoting infima resp. suprema in $\langle U, {}^1\!\preceq\rangle$.

For two **L**-ordered sets $\mathbf{U} = \langle\langle U, \approx_U\rangle, \preceq_U\rangle$ and $\mathbf{V} = \langle\langle V, \approx_V\rangle, \preceq_V\rangle$, a mapping $f : U \to V$ is isotone, if $(u_1 \preceq_U u_2) \leqslant (f(u_1) \preceq_V f(u_2))$, and an embedding, if $(u_1 \preceq_U u_2) = (f(u_1) \preceq_V f(u_2))$, for any $u_1, u_2 \in V$.

A mapping $f : U \to V$ is called an isomorphism of **U** and **V**, if it is both, a bijection and an embedding. **U** and **V** are then called isomorphic.

An antitone mapping and dual embedding are defined by $(u_1 \preceq_U u_2) \leqslant (f(u_2) \preceq_V f(u_1))$ and $(u_1 \preceq_U u_2) = (f(u_2) \preceq_V f(u_1))$, respectively. A dual isomorphism is a bijection which is a dual embedding.

Let **U** be an **L**-ordered set. For any $W \in L^U$ and $w \in U$ we set

$$\mathcal{L}W(w) = \bigwedge_{u \in U} W(u) \to (w \preceq u), \qquad \mathcal{U}W(w) = \bigwedge_{u \in U} W(u) \to (u \preceq w). \tag{24}$$

The right-hand side of the first equation is the degree of "For each $u \in U$, if u is in W, then w is less than or equal to u", and similarly for the second equation. Thus, $\mathcal{L}W(w)$ ($\mathcal{U}W(w)$) can be seen as the degree to which w is less (greater) than or equal to each element of W. The **L**-set $\mathcal{L}W$ (resp. $\mathcal{U}W$) is called the lower cone (resp. the upper cone) of W.

For $u, v \in U$, $u \leqslant v$, the **L**-set $[\![u, v]\!] = \mathcal{U}\{u\} \cap \mathcal{L}\{v\}$ is called an **L**-interval with bounds u and v. We have

$$[\![u, v]\!](w) = (u \preceq w) \wedge (w \preceq v). \tag{25}$$

Let **U** be an **L**-ordered set. For any **L**-set $W \in L^U$ there exists at most one element $u \in U$ such that $\mathcal{L}W(u) \wedge \mathcal{U}(\mathcal{L}W)(u) = 1$ (resp. $\mathcal{U}W(u) \wedge \mathcal{L}(\mathcal{U}W)(u) = 1$) [9, 8]. If there is such an element, we call it the infimum of W (resp. the supremum of W) and denote $\inf W$ (resp. $\sup W$); otherwise we say that the infimum (resp. supremum) does not exist.

U is called completely lattice **L**-ordered, if for each $W \in L^U$, both $\inf W$ and $\sup W$ exist.

An important example of a completely lattice **L**-ordered set is the tuple $\mathbf{L}^X = \langle\langle L^X, \approx^X\rangle, S\rangle$, where X is an arbitrary set and $\approx^X$ and S are given by (6) and (5), respectively. Infima and suprema in $\mathbf{L}^X$ are intersections and unions: for any $M \in L^{L^X}$ we have

$$\inf M = \bigcap M, \qquad \sup M = \bigcup M. \tag{26}$$

2.6 Formal L-Concept Analysis

As we have now introduced all essential mathematical notions, we can finally turn our attention to the formal **L**-concept analysis. Many ways to generalize FCA can be found in the literature [8, 9, 54, 47, 38, 22] (see also [53] and references therein). From here to the end of Section 2 we present the approach of Belohlavek and Pollandt [8, 9, 54].

An **L**-context is a triplet $\langle X, Y, I\rangle$ where X and Y are (ordinary nonempty) sets and $I \in L^{X\times Y}$ is an **L**-relation between X and Y. Elements of X are called objects, elements of Y are called attributes, I is called an incidence relation. $I(x, y) = a$ is read:

> "the object x has the attribute y *at least* to degree a"

or

> "the object x has the attribute y *at most* to degree a"

depending on whether the incidence between x and y is seen as an affirmation or denial.

We consider the following pairs of operators, called concept-forming operators, induced by an **L**-context $\langle X, Y, I\rangle$. First, the pair $\langle^\uparrow, ^\downarrow\rangle$ of standard concept-forming operators $^\uparrow : L^X \to L^Y$ and $^\downarrow : L^Y \to L^X$ is defined, for all $A \in L^X$ and $B \in L^Y$, by

$$\begin{aligned} A^\uparrow(y) &= \bigwedge_{x\in X}(A(x) \to I(x, y)),\\ B^\downarrow(x) &= \bigwedge_{y\in Y}(B(y) \to I(x, y)). \end{aligned} \tag{27}$$

	α	β	γ
A	0.5	0	1
B	1	0.5	1
C	0	0.5	0.5
D	0.5	0.5	1

Figure 4: Example of **L**-context with objects A,B,C,D and attributes α, β, γ; **L** is a chain $0 < 0.5 < 1$ with Łukasiewicz operations.

In words, the operator $^{\uparrow}$ assign to an **L**-set A of objects the **L**-set $A^{\uparrow}$ of attributes which are shared by all the objects in A. Analogously, the operator $^{\downarrow}$ assign to an **L**-set B of attributes the **L**-set $B^{\downarrow}$ of objects which have all the attributes in B.

Second, the pair $\langle {}^{\cap}, {}^{\cup} \rangle$ of attribute-oriented concept-forming operators $^{\cap} : L^X \to L^Y$ and $^{\cup} : L^Y \to L^X$ is defined by

$$\begin{aligned} A^{\cap}(y) &= \bigvee_{x \in X} (A(x) \otimes I(x,y)), \\ B^{\cup}(x) &= \bigwedge_{y \in Y} (I(x,y) \to B(y)). \end{aligned} \tag{28}$$

In words, the operator $^{\cap}$ assign to an **L**-set A of objects the **L**-set $A^{\cap}$ of attributes which at least one object in A has. The operator $^{\cup}$ assign to an **L**-set B of attributes the **L**-set $B^{\cup}$ of objects which have no other attributes than those in B.

Additionally, dual operators to attribute-oriented concept-forming will be sometimes considered. Specifically, a pair of operators $^{\wedge} : L^X \to L^Y$ and $^{\vee} : L^Y \to L^X$

$$\begin{aligned} A^{\wedge}(y) &= \bigwedge_{x \in X} (I(x,y) \to A(x)), \\ B^{\vee}(x) &= \bigvee_{y \in Y} (B(y) \otimes I(x,y)). \end{aligned} \tag{29}$$

The operators $\langle {}^{\wedge}, {}^{\vee} \rangle$ are called object-oriented concept-forming operators.

When we need to emphasize which **L**-relation induces the concept-forming operators, we use an additional subscript; for example, we write $^{\uparrow_I}$ instead of just $^{\uparrow}$.

Example 1. Consider the data (**L**-context) in Fig. 4, the objects represent employees, and the attributes represent skills.

(a) One can handle the incidences in the **L**-context as affirmations and form concepts based on having the same skills at least in some degree; such concepts are standard concepts formed by $\langle\uparrow,\downarrow\rangle$. Extents of the concepts can be interpreted as collections of employees able to fulfill a task which requires particular skill set. For example, the collection of employees able to fulfill a task which requires the skill α in full degree and the skill β in half degree can be found as $\{\alpha, {}^{0.5}\!/\beta\}^{\downarrow}$.

(b) Or he can handle the incidences as denials and form concept based on having the same skills at most in some degree; such concepts are are standard concepts formed by isotone concept-forming operators $\langle\cap,\cup\rangle$. Extents of the concepts can be interpreted as collections of employees who lack the same skills and need some training in them. For example, the collection of employees who lack the skill α and have the skill β at most in degree is can be found as $\{{}^{0.5}\!/\beta, \gamma\}^{\cup}$.

Remark 2. *Notice that the three pairs of concept-forming operators can be interpreted as compositions relations. Applying the isomorphisms $L^{1\times X} \cong L^X$ and $L^{Y\times 1} \cong L^Y$ whenever necessary, one could write them, alternatively, as follows:*

$$\begin{aligned} A^{\uparrow} &= A \triangleleft I & A^{\cap} &= A \circ I & A^{\wedge} &= A \triangleright I \\ B^{\downarrow} &= I \triangleright B & B^{\cup} &= I \triangleleft B & B^{\vee} &= I \circ B \end{aligned}$$

The concept-forming operators induced by **L**-contexts are in correspondence with an antitone and isotone **L**-Galois connection:

Theorem 4 ([5])**.** *Let $\langle X, Y, I\rangle$ be an **L**-context, $\langle f, g\rangle$ be an antitone **L**-Galois connection between X and Y. Then*

(i) $\langle {}^{\uparrow_I}, {}^{\downarrow_I}\rangle$ is a Galois connection.

(ii) $I_{\langle f,g\rangle}$ defined by

$$I_{\langle f,g\rangle}(x, y) = f(\{1/x\})(y) \tag{30}$$

*is an **L**-relation between X and Y and we have*

(iii) $\langle f, g\rangle = \langle {}^{\uparrow_{I_{\langle f,g\rangle}}}, {}^{\downarrow_{I_{\langle f,g\rangle}}}\rangle$ and $I = I_{\langle \uparrow_I, \downarrow_I\rangle}$.

Theorem 5. *Let $\langle X, Y, I\rangle$ be an **L**-context, $\langle f, g\rangle$ be an isotone **L**-Galois connection between X and Y. Then*

*(i) $\langle {}^{\cap_I}, {}^{\cup_I}\rangle$ is an isotone **L**-Galois connection.*

(ii) $I_{\langle f,g\rangle}$ defined by

$$I_{\langle f,g\rangle}(x, y) = f(\{1/x\})(y) \tag{31}$$

*is an **L**-relation between X and Y and we have* *(iii)* $\langle f, g\rangle = \langle {}^{\cap}I_{\langle f,g\rangle}, {}^{\cup}I_{\langle f,g\rangle}\rangle$ *and* $I = I_{\langle {}^{\cap}I, {}^{\cup}I\rangle}$.

Remark 3.

(a) The standard concept-forming operators represent a direct generalization of the concept-forming operators in the ordinary setting and they become the concept-forming operators in the ordinary setting when $\mathbf{L} = \mathbf{2}$.

(b) The two additional pairs of concept-forming operators are not separately studied in the crisp setting, since there they are easily convertible to the standard pair of concept-forming operators due to the double negation law.

(c) For an **L***-set* $A \in L^X$*, the truth degrees in which objects (fully) in* A *have attribute* y *are all in the upper cone of* $A^{\uparrow}(y)$ *in* **L** *(Fig. 5 (left)). In the case* $A^{\uparrow}(y) = 0$*, objects (fully) in* A *may have the attribute* y *in any degree (Fig. 5 (middle)). In the case* $A^{\uparrow}(y) = 1$*, objects (fully) in* A *have the attribute* y *in full degree (Fig. 5 (right)). As positive information (having an attribute) is absolute in this setting, we say that the pair of concept-forming operators* $\langle {}^{\downarrow}, {}^{\uparrow}\rangle$ *considers attributes in a* positive *way – as affirmations. On the contrary, the truth degrees in which objects (fully) in* A *have attribute* y *are all in the lower cone of* $A^{\cap}(y)$ *in* **L** *(Fig. 6 (left)). In the case* $A^{\cap}(y) = 0$*, objects (fully) in* A *do not have the attribute* y*; i.e. they have it in degree 0. (Fig. 6 (middle)). In the case* $A^{\cap}(y) = 1$*, objects (fully) in* A *may have the attribute* y *in any degree (Fig. 6 (right)). As negative information (not having an attribute) is absolute in this setting, we say that the pair of concept-forming operators* $\langle {}^{\cup}, {}^{\cap}\rangle$ *considers attributes in a* negative *way – as denials.*

2.7 L-Concept Lattices

The pairs $\langle A, B\rangle \in L^X \times L^Y$, such that $A^{\uparrow} = B$ and $B^{\downarrow} = A$, are called standard **L**-concepts. Analogously, the pairs $\langle A, B\rangle \in L^X \times L^Y$, such that $A^{\cap} = B$ and $B^{\cup} = A$, are called attribute-oriented **L**-concepts. The components A and B in standard or attribute-oriented **L**-concept $\langle A, B\rangle$ are called extent and intent respectively.

The set of all formal concepts (along with set inclusion) forms a complete lattice, called **L**-concept lattice. We denote the sets of all concepts (as well as the corresponding **L**-concept lattice) by $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$ and $\mathcal{B}^{\cap\cup}(X, Y, I)$, i.e.

$$\mathcal{B}^{\uparrow\downarrow}(X, Y, I) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\uparrow} = B,\ B^{\downarrow} = A\},$$
$$\mathcal{B}^{\cap\cup}(X, Y, I) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\cap} = B,\ B^{\cup} = A\}.$$

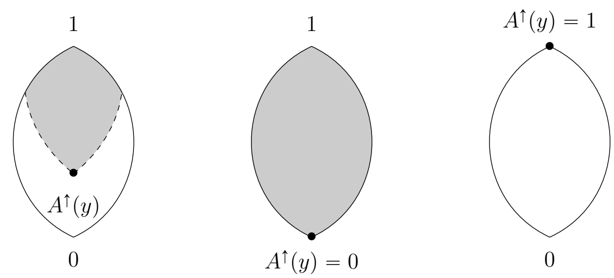


Figure 5: The truth degrees in which objects (fully) in A may have attribute y (gray area); general case (left), extreme cases $A^{\uparrow}(y) = 0$ and $A^{\uparrow}(y) = 1$ (middle and right, respecively).

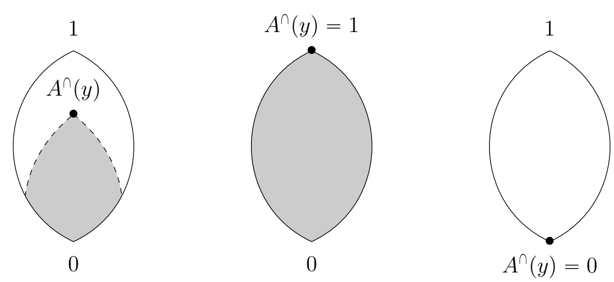
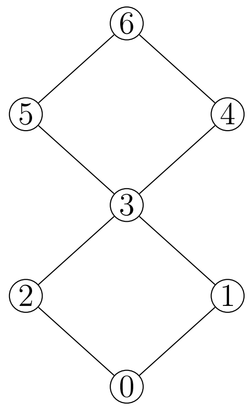


Figure 6: The truth degrees in which objects (fully) in A may have attribute y (gray area); general case (left), extreme cases $A^{\cap}(y) = 0$ and $A^{\cap}(y) = 1$ (middle and right, respecively).



	extent	intent
⓪	$\{{}^{0.5}/\mathrm{B}, {}^{0.5}/\mathrm{D}\}$	$\{\alpha, \beta, \gamma\}$
①	$\{{}^{0.5}/\mathrm{B}, {}^{0.5}/\mathrm{C}, {}^{0.5}/\mathrm{D}\}$	$\{\alpha, {}^{0.5}/\beta, \gamma\}$
②	$\{{}^{0.5}/\mathrm{A}, \mathrm{B}, {}^{0.5}/\mathrm{D}\}$	$\{{}^{0.5}/\alpha, \beta, \gamma\}$
③	$\{{}^{0.5}/\mathrm{A}, \mathrm{B}, {}^{0.5}/\mathrm{C}, \mathrm{D}\}$	$\{{}^{0.5}/\alpha, {}^{0.5}/\beta, \gamma\}$
④	$\{{}^{0.5}/\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$	$\{{}^{0.5}/\beta, {}^{0.5}/\gamma\}$
⑤	$\{\mathrm{A}, \mathrm{B}, {}^{0.5}/\mathrm{C}, \mathrm{D}\}$	$\{{}^{0.5}/\alpha, \gamma\}$
⑥	$\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$	$\{{}^{0.5}/\gamma\}$

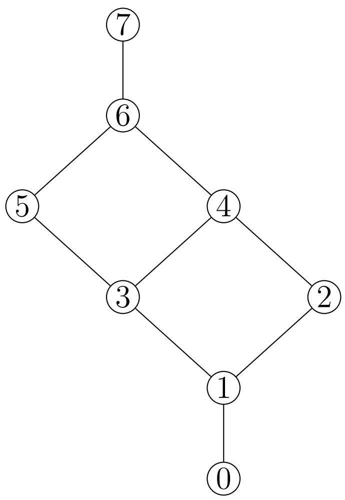
$\preceq$	⓪	①	②	③	④	⑤	⑥
⓪	1	1	1	1	1	1	1
①	0.5	1	0.5	1	1	1	1
②	0.5	0.5	1	1	1	1	1
③	0.5	0.5	0.5	1	1	1	1
④	0	0.5	0	0.5	1	0.5	1
⑤	0	0	0.5	0.5	0.5	1	1
⑥	0	0	0	0.5	0.5	0.5	1

Figure 7: **L**-concept lattice $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$ (top left) of the **L**-context in Fig. 4, description of its **L**-concepts and the **L**-order $\preceq$ (bottom) .

For an **L**-concept lattice $\mathcal{B}^{\#}(X, Y, I)$, where $\mathcal{B}^{\#}$ is either $\mathcal{B}^{\uparrow\downarrow}$ or $\mathcal{B}^{\cap\cup}$, denote the corresponding sets of extents and intents by $\mathrm{Ext}^{\#}(X, Y, I)$ and $\mathrm{Int}^{\#}(X, Y, I)$. That is,

$$\mathrm{Ext}^{\#}(X, Y, I) = \{A \in L^X \mid \langle A, B\rangle \in \mathcal{B}^{\#}(X, Y, I) \text{ for some } B\},$$
$$\mathrm{Int}^{\#}(X, Y, I) = \{B \in L^Y \mid \langle A, B\rangle \in \mathcal{B}^{\#}(X, Y, I) \text{ for some } A\}.$$

See examples of standard and attribute oriented **L**-concept lattices depicted in Fig. 7 and Fig. 8).



	extent	intent
⓪	$\{{}^{0.5}/\mathrm{C}\}$	$\varnothing$
①	$\{{}^{0.5}/\mathrm{A}, {}^{0.5}/\mathrm{C}, {}^{0.5}/\mathrm{D}\}$	$\{{}^{0.5}/\gamma\}$
②	$\{{}^{0.5}/\mathrm{A}, \mathrm{C}, {}^{0.5}/\mathrm{D}\}$	$\{{}^{0.5}/\beta, {}^{0.5}/\gamma\}$
③	$\{{}^{0.5}/\mathrm{A}, {}^{0.5}/\mathrm{B}, {}^{0.5}/\mathrm{C}, {}^{0.5}/\mathrm{D}\}$	$\{{}^{0.5}/\alpha, {}^{0.5}/\gamma\}$
④	$\{{}^{0.5}/\mathrm{A}, {}^{0.5}/\mathrm{B}, \mathrm{C}, {}^{0.5}/\mathrm{D}\}$	$\{{}^{0.5}/\alpha, {}^{0.5}/\beta, {}^{0.5}/\gamma\}$
⑤	$\{\mathrm{A}, {}^{0.5}/\mathrm{B}, {}^{0.5}/\mathrm{C}, {}^{0.5}/\mathrm{D}\}$	$\{{}^{0.5}/\alpha, \gamma\}$
⑥	$\{\mathrm{A}, {}^{0.5}/\mathrm{B}, \mathrm{C}, \mathrm{D}\}$	$\{{}^{0.5}/\alpha, {}^{0.5}/\beta, \gamma\}$
⑦	$\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$	$\{\alpha, {}^{0.5}/\beta, \gamma\}$

$\preceq$	⓪	①	②	③	④	⑤	⑥	⑦
⓪	1	1	1	1	1	1	1	1
①	0.5	1	0.5	1	1	1	1	1
②	0.5	0.5	1	0.5	1	0.5	1	1
③	0.5	0.5	0.5	1	1	1	1	1
④	0.5	0.5	0.5	0.5	1	0.5	1	1
⑤	0	0.5	0.5	0.5	0.5	1	1	1
⑥	0	0.5	0.5	0.5	0.5	0.5	1	1
⑦	0	0	0	0.5	0.5	0.5	0.5	1

Figure 8: **L**-concept lattice $\mathcal{B}^{\cap\cup}(X, Y, I)$ (top left) of the **L**-context in Fig. 4, description of its **L**-concepts and the **L**-order $\preceq$ (bottom) .

2.8 Parameterization with Truth-Stressing Hedges

The standard concept-forming operators parameterized with the truth-stressing hedges were studied in [7, 12, 18][2]. The parametrization goes as follows: let $\langle X, Y, I\rangle$ be an $\mathbf{L}$-context and let $*, \bullet$ be truth-stressing hedges on $\mathbf{L}$. The standard concept-forming operators parameterized by $*$ and $\bullet$ induced by I are defined as

$$\begin{aligned} A^{\Uparrow}(y) &= \bigwedge_{x\in X}(A(x)^{*} \to I(x,y)) \\ B^{\Downarrow}(x) &= \bigwedge_{y\in Y}(B(y)^{\bullet} \to I(x,y)) \end{aligned} \tag{32}$$

for all $A \in L^X, B \in L^Y$.

The two boundary instances of hedges, namely $*$ being identity and globalization, are particularly important: With both truth-stressing hedges being identity, one obtains the standard fuzzy concept lattices of [9, 54], while for one of the truth-stressing hedge being globalization and the other being identity, one obtains the one-sided, or crisply generated, fuzzy concept lattices [19, 67, 47].

The meaning of $A^{\Uparrow}$ and $B^{\Downarrow}$ is essentially the same as that of their unhedged version. The difference is in that parts of the verbal description of hedged version contains "very true" and "slightly true" respectively, compared to that of $A^{\uparrow}$ and $B^{\downarrow}$. For example, $A^{\uparrow}(y)$ is the truth degree of "all x for which it is *very true* that it belongs to A have attribute y".

Standard $\mathbf{L}$-concepts with hedges $*, \bullet$ are pairs $\langle A, B\rangle \in L^X \times L^Y$ which satisfy $A^{\Uparrow} = B$ and $B^{\Downarrow} = A$. The set of all such concepts is denoted $\mathcal{B}^{\uparrow\downarrow}{}_{*,\bullet}(X, Y, I)$. The following theorem is an analogy to the main theorem on concept lattices.

Theorem 6. *1. $\mathcal{B}^{\uparrow\downarrow}{}_{*,\bullet}(X, Y, I)$ equipped with $\leqslant$, defined by*

$$\langle A_1, B_1\rangle \leqslant \langle A_2, B_2\rangle \text{ iff } A_1 \subseteq A_2,$$

is a complete lattice where the infima and suprema are given by

$$\begin{aligned} \bigwedge_{j\in J}\langle A_j, B_j\rangle &= \langle(\bigcap_{j\in J} A_j)^{\uparrow\downarrow}, (\bigcup_{j\in J} B_j^{\bullet})^{\downarrow\uparrow}\rangle, \\ \bigvee_{j\in J}\langle A_j, B_j\rangle &= \langle(\bigcap_{j\in J} A_j^{*})^{\uparrow\downarrow}, (\bigcap_{j\in J} B_j)^{\downarrow\uparrow}\rangle. \end{aligned}$$

[2] Parameterization of attribute-oriented concept-forming operators is one of the contribution of this thesis, see Section A.

2. *Moreover, an arbitrary complete lattice* $\mathbf{K} = \langle K, \leqslant \rangle$ *is isomorphic to* $\mathcal{B}^{\uparrow\downarrow}{}_{*,\bullet}(X,Y,I)$ *iff there are mappings* $\mu : \mathrm{fix}(^*) \times X \to K$, $\nu : \mathrm{fix}(^\square) \times Y \to K$ *such that*

   (a) $\mu(\mathrm{fix}(^*) \times X)$ *is* $\bigvee$*-dense in* K, $\nu(\mathrm{fix}(^\bullet) \times Y)$ *is* $\bigwedge$*-dense in* K.

   (b) $\mu(a,x) \leqslant \nu(b,y)$ *iff* $a \otimes b \leqslant I(x,y)$.

The reason for this parameterization is to have a tool to influence size of the number of concept lattice.

2.9 L-Attribute Implications

Attribute implications in the fuzzy setting with semantics corresponding to standard concept-forming operators were thoroughly studied in [20, 21].

Each expression of the form $A \Rightarrow B$, in which A and B are **L**-sets of attributes (i.e. $A, B \in L^Y$) is called a *fuzzy attribute implication* (FAI) over Y. Their intended meaning the same as in the ordinary case, that is:

if an object has all attributes in A it has also all attributes in B.

Since in a fuzzy setting, object-attribute incidence is a matter of degree, validity of our formulas is a matter of degree as well.

Let x denote an object and $M \in L^Y$ an **L**-set representing the attributes of x, i.e. for each $y \in Y$ the degree to which object x has attribute y is M. For the notion of validity, Belohlavek and Vychodil [20, 21] provide a general definition which subsumes two particular cases, one for bivalent and one for graded inclusion. For the bivalent inclusion, the fact that $A \Rightarrow B$ is fully true in M (in symbols $\|A \Rightarrow B\|_M = 1$) means:

$$\text{if } A \subseteq M \text{ then } B \subseteq M. \tag{33}$$

For the graded inclusion, the fact that $A \Rightarrow B$ is fully true in M means:

$$S(A, M) \leqslant S(B, M), \tag{34}$$

i.e. a degree of inclusion of A in M is less than or equal to the degree of inclusion of B in A, cf. (5). Both the approaches can be obtained as particular cases of the following definition.

$$\|A \Rightarrow B\|_M = S(A, M)^* \to S(B, M). \tag{35}$$

where the truth-stressing hedge * is used as a parameter.

For a collection $\mathcal{M}$ of fuzzy sets M of attributes in Y, we define the degree to which $A \Rightarrow B$ is valid in $\mathcal{M}$ as follows:

$$\|A \Rightarrow B\|_{\mathcal{M}} = \bigwedge_{M \in \mathcal{M}} \|A \Rightarrow B\|_M. \tag{36}$$

For an **L**-context $\langle X, Y, I\rangle$, we define the degree to which $A \Rightarrow B$ is valid in $\langle X, Y, I\rangle$ by

$$\|A \Rightarrow B\|_{\langle X,Y,I\rangle} = \|A \Rightarrow B\|_{\{I_x | x \in X\}}, \tag{37}$$

where I_x denotes an **L**-set representing the row corresponding to object x, i.e. $I_x(y) = I(x, y)$ for each $y \in Y$.

For a fuzzy attribute implication $A \Rightarrow B$ and a fuzzy set M of attributes (of some object x) we define the *degree* $\|A \Rightarrow B\|_M \in L$ *to which* $A \Rightarrow B$ *is valid in* M as follows:

$$\|A \Rightarrow B\|_M = S(A, M)^* \to S(B, M). \tag{38}$$

One easily verifies that if * is globalization and identity, respectively, (42) meets the above cases corresponding to bivalent and graded inclusion, (40) and (41), respectively.

Given an **L**-context $\langle X, Y, I\rangle$ and a FAI $A \Rightarrow B$ over Y, we have

$$\|A \Rightarrow B\|_{\langle X,Y,I\rangle} = \|A \Rightarrow B\|_{\mathrm{Int}^{\uparrow\downarrow}{}_*(X,Y,I)} \quad = S(B, A^{\downarrow\uparrow}). \tag{39}$$

A *theory* (over Y) is any set T of FAIs (over Y). The set $\mathrm{Mod}(T)$ of all *models* of a given theory T is then defined as

$$\mathrm{Mod}(T) = \{M \in \mathbf{L}^Y \mid \text{for each } A, B \in \mathbf{L}^Y: \\ T(A \Rightarrow B) \leqslant \|A \Rightarrow B\|_M\}.$$

$\mathrm{Mod}(T)$ is an **L**-closure system.

We say that an FAI $A \Rightarrow B$ *semantically follows* from theory T, written $T \Vdash A \Rightarrow B$, if $A \Rightarrow B$ is valid in every model of T.

Bases We say that a theory is called

- *complete* in $\langle X, Y, I\rangle$ if for any FAI $A \Rightarrow B$ we have

$$\|A \Rightarrow B\|_{\langle X,Y,I\rangle} = 1 \text{ iff } T \Vdash A \Rightarrow B;$$

- *non-redundant* if for any $A \Rightarrow B \in T$ we have $T - \{A \Rightarrow B\} \Vdash A \Rightarrow B$;

- *basis* of $\langle X, Y, I\rangle$ if it is complete in $\langle X, Y, I\rangle$ and non-redundant.

We call a system $\mathcal{P}$ of fuzzy sets in Y a *system of pseudo-intents* (w.r.t. $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$) of **L**-context $\langle X, Y, I\rangle$ if for every **L**-set $P \in L^Y$ the following holds: $P \in \mathcal{P}$ iff $P \neq P^{\downarrow\uparrow}$ and for each $Q \in \mathcal{P}$ with $Q \neq P$ we have $\|Q \Rightarrow Q^{\downarrow\uparrow}\|_P = 1$.

Theorem 7. *Let $\langle X, Y, I\rangle$ be a formal context and $\mathcal{P}$ be a system of pseudo-intents. Then the theory*

$$\{P \Rightarrow P^{\downarrow\uparrow} \mid P \in \mathcal{P}\}$$

is a basis of $\langle X, Y, I\rangle$.

The basis defined in Theorem 7 is called the *Guigues-Duquenne basis* [33, 29]. The main features of the Guigues-Duquenne basis in the ordinary setting are that it is unique (as exactly one system of pseudo-intents exist in the context), computationally tractable, and it is optimal in terms of its size; i.e. no other basis is smaller in terms of the number of FAIs it contains. It keeps these properties in the graded setting when globalization is used as the truth-stressing hedge.

3 Contributions of the Thesis

A Isotone Fuzzy Galois Connections with Hedges

[43] Jan Konecny. Isotone fuzzy Galois connections with hedges. *Information Sciences*, 181(10):1804–1817, 2011.

The paper is a thorough study of attribute-oriented concept-forming operators and concept lattices which are used to handle denials in our setting. Specifically, we consider their parameterization with hedges in a similar way as their standard counterparts are parameterized in Section 2.8; however, some of the results can be considered new even for the "unhedged" operators.

While the standard concept-forming operators parameterized with the truth-stressing hedges have been extensively studied [7, 12, 18], the attribute-oriented concept-forming operators received attention only in our works [2, 43].

For a formal **L**-context $\langle X, Y, I\rangle$ we define a pair $\langle {}^\cap, {}^\cup\rangle$ of mappings ${}^\cap : L^X \to L^Y$ and ${}^\cup : L^Y \to L^X$ by

$$A^\cap(y) = \textstyle\bigvee_{x\in X}(A(x)^* \otimes I(x,y)),$$
$$B^\cup(x) = \textstyle\bigwedge_{y\in Y}(I(x,y) \to B(y)^\square).$$

The meaning of $A^\cap$ and $B^\cup$ is essentially the same as that of their unhedged version. The difference is that parts of the verbal description of the hedged version contains "very true" and "slightly true" respectively, compared to that of $A^\cap$ and $B^\cup$. For example, $A^\cap(y)$ is the truth degree of "there exists x for which it is very true that it belongs to A and which has y".

We study formal concepts and concept lattices $\mathcal{B}^{\cap\cup}{}_{*,\square}(X, Y, I)$ formed by the operators with hedges and provide an analogy of the main theorem for them.

We show that hedges enable us to control the number of formal **L**-concepts in the associated **L**-concept lattice. The whole point of generalizing the attribute-oriented concept-forming operators $\langle {}^\cap, {}^\cup\rangle$ by using a truth-stressing and truth-depressing hedge is to gain control over the size of the resulting **L**-concept lattice. In the case of the unhedged attribute-oriented concept-forming operators, the number of formal **L**-concepts can be inconveniently big.

In our previous work [2], we have studied a version of attribute-oriented concept-forming operators parameterized with truth-stressing hedges, specifically the pair $\langle ⋒, ⋓\rangle$ given by

$$A^{⋒}(y) = \textstyle\bigvee_{x\in X}(A(x)^* \otimes I(x,y)),$$
$$B^{⋓}(x) = \textstyle\bigwedge_{y\in Y}(I(x,y) \to B(y)^\bullet)$$

where $*$ and $\bullet$ are both truth-stressing hedges. However, we demonstrated that the pair $\langle \cap, \cup \rangle$ provided better reduction of size than $\langle \Cap, \Cup \rangle$ as the reduction with the latter was too drastic and often led to a trivial two-element concept lattice.

Additionally, we provide a reduction theorem which enables us to elevate particular results valid in the ordinary setting into the graded setting with hedges.

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Isotone fuzzy Galois connections with hedges ☆

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ABSTRACT

We study isotone fuzzy Galois connections and concept lattices parameterized by particular unary operators. The operators represent linguistic hedges such as "very", "rather", "more or less", etc. Isotone fuzzy Galois connections and concept lattices provide an alternative to their antitone counterparts which are the fundamental structures behind formal concept analysis of data with fuzzy attributes. We show that hedges enable us to control the number of formal concepts in the associated concept lattice. We also describe the structure of the concept lattice and provide a counterpoint to the main theorem of concept lattices.

1. Introduction

Antitone fuzzy Galois connections, fuzzy closure operators and concept lattices are fundamental structures behind formal concept analysis of data with fuzzy attributes, see [6,3,2]. In [7,9], these structures were generalized using particular unary operations called truth-stressing hedges. These operations are used as parameters which influence the size of concept lattices. In [15,20], the authors developed *isotone fuzzy Galois connections*—an alternative approach to antitone fuzzy Galois connections. In classical setting, there is a bijective correspondence between antitone and isotone Galois connections. However, in a fuzzy setting, this is no longer the case. In [1] we extended the alternative approach by two truth-stressing hedges. In this paper we investigate another generalization which uses a truth-stressing hedge and a truth-depressing hedge to parameterize isotone fuzzy Galois connections. Our motivation is to have a parameterized version of isotone Galois connections where the parameters control the number of fixed points, i.e. clusters extracted from data. We show properties of the generalized concept-forming operators, investigate the associated concept lattices and provide illustrative examples. In particular, Section 2 provides preliminaries from fuzzy logic, fuzzy sets and formal concept analysis. In Section 3 we study isotone fuzzy Galois connections with hedges. Section 4 provides an illustrative example. Further issues and conclusions are summarized in Section 5.

2. Preliminaries

We recall basic facts of residuated lattices, truth-stressing and truth-depressing hedges, and fuzzy sets.

2.1. Residuated lattices and fuzzy sets

We use complete residuated lattices as basic structures of truth degrees. A complete residuated lattice [5,16,22] is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$ such that

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(i) $\langle L,\wedge,\vee,0,1\rangle$ is a complete lattice, i.e. a partially ordered set in which arbitrary infima and suprema exist;
(ii) $\langle L,\otimes,1\rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation which is commutative, associative, and $a\otimes 1 = a$ for each $a\in L$;
(iii) $\otimes$ and $\rightarrow$ satisfy adjointness, i.e. $a\otimes b\leqslant c$ iff $a\leqslant b\rightarrow c$.

0 and 1 denote the least and greatest elements. The partial order of **L** is denoted by $\leqslant$. Throughout the paper, **L** denotes an arbitrary complete residuated lattice.

Elements a of L are called truth degrees. $\otimes$ and $\rightarrow$ (truth functions of) "fuzzy conjunction" and "fuzzy implication".

Common examples of complete residuated lattices include those defined on a unit interval, (i.e. $L = [0,1]$) or on a finite chain in a unit interval, e.g. $L = \{0,\frac{1}{n},\ldots,\frac{n-1}{n},1\}$, $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\rightarrow$. The three most important pairs of adjoint operations on the unit interval are

$$\text{ukasiewicz}: \begin{array}{l} a\otimes b = \max(a+b-1,0), \\ a \rightarrow b = \min(1-a+b,1), \end{array}$$

$$\text{Godel}: \begin{array}{l} a\otimes b = \min(a,b), \\ a \rightarrow b = \begin{cases} 1 & a\leqslant b, \\ b & \text{otherwise}, \end{cases} \end{array}$$

$$\text{Goguen(product)}: \begin{array}{l} a\otimes b = a\cdot b, \\ a \rightarrow b = \begin{cases} 1 & a\leqslant b, \\ \frac{b}{a} & \text{otherwise}. \end{cases} \end{array}$$

An **L**-set (or fuzzy set) A in a universe set X is a mapping assigning to each $x\in X$ some truth degree $A(x)\in L$ where L is a support of a complete residuated lattice. The set of all **L**-sets in a universe X is denoted $\mathbf{L}^X$.

The operations with **L**-sets are defined componentwise. For instance, the intersection of **L**-sets $A, B\in\mathbf{L}^X$ is an **L**-set $A\cap B$ in X such that $(A\cap B)(x) = A(x)\wedge B(x)$ for each $x\in X$, etc. An **L**-set $A\in\mathbf{L}^X$ is also denoted $\{^{A(x)}/x|x\in X\}$. If for all $y\in X$ distinct from $x_1, x_2, \ldots, x_n$ we have $A(y) = 0$, we also write $\{^{A(x_1)}/x_1,{}^{A(x_2)}/x_1,\ldots,{}^{A(x_n)}/x_n\}$. If there is exactly one $x\in X$ s.t. $A(x) > 0$ (i.e. $A = \{^{A(x)}/x\}$) we call A a singleton.

Binary **L**-relations (binary fuzzy relations) between X and Y can be thought of as **L**-sets in the universe $X\times Y$. That is, a binary **L**-relation $I\in\mathbf{L}^{X\times Y}$ between a set X and a set Y is a mapping assigning to each $x\in X$ and each $y\in Y$ a truth degree $I(x,y)\in L$ (a degree to which x and y are related by I). An **L**-set $A\in\mathbf{L}^X$ is called crisp if $A(x)\in\{0,1\}$ for each $x\in X$. Crisp **L**-sets can be identified with ordinary sets. For a crisp A, we also write $x\in A$ for $A(x) = 1$ and $x\notin A$ for $A(x) = 0$. An **L**-set $A\in\mathbf{L}^X$ is called empty (denoted by $\emptyset$) if $A(x) = 0$ for each $x\in X$. For $a\in L$ and $A\in\mathbf{L}^X$, $a\otimes A\in\mathbf{L}^X$ and $a\rightarrow A\in\mathbf{L}^X$ are defined by

$$(a\otimes A)(x) = a\otimes A(x) \quad \text{and } (a \rightarrow A)(x) = a \rightarrow A(x).$$

For universe X we define **L**-relation *graded subsethood* $L^X\times L^X\rightarrow L$ by:

$$S(A,B) = \bigwedge_{x\in X} A(x) \rightarrow B(x). \tag{1}$$

Graded subsethood generalizes the classical subsethood relation $\subseteq$ (note that unlike $\subseteq$, S is a binary **L**-relation on L^X). Described verbally, $S(A,B)$ represents a degree to which A is a subset of B. In particular, we write $A\subseteq B$ iff $S(A,B) = 1$. As a consequence, we have $A\subseteq B$ iff $A(x)\leqslant B(x)$ for each $x\in X$. In the following we use well-known properties of residuated lattices and fuzzy structures which can be found e.g. in [5,16].

2.2. Linguistic hedges

We use unary operations called truth-stressing and truth-depressing hedges. Truth-stressing hedges were studied from the point of fuzzy logic as logical connectives "very true", see [17]. Our approach is close to that in [17]. A *truth-stressing hedge* is a mapping $*: L\rightarrow L$ satisfying the following conditions

$$1^* = 1, \tag{2}$$

$$a^* \leqslant a, \tag{3}$$

$$(a\rightarrow b)^* \leqslant a^*\rightarrow b^*, \tag{4}$$

$$a^{**} = a^* \tag{5}$$

for each $a, b\in L$. Truth-stressing hedges were used to parameterize antitone **L**-Galois connections e.g. in [4,7,9], and also to parameterize antitone **L**-Galois connections in [1].

On every complete residuated lattice **L**, there are two important truth-stressing hedges:

(i) identity, i.e. $a* = a (a\in L)$;

(ii) globalization, i.e.

$$a^* = \begin{cases} 1, & \text{if } a = 1, \\ 0, & \text{otherwise}. \end{cases} \quad (6)$$

Fig. 1 shows examples of truth-stressing hedges on 5-element chain with Łukasiewicz operations $\mathbf{L} = \langle \{0, 0.25, 0.5, 0.75, 1\}, \min, \max, \otimes, \to, 0, 1\rangle$. The left-most truth-stressing hedge $^{\mathrm{id}_L}$ is identity; the right-most truth-stressing hedge ^{*_G} is a globalization.

A *truth-depressing hedge* with respect to truth-stressing hedge $*$ is a mapping $^\square: L \to L$ such that following conditions are satisfied

$$0^\square = 0, \quad (7)$$

$$a \leqslant a^\square, \quad (8)$$

$$(a \to b)^* \leqslant a^\square \to b^\square, \quad (9)$$

$$a^{\square\square} = a^\square \quad (10)$$

for each $a, b \in L$. A truth-depressing hedge is a (truth function of) logical connective "slightly true", see [21].

On every complete residuated lattice $\mathbf{L}$, there are two important truth-depressing hedges:

(i) identity, i.e. $a^\square = a (a \in L)$;
(ii) antiglobalization, i.e.

$$a^\square = \begin{cases} 0, & \text{if } a = 0, \\ 1, & \text{otherwise}. \end{cases} \quad (11)$$

Fig. 2 shows all truth-depressing hedges on 5-element chain with Łukasiewicz operations $\mathbf{L} = \langle \{0, 0.25, 0.5, 0.75, 1\}, \min, \max, \otimes, \to, 0, 1\rangle$. In parentheses are listed the truth-stressing hedges for which the truth-depressing hedge satisfies (9). The left-most truth-depressing hedge in upper row $^{\mathrm{id}_L}$ is identity; the right-most truth-depressing hedge in lower row $^{\square_{AG}}$ is antiglobalization.

Remark 1

(a) Note that from (4) follows that any truth-stressing hedge is monotone. If $a \leqslant b$ then $(a \to b)* = 1$. From (4) we have $1 \leqslant a* \to b*$, i.e. $a \leqslant b$ implies $a* \leqslant b*$. Similarly, from (9) we have monotony of truth-depressing hedge.
(b) The identity is a truth-depressing hedge with respect to any truth-stressing hedge.
(c) If $^\square$ is truth-depressing hedge w.r.t truth-stressing hedge $*$ then $^\square$ is truth-depressing hedge w.r.t. globalization ^{*_G} (since $(a \to b)^{*_G} \leqslant (a \to b)^* \leqslant a^\square \to b^\square$). For that reason we do not declare the truth-stressing hedge for which the truth-depressing hedge satisfies (9), if it is not important.

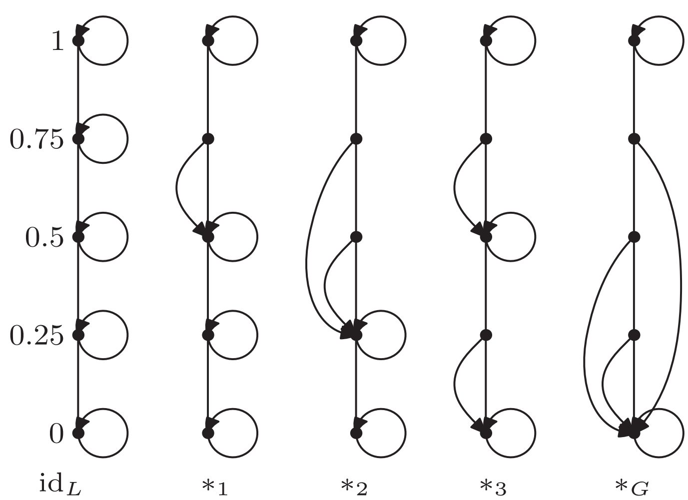


Fig. 1. Truth-stressing hedges.

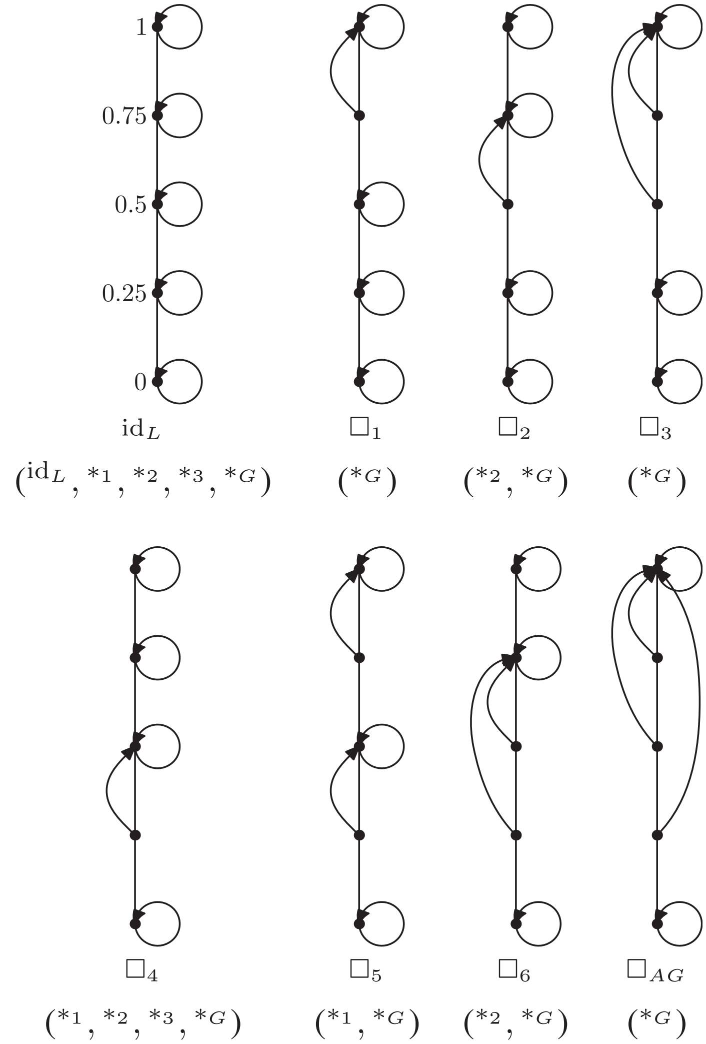
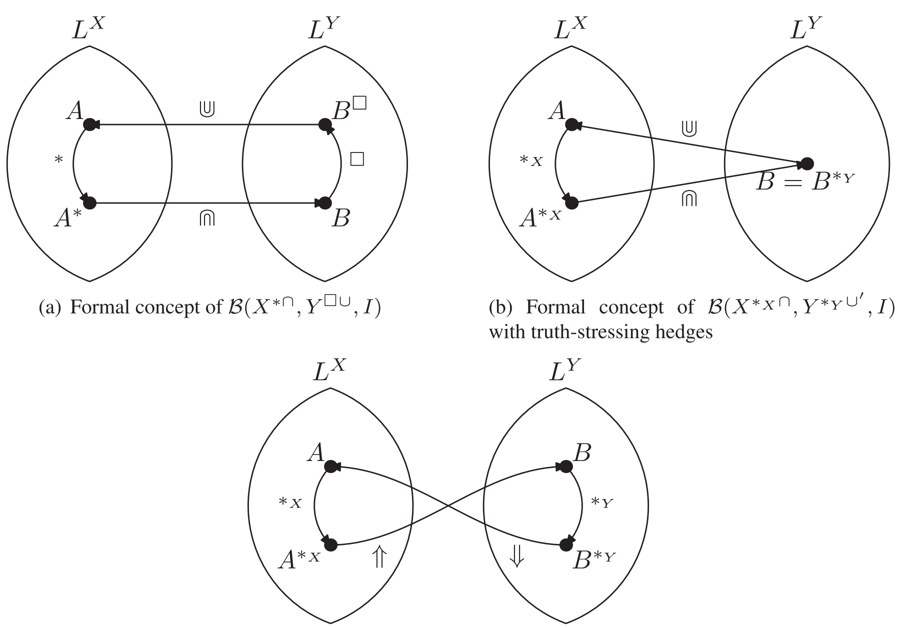


Fig. 2. Truth-depressing hedges.



(a) Formal concept of $\mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$

(b) Formal concept of $\mathcal{B}(X^{*_X\cap}, Y^{*_Y\cup'}, I)$ with truth-stressing hedges

(c) Formal concept of $\mathcal{B}(X^{*_X\uparrow}, Y^{*_Y\downarrow}, I)$

Fig. 3. Formal concept w.r.t. various concept-forming operators; arrows in (a), (b), and (c) represent mappings. For example in (a), mapping $*: L^X \to L^X$ is represented by arrow between A and $A*$ inside L^X; $A*$ placed under A means $A* \subseteq A$.

We need following lemmas.

Lemma 1 [7]. *A truth-stressing hedge* $*$ *satisfies* $(\bigvee_{i\in I} a_i^*)^* = \bigvee_{i\in I} a_i^*$.

Lemma 2. *A truth-depressing hedge* $^\square$ *satisfies* $(\bigwedge_{i\in I} a_i^\square)^\square = \bigwedge_{i\in I} a_i^\square$.

Proof. "$\geqslant$" Obvious from the definition of truth-depressing hedge.

"$\leqslant$" From $(\bigwedge_{i\in I} a_i^\square) \leqslant a_i^\square$ and monotony of $^\square$ (see Remark 1(a)) we have $(\bigwedge_{i\in I} a_i^\square)^\square \leqslant a_i^{\square\square} = a_i^\square$. Hence $(\bigwedge_{i\in I} a_i^\square)^\square \leqslant \bigwedge_{i\in I} a_i^\square$. □

2.3. Formal concept analysis

In this part we recall basics of formal concept analysis (FCA). The main aim in FCA is to extract interesting clusters (called formal concepts) from tabular data. A partially ordered collection of all formal concept is called a concept lattice. In the basic setting, the input data to FCA is organized in a table (formal context) such as the one in Table 1.

A *formal context* is a triplet $\langle X,Y,I\rangle$, where X and Y are sets of objects and attributes, respectively, and $I \subseteq X \times Y$ is a relation between X and Y. The fact that $\langle x,y\rangle \in I$ is interpreted as "object x has an attribute y".

A formal context $\langle X,Y,I\rangle$ *induces operators* $^{\Uparrow_I} : \mathbf{2}^X \to \mathbf{2}^Y$ and $^{\Downarrow_I} : \mathbf{2}^Y \to \mathbf{2}^X$:

$$A^{\Uparrow_I} = \{y | \langle x,y\rangle \in I \quad \text{for each } x \in A\}, \tag{12}$$

$$B^{\Downarrow_I} = \{x | \langle x,y\rangle \in I \quad \text{for each } y \in B\}. \tag{13}$$

In words, we can describe the induced operators as follows: $A^{\Uparrow_I}$ is a set of all attributes shared by all objects from A. $B^{\Downarrow_I}$ is a set of all objects sharing all attributes from B.

A *formal concept* of $\langle X,Y,I\rangle$ is a pair $\langle A,B\rangle$ such that

$$A^{\Uparrow_I} = B \quad \text{and } B^{\Downarrow_I} = A. \tag{14}$$

The *set of all formal concepts* of $\langle X,Y,I\rangle$ is denoted $\mathcal{B}(X^\Uparrow, Y^\Downarrow, I)$:

$$\mathcal{B}(X^\Uparrow, Y^\Downarrow, I) = \{\langle A,B\rangle | A^{\Uparrow_I} = B \text{ and } B^{\Downarrow_I} = A\}. \tag{15}$$

A *subconcept-superconcept hierarchy* of formal concepts is a partial order $\leqslant$ defined as follows

$$\langle A_1,B_1\rangle \leqslant \langle A_2,B_2\rangle \quad \text{iff } A_1 \subseteq A_2, \tag{16}$$

$$(\text{or, equivalently, iff } B_2 \subseteq B_1) \tag{17}$$

for each $\langle A_1,B_1\rangle, \langle A_2,B_2\rangle \in \mathcal{B}(X^\Uparrow, Y^\Downarrow, I)$.

$\mathcal{B}(X^\Uparrow, Y^\Downarrow, I)$ with $\leqslant$ forms a complete lattice:

Theorem 3 (Main theorem of concept lattices, [14]). *Let* $\langle X,Y,I\rangle$ *be formal context. Then* $\mathcal{B}(X^\Uparrow, Y^\Downarrow, I)$ *is complete lattice whose infima and suprema are defined as follows:*

$$\bigwedge\nolimits_{j\in J}\langle A_j,B_j\rangle = \left\langle \bigcap_{j\in J} A_j, \left(\bigcup_{j\in J} B_j\right)^{\Downarrow\Uparrow}\right\rangle, \tag{18}$$

$$\bigvee\nolimits_{j\in J}\langle A_j,B_j\rangle = \left\langle \left(\bigcup_{j\in J} A_j\right)^{\Uparrow\Downarrow}, \bigcap_{j\in J} B_j,\right\rangle. \tag{19}$$

Moreover, an arbitrary complete lattice $\mathbf{K} = \langle K,\leqslant\rangle$ *is isomorphic to* $\mathcal{B}(X^\Uparrow, Y^\Downarrow, I)$ *iff there are mappings* $\mu: X \to K$, $\nu: Y \to K$ *such that*

1. $\mu(X)$ *is* $\bigvee$*-dense in* K, $\nu(Y)$ *is* $\bigwedge$*-dense in* K;
2. $\mu(x) \leqslant \nu(y)$ *iff* $\langle x,y\rangle \in I$.

Table 1
Formal context describing objects x_1, x_2, x_3 and their yes/no attributes y_1, y_2, y_3, y_4.

	y_1	y_2	y_3	y_4
x_1	1	1	0	0
x_2	0	1	1	0
x_3	0	0	1	1

[18] showed that operators $\langle {}^{\Uparrow_I}, {}^{\Downarrow_I} \rangle$ are in one-to-one correspondence with so-called *antitone Galois connections*. A pair $\langle {}^{\Uparrow}, {}^{\Downarrow} \rangle$ of mappings ${}^{\Uparrow}$: $2^X \to 2^Y$, ${}^{\Downarrow}$: $2^Y \to 2^X$ is said to form *antitone Galois connection* between sets X and Y if the following the conditions are satisfied:

(i) if $A_1 \subseteq A_2$ then $A_2^{\Uparrow} \subseteq A_1^{\Uparrow}$,
(ii) if $B_1 \subseteq B_2$ then $B_2^{\Downarrow} \subseteq B_1^{\Downarrow}$,
(iii) $A \subseteq A^{\Uparrow\Downarrow}$,
(iv) $B \subseteq B^{\Uparrow\Downarrow}$

for A, A_1, $A_2 \in 2^X$ and B, B_1, $B_2 \in 2^Y$.

The following theorem explains the correspondence:

Theorem 4 ([18]). *Let $\langle X, Y, I \rangle$ be formal context, $\langle {}^{\Uparrow}, {}^{\Downarrow} \rangle$ be an antitone Galois connection between X and Y. Then*

(i) $\langle {}^{\Uparrow_I}, {}^{\Downarrow_I} \rangle$ *is an antitone Galois connection.*
(ii) $I_{\langle \Uparrow, \Downarrow \rangle}$ *defined by*

$$I_{\langle \Uparrow, \Downarrow \rangle}(x, y) \quad \textit{iff} \;\; y \in \{{}^1/x\}^{\Uparrow} \tag{20}$$

is a relation between X and Y and we have

(iii) $\langle {}^{\Uparrow}, {}^{\Downarrow} \rangle = \left\langle {}^{\Uparrow_{I_{\langle \Uparrow, \Downarrow \rangle}}}, {}^{\Downarrow_{I_{\langle \Uparrow, \Downarrow \rangle}}} \right\rangle$ *and* $I = I_{\langle \Uparrow_I, \Downarrow_I \rangle}$.

3. Isotone Galois connections with hedges

3.1. Definition

We start by recalling the definition of and basic facts about the isotone fuzzy Galois connections [15,20]:

Definition 1. An isotone **L**-Galois connection between sets X and Y is a pair $\langle {}^{\Cap}, {}^{\Cup} \rangle$ of mappings ${}^{\Cap}$: $L^X \to L^Y$ and ${}^{\Cup}$: $L^Y \to L^X$ satisfying $S(A, B^{\Cup}) = S(A^{\Cap}, B)$. S is the graded subsethood (1).

Isotone **L**-Galois connections are sometimes called isotone fuzzy Galois connections. The following theorem provides an alternative definition using perhaps more comprehensible conditions [15].

Theorem 5. *A pair $\langle {}^{\Cap}, {}^{\Cup} \rangle$ of mappings ${}^{\Cap}$: $L^X \to L^Y$ and ${}^{\Cup}$: $L^Y \to L^X$ is an isotone Galois connection iff ${}^{\Cap}$ and ${}^{\Cup}$ satisfy*

$$S(A_1, A_2) \leqslant S(A_1^{\Cap}, A_2^{\Cap}), \tag{21}$$
$$S(B_1, B_2) \leqslant S(B_1^{\Cup}, B_2^{\Cup}), \tag{22}$$
$$A \subseteq A^{\Cap\Cup}, \tag{23}$$
$$B \supseteq B^{\Cup\Cap}. \tag{24}$$

The importance of Galois connections, both antitone and isotone, derives from the fact that they are induced in a natural way from binary relations and that the fixpoints (i.e. pairs s.t. $\langle A, B \rangle$ s.t. $A^{\Cap} = B$ and $B^{\Cup} = A$) of Galois connections have natural meaning. A canonical way an isotone Galois connection $\langle {}^{\Cap}, {}^{\Cup} \rangle$ arises from a binary fuzzy relation I between sets X and Y is described by:

$$A^{\Cap}(y) = \bigvee_{x \in X} A(x) \otimes I(x, y), \tag{25}$$
$$B^{\Cup}(x) = \bigwedge_{y \in Y} I(x, y) \to B(y). \tag{26}$$

If X and Y are interpreted as the set of objects and attributes and $I(x,y)$ as a degree to which object $x \in X$ has attribute $y \in Y$, then $A^{\Cap}(y)$ is just the truth degree of "there exists x in A which has y" and $B^{\Cup}(x)$ is the truth degree of "for all y: if x has y then y belongs to B". That is, $A^{\Cap}$ is the fuzzy set of attributes shared by at least one object from A and $B^{\Cup}$ is the fuzzy set of objects whose attributes are all in B.

Note that in the bivalent case, i.e. when I is an ordinary relation and A and B are ordinary sets, the operators defined by (25) and (26) are studied in [13]. The operators studied in this paper extend those from [13] in that we assume that I is a fuzzy relation and A and B are fuzzy sets with truth degrees taken from a complete residuated lattice L. If L is the two-element Boolean algebra, operators (25) and (26) as well as their parameterized versions (27) and (28) introduced below studied coincide with those from [13]. Note also that the pairs of mappings (25) and (26) appear in [12,15,20] and also in [19]. In what follows, we present and study operators which generalize (25) and (26) in that we parameterize (25) and (26). Technically, we parameterize (25) and (26) by inserting hedges at particular places in (25) and (26). Throughout the rest of the paper, we assume that * is truth-stressing hedge on **L** and $^\square$ is truth-depressing hedge on **L** (which does not need to be a truth-depressing hedge w.r.t. *).

Let X, Y be sets of objects and attributes respectively, I be an $\mathbf{L}$-relation between X and Y, i.e. I is a mapping $I: X \times Y \to L$. $\langle X, Y, I\rangle$ is called a *formal fuzzy context*.

For a formal fuzzy context $\langle X, Y, I\rangle$ we define a pair $\langle^{\cap}, ^{\cup}\rangle$ of mappings $^{\cap}: L^X \to L^Y$ and $^{\cup}: L^Y \to L^X$ by

$$A^{\cap}(y) = \bigvee_{x\in X} A(x)^{*} \otimes I(x,y), \tag{27}$$

$$B^{\cup}(x) = \bigwedge_{y\in Y} I(x,y) \ \to \ B(y)^{\square}. \tag{28}$$

These mappings play a crucial role in our paper. The meaning of $A^{\cap}$ and $B^{\cup}$ is essentially the same as that of $A^{\Cap}$ and $B^{\Cup}$. The difference is in that parts of the verbal description of $A^{\cap}$ and $B^{\cup}$ contain "very true" and "slightly true" respectively, compared to that of $A^{\Cap}$ and $B^{\Cup}$. For example, $A^{\cap}(y)$ is the truth degree of "there exists x for which it is very true that it belongs to A and which has y".

The fixed points of $\langle^{\cap}, ^{\cup}\rangle$ (i.e. pairs $\langle A, B\rangle$ such that $A^{\cap} = B$ and $B^{\cup} = B$) are called *formal (fuzzy) concepts*. Operators induced by formal fuzzy context are usually called *concept-forming operators*. The set of all formal concepts of $\langle X, Y, I\rangle$ is denoted $\mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$.

For formal concepts $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$ we define

$$\langle A_1, B_1\rangle \leqslant \langle A_2, B_2\rangle \quad \text{iff } A_1 \subseteq A_2 \quad (\text{iff } B_1 \subseteq B_2). \tag{29}$$

As we show later, $\mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$ with $\leqslant$ forms a complete lattice.

3.2. Basic properties

This section describes the generalization $\langle^{\cap}, ^{\cup}\rangle$ of concept-forming operators $\langle^{\Cap}, ^{\Cup}\rangle$ from [15] and shows basic properties of $\langle^{\cap}, ^{\cup}\rangle$.

Theorem 6. *Mappings* $^{\cap}, ^{\cup}$ *defined by (27) and (28) satisfy the following properties:*

(i) $A^{\cap} = A^{*\Cap}$ *and* $B^{\cup} = B^{\square\Cup}$
(ii) $A^{\cap} = A{*}^{\cap}$ *and* $B^{\cup} = B^{\square\cup}$
(iii) $A^{\cap} \subseteq A^{\Cap}$ *and* $B^{\Cup} \subseteq B^{\cup}$
(iv) $S(A_1, A_2)^{*} \leqslant S(A_1^{*}, A_2^{*}) \leqslant S(A_1^{\cap}, A_2^{\cap})$
$S(B_1, B_2)^{*_Y} \leqslant S(B_1^{\square}, B_2^{\square}) \leqslant S(B_1^{\cup}, B_2^{\cup})$ *where* ^{*_Y} *is a truth-stressing hedge for which (9) is satisfied.*
(v) $A{*} \subseteq A^{\cap\cup}$ *and* $B^{\cup\cap} \subseteq B^{\square}$
(vi) $A_1 \subseteq A_2$ *implies* $A_1^{\cap} \subseteq A_2^{\cap}$
$B_1 \subseteq B_2$ *implies* $B_1^{\cup} \subseteq B_2^{\cup}$
(vii) $S(A{*}, B^{\cup}) = S(A^{\cap}, B^{\square})$
(viii) $(\bigcup_{i\in I} A_i^{*})^{\cap} = \bigcup_{i\in I} A_i^{\cap}$ *and* $(\bigcap_{i\in I} B_i^{\square})^{\cup} = \bigcap_{i\in I} B_i^{\cup}$
(ix) $A^{\cap\cup\cap\cup} = A^{\cap\cup}$ *and* $B^{\cap\cup\cap\cup} = B^{\cap\cup}$

Proof.

(i), (ii) follow immediately from definition of $^{\cap}$ and $^{\cup}$ and from properties of hedges.
(iii) follows from the fact, that $\otimes$ is monotone and $\to$ is isotone in the second argument.
(iv) $S(A_1, A_2)^{*} \leqslant S(A_1^{*}, A_2^{*})$ and $S(B_1, B_2)^{*_Y} \leqslant S(B_1^{\square}, B_2^{\square})$ follow from definitions of the truth-stressing and truth-depressing hedges and Lemmas 1 and 2. $S(A_1^{\cap}, A_2^{\cap}) = S(A_1^{*\Cap}, A_2^{*\Cap}) = S(A_1^{*}, A_2^{*\Cap\Cup}) \geqslant S(A_1^{*}, A_2^{*})$. The second assertion is similar.
(v) $A{*} \subseteq A{*}^{\Cap\Cup} = A^{\cap\Cup} \subseteq A^{\cap\cup}$; $B^{\cup\cap} \subseteq B^{\cup\Cap} = B^{\square\Cup\Cap} \subseteq B^{\square}$.
(vi) $A_1 \subseteq A_2$ implies $1 = S(A_1, A_2)^{*} \leqslant S(A_1^{\cap}, A_2^{\cap})$. The second claim is similar.
(vii) $S(A{*}, B^{\cup}) = S(A{*}, B^{\square\Cup}) = S(A{*}^{\cap}, B^{\square}) = S(A^{\cap}, B^{\square})$.
(viii) Using Lemma 1, we have

$$\left(\bigcup_{i\in I} A_i^{*}\right)^{\cap}(y) = \bigvee_{x\in X}\left(\bigvee_{i\in I}(A_i^{*}(x))^{*} \otimes I(x,y)\right) = \bigvee_{x\in X}\left(\left(\bigvee_{i\in I} A_i^{*}(x)\right) \otimes I(x,y)\right) = \bigvee_{i\in I}\left(\bigvee_{x\in X} A_i^{*}(x) \otimes I(x,y)\right) = \bigvee_{i\in I} A_i^{\cap}(y).$$

Similarly, using Lemma 2, we have

$$\left(\bigcap_{i\in I} B_i^{\square}\right)^{\cup}(x) = \bigwedge_{y\in Y}\left(I(x,y) \to \left(\bigwedge_{i\in I} B_i^{\square}(y)\right)^{\square}\right) = \bigwedge_{y\in Y}\left(I(x,y) \to \left(\bigwedge_{i\in I} B_i^{\square}(y)\right)\right) = \bigwedge_{i\in I}\left(\bigwedge_{y\in Y} I(x,y) \to B_i^{\square}(y)\right) = \bigwedge_{i\in I} B_i^{\cup}(x).$$

(ix) Using (v) $A \subseteq A^{\cap\cup}$ and (vi) two times we get $A^{\cap\cup} \subseteq A^{\cap\cup\cap\cup}$. Using (v) with $B = A^{\cap}$ we have $A^{\cap\cup\cap} \subseteq A^{\cap\square}$. Using (vi) we get the first claim. The second claim is similar. $\square$

Remark 2. Note that the induced concept-forming operators with $*$, $^{\Box}$ have very similar properties to those defined by (33) and (34) which were introduced in [1]. The following list sums up properties of these operators which are analogous to those from Theorem 6.

(i) $A^{\cap} = A^{*_X \Cap}$ and $B^{\cup} = B^{*_Y \Cup}$,
(ii) $A^{\cap} = A^{*_X \cap}$ and $B^{\cup} = B^{*_Y \cup}$,
(iii) $A^{\cap} \subseteq A^{\Cap}$ and $B^{\cup} \subseteq B^{\Cup}$,
(iv) $S(A_1, A_2)^{*_X} \leqslant S(A_1^{*_X}, A_2^{*_X}) \leqslant S(A_1^{\cap}, A_2^{\cap})$,
$S(B_1, B_2)^{*_Y} \leqslant S(B_1^{*_Y}, B_2^{*_Y}) \leqslant S(B_1^{\cup}, B_2^{\cup})$,
(v) $B^{\cup\cap} \subseteq B^{*_Y}$,
(vi) $A_1 \subseteq A_2$ implies $A_1^{\cap} \subseteq A_2^{\cap}$,
$B_1 \subseteq B_2$ implies $B_1^{\cup} \subseteq B_2^{\cup}$,
(vii) $S(A^{*_X}, B^{\cup}) = S(A^{\cap}, B^{*_Y})$,
(viii) $(\bigcup_{i\in I} A_i^{*_X})^{\cap} = \bigcup_{i\in I} A_i^{\cap}$ and $(\bigcap_{i\in I} B_i)^{\cup} = (\bigcap_{i\in I} B_i^{*_Y})^{\cup}$,
(ix) $A^{\cap\cup\cap} \subseteq A^{\cap *_Y}$ and $B^{\cup\cap\cup} \subseteq B^{*_Y \cup}$.

We have extended (25) and (26) and made them parameterizable using truth-stressing hedge and truth-depressing hedge while we have kept most of their basic properties. In particular we have lost properties $A^{\Cap\Cup\Cap} = A^{\Cap}$ and $B^{\Cap\Cup\Cap} = B^{\Cap}$ and replaced them by the property (ix) in Theorem 6.

3.3. Axiomatization

We now turn to the problem of axiomatization of the mappings defined by (27) and (28). We present characteristic properties of these mappings.

In this part, we use subscription I to denote operations induced by context $\langle X, Y, I\rangle$ ($^{\cap_I}$ and $^{\cup_I}$) to distinguish them from operators introduced in Definition 2. At the end of this part, we show that these operations are the same, thus we do not need to distinguish them in later parts of this paper.

Isotone Galois connections were axiomatized in [15]. We generalize the approach of [15] as follows:

Definition 2. Let X, Y be two universes. A pair of mappings $\langle^{\cap}, ^{\cup}\rangle$, $^{\cap}$: $L^X \to L^Y$, $^{\cup}$: $L^Y \to L^X$ is called *isotone* **L**-*Galois connection* between X and Y if

$$S(A^*, B^{\cup}) = S(A^{\cap}, B^{\Box}), \tag{30}$$

$$\left(\bigcup_{i\in I} A_i^*\right)^{\cap} = \bigcup_{i\in I} A_i^{\cap}, \tag{31}$$

$$a^* \otimes \{^1/x\}^{\cap}(y) = \{a/x\}^{\cap}(y) \tag{32}$$

Lemma 7. *Operators $^{\cap_I}$ and $^{\cup_I}$ defined by (27) and (28) form an isotone* **L**-*Galois connection $\langle^{\cap_I}, ^{\cup_I}\rangle$ with hedges $*$ and $^{\Box}$.*

Proof. Due to Theorem 6 (vii) and (viii), it is enough to show that (32) is satisfied. Indeed,

$$a^* \otimes \{^1/x\}^{\cap_I}(y) = a^* \otimes \bigvee_{x\in X} 1 \otimes I(x,y) = \bigvee_{x\in X} a^* \otimes I(x,y) = \{^a/x\}^{\cap}(y). \qquad \Box$$

Lemma 8. *For every mapping $^{\cap}$: $L^X \to L^Y$ there exist at most one mapping $^{\cup}$: $L^Y \to L^X$ satisfying $S(A*, B^{\cup}) = S(A^{\cap}, B^{\Box})$ for every $A \in L^X$ and $B \in L^Y$.*

Proof. If $^{\cup'}$ is another such mapping, we have $S(A^*, B^{\cup'}) = S(A^{\cap}, B^{\Box})$ for any A and B. Take any $x \in X$ and put $A = \{^1/x\}$. Then

$$B^{\cup}(x) = S(A^*, B^{\cup}) = S(A^*, B^{\cup'}) = B^{\cup'}(x).$$

Therefore, $^{\cup}$ coincides with $^{\cup'}$. $\Box$

Lemma 9. *Let $\langle^{\cap}, ^{\cup}\rangle$ be an isotone* **L**-*Galois connection with hedges * and $^{\Box}$. Then there exists an* **L**-*relation I between X and Y such that $\langle^{\cap}, ^{\cup}\rangle = \langle^{\cap_I}, ^{\cup_I}\rangle$.*

Proof. We need to find I such that $A^{\cap} = A^{\cap_I}$ and $B^{\cup} = B^{\cup_I}$ for all $A \in L^X$, $B \in L^Y$. Due to Lemma 8, it is sufficient to find I for which $A^{\cap} = A^{\cap_I}$. Namely, $\langle^{\cap_I}, ^{\cup_I}\rangle$ satisfy $S(A*, B^{\cup}) = S(A^{\cap}, B^{\Box})$ by Lemma 7. Hence, $^{\cup_I}$ coincides with $^{\cup}$ due to Lemma 8.

Define I by $I(x,y) = \{^1/x\}^{\cap}(y)$. Then we get

$$A^{\cap}(y) = A^{*\cap}(y) = \bigvee_{x\in X, y\in Y}\{^{A^*(x)}/x\}^{\cap}(y) = \bigvee_{x\in X}\bigvee_{y\in Y}A^*(x)\otimes\{^1/x\}^{\cap}(y) = \bigvee_{x\in X}A^*(x)\otimes I(x,y) = A^{\cap_I}(y).$$

This finishes the proof. $\square$

Theorem 10. *Let $\langle X, Y, I\rangle$ be formal fuzzy context, $\langle^{\cap},^{\cup}\rangle$ be an isotone **L**-Galois connection with hedges $*$ and $^{\square}$. Then*

- $\langle^{\cap_I},^{\cup_I}\rangle$ *is isotone **L**-Galois connection with hedges $*$ and $^{\square}$.*
- $I_{\langle\cap,\cup\rangle}$ *defined by $I_{\langle\cap,\cup\rangle}(x,y) = \{^1/x\}^{\cap}(y)$ is an **L**-relation between X and Y and we have*
- $\langle^{\cap},^{\cup}\rangle = \left\langle^{\cap_{I_{\langle\cap,\cup\rangle}}},^{\cup_{I_{\langle\cap,\cup\rangle}}}\right\rangle$ *and $I = I_{\langle\cap,\cup\rangle}$.*

Proof. Due to Lemmas 7 and 9, it suffices to show that $I = I_{\langle\cap,\cup\rangle}$. We have

$$I_{\langle\cap,\cup\rangle}(x,y) = \{^1/x\}^{\cap_I}(y) = \bigvee_{z\in X}\{^{1^*}/x\}(z)\otimes I(z,y) = I(x,y) \qquad \square$$

Remark 3. Note that we need only $^{\cap}$ to define $I_{\langle\cap,\cup\rangle}$ and we need $^{\cap}$ to satisfy only (31) and (32). Having such an operation, we can use Theorem 10 to find corresponding $^{\cup}$ as $^{\cup} = {}^{\cup_{I_{\langle\cap,\cup\rangle}}}$.

3.4. Why we use a truth-depressing hedge?

In [1] we introduced the following concept-forming operators:

$$A^{\cap}(y) = \bigvee_{x\in X}A(x)^{*_X}\otimes I(x,y), \tag{33}$$

$$B^{\cup'}(x) = \bigwedge_{y\in Y}I(x,y) \rightarrow B(y)^{*_Y}, \tag{34}$$

where ^{*_X} and ^{*_Y} are truth-stressing hedges. Note that the only difference from the concept-forming operators defined by (27) and (28) is that a truth-stressing hedge ^{*_Y} is used in (34) while a truth-depressing hedge $^{\square}$ is used in (28). In this part we argue that the use of a truth-depressing hedge is more convenient.

Let us take a look at a geometric interpretation of a formal concept as a fixpoint of isotone **L**-Galois connection with hedges $*$ and $^{\square}$ (see Fig. 3(a)).

If a truth-stressing hedge ^{*_Y} is used we have the situation depicted in Fig. 3(b). B and B^{*_Y} degenerate into one point, as described by the following theorem.

Theorem 11. *Let $\mathcal{B}(X^{*_X\cap}, Y^{*_Y\cup}, I)$ denote the set of all fixpoints of the operators by (27) and (28) and $\mathcal{B}(X^{*_X\cap}, Y^{\cup}, I)$ set of all fixpoints of the same operators for ^{*_Y} being identity $^{\mathrm{id}_L}$. Then we have*

$$\mathcal{B}(X^{*_X\cap}, Y^{*_Y\cup}, I) = \{\langle A, B\rangle \in \mathcal{B}(X^{*_X\cap}, Y^{\cup}, I) | B = B^{*_Y}\}. \tag{35}$$

Proof. "$\subseteq$": $B = B^{\cup'\cap} \subseteq B^{*_Y} \subseteq B$ proves, that $B = B^{*_Y}$. For intent B we have $B = B^{\cup'\cap} = B^{*_Y \Cup *_X \Cap} = B^{\Cup *_X \Cap}$ proving that $B \in \mathrm{Int}(X^{*_X\cap}, Y^{\cup}, I)$.

"$\supseteq$": $B = B^{\Cup *_X \Cap} = B^{*_Y \Cup *_X \Cap} = B^{\cup'\cap}$, thus $B \in \mathrm{Int}(X^{*_X\cap}, Y^{*_Y\cup}, I)$. $\square$

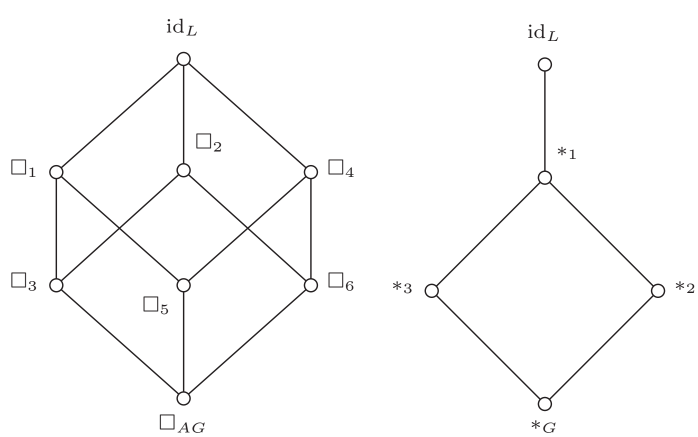


Fig. 4. Truth-depressing hedges from Fig. 2 with ⩽ (left) and truth-stressing hedges from Fig. 1 with ⩽ (right).

Note that Theorem 11 says that using ^{*_Y} brings just trivial selection of formal concepts.

The use of truth-depressing hedge brings us to analogy of the geometrical interpretation of a formal concept of $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$, which is depicted in Fig. 3(c).

In the case of concept-forming operators ${}^{\uparrow}$, ${}^{\downarrow}$, we have both composition ${}^{\Uparrow\Downarrow}$ and ${}^{\Downarrow\Uparrow}$ being closure operators. With truth-stressing hedges ^{*_Y} and ^{*_X} the compositions ${}^{\Uparrow *_Y \Downarrow}$ and ${}^{\Downarrow *_X \Uparrow}$ keep to be closure operators. On the other hand, the truth-stressing hedges ^{*_X} and ^{*_Y} are interior operators.[1] Similarly, in the case of concept-forming operators ${}^{\cap}$, ${}^{\cup}$, we have the composition ${}^{\Cap\Cup}$ being a closure operator. With a truth-depressing hedge ${}^{\square}$ the composition ${}^{\Cap\square\Cup}$ keeps to be a closure operator. A truth-stressing hedge $*$ works opposite way to the composition ${}^{\Cap\square\Cup}$. Dually, the compositions ${}^{\Cup\Cap}$ and ${}^{\Cup *\Cap}$ are interior operators, while ${}^{\square}$ is a closure operator.

The main benefits of using truth-depressing hedge in (28) are:

- According to Theorem 6 for any isotone **L**-Galois connection with * and ${}^{\square}$ we have convenient properties $A^{\cap\cup\cap} \subseteq A^{\cap *_Y}$ and $B^{\cup\cap\cup} \subseteq B^{*_Y\cup}$. Analogous properties do not generally hold true for isotone **L**-Galois connection with ^{*_X} and ^{*_Y}.
- By Theorem 11, using a truth-stressing hedge ^{*_Y} in (34) turns to be a selection of formal concepts from $\mathcal{B}(X^{*_X\cap}, Y^{\cup}, I)$ based on membership degrees in their intents. Particulary, all concepts from $\mathcal{B}(X^{*_X\cap}, Y^{\cup}, I)$ whose intents contain attributes in other truth degrees than fix(^{*_Y}) are filtered out. Objects can even completely 'disappear' from resulting concept lattice not being present in any of its extents. This kind of selection does not seem to be reasonable.
- The reduction of the size of the associated concept lattice with two truth-stressing hedges is too drastic [1]. Especially when using ${}^{*_Y} = {}^{*_G}$, the resulting concept lattice commonly happens to be a trivial lattice containing no interesting information. Reduction with truth-stressing hedge and truth-depressing hedge (see Section 4) seems to be more natural in comparison with the previous one.

3.5. Main theorem on the structure of $\mathcal{B}(X^{*\cap}, X^{\square\cup}, I)$

In this part we show that concept lattice $\mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$ is isomorphic to a concept lattice of a particular ordinary formal context with ${}^{\Uparrow}$,${}^{\Downarrow}$. Moreover, we provide a variant of the main theorem of concept lattices for $\mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$. The content of this part is inspired by Belohlavek [4], Belohlavek and Vychodil [8].

We need the following notions:

Definition 3. For $A \in L^X$ we define $\lfloor A \rfloor_{\vee} \in 2^{X\times L}$ and $\lfloor A \rfloor_{\wedge} \in 2^{X\times L}$ by

$$\lfloor A \rfloor_{\vee} = \{\langle x, a\rangle | a \leqslant A(x)\}, \tag{36}$$

$$\lfloor A \rfloor_{\wedge} = \{\langle x, a\rangle | A(x) \leqslant a\}. \tag{37}$$

Described verbally, $\lfloor A \rfloor_{\vee}$ can be considered as an area in $X \times L$ under the membership function A: $X \to L$ and $\lfloor A \rfloor_{\wedge}$ as an area in $X \times L$ above the membership function A: $X \to L$.

For $A' \in \mathbf{2}^{X\times L}$ we define $\lceil A' \rceil_{\vee} \in L^X$ and $\lceil A' \rceil_{\wedge} \in L^X$ by

$$\lceil A' \rceil_{\vee}(x) = \bigvee \{a | \langle x, a\rangle \in A'\}, \tag{38}$$

$$\lceil A' \rceil_{\wedge}(x) = \bigwedge \{a | \langle x, a\rangle \in A'\} \tag{39}$$

for each $x \in X$.

Definition 4. For $A' \subseteq X \times L$ and (truth-stressing or truth-depressing) hedge ${}^{\bullet}$: $L \to L$, define $A'^{\bullet} = \{\langle x, a^{\bullet}\rangle | \langle x, a\rangle \in A'\}$.

Lemma 12 ([9]). *For $A \subseteq X \times$ fix $(^{*})$ we have $A \subseteq \lfloor \lceil A \rceil_{\vee}^{*} \rfloor_{\vee}^{*}$.*

Analogously, we have:

Lemma 13. *For $B \subseteq$ fix $(^{\square}) \times Y$ we have $B \subseteq \lfloor \lceil B \rceil_{\wedge}^{\square} \rfloor_{\wedge}^{\square}$.*

Proof. Let $\langle y, b\rangle \in B$. Then $b \geqslant \lceil B \rceil_{\wedge}$. Since $b \in$ fix $(^{\square})$, we have $b \geqslant \lceil B \rceil_{\wedge}^{\square}$. Thus $\langle y, b\rangle \in \lfloor \lceil B \rceil_{\wedge}^{\square} \rfloor_{\wedge}$. Finally $\langle y, b\rangle \in \lfloor \lceil B \rceil_{\wedge}^{\square} \rfloor_{\wedge}^{\square}$ since $b \in$ fix $(^{\square})$. □

Define mappings ${}^{\uparrow_\times}: X \times \text{fix}(^{*}) \to Y \times \text{fix}(^{\square})$ and ${}^{\downarrow_\times}: Y \times \text{fix}(^{\square}) \to X \times \text{fix}(^{*})$ by

$$A^{\uparrow_\times} = \lfloor \lceil A \rceil_{\vee}^{\cap} \rfloor_{\wedge}^{\square} \quad \text{and } B^{\downarrow_\times} = \lfloor \lceil B \rceil_{\wedge}^{\cup} \rfloor_{\vee}^{*}. \tag{40}$$

[1] Further description of $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$ is out of scope of this paper, see f.e. [4,7,9] for this topic.

Lemma 14. *The pair $\langle {}^{\uparrow_\times}, {}^{\downarrow_\times} \rangle$ forms an antitone Galois connection between sets $X \times \mathrm{fix}(^*)$ and $Y \times \mathrm{fix}(^\square)$.*

Proof. Antitony: $A_1 \subseteq A_2$ implies $\lceil A_1 \rceil_\vee \subseteq \lceil A_2 \rceil_\vee$ which implies $\lceil A_1 \rceil_\vee{}^\cap \subseteq \lceil A_2 \rceil_\vee{}^\cap$ which implies $\lfloor \lceil A_2 \rceil_\vee^\cap \rfloor_\wedge \subseteq \lfloor \lceil A_1 \rceil_\vee^\cap \rfloor_\wedge$. Similarly $B_1 \subseteq B_2$ implies $\lceil B_2 \rceil_\wedge \subseteq \lceil B_1 \rceil_\wedge$ which implies $\lceil B_1 \rceil_\wedge{}^\cup \subseteq \lceil B_2 \rceil_\wedge{}^\cup$ which implies $\lfloor \lceil B_2 \rceil_\wedge^\cup \rfloor_\vee \subseteq \lfloor \lceil B_1 \rceil_\wedge^\cup \rfloor_\vee$.

Extensivity: Using Lemma 12, $A^{\uparrow_\times \downarrow_\times} = \lfloor \lceil \lfloor \lceil A \rceil_\vee^\cap \rfloor_\vee^\square \rceil_\wedge^\cup \rfloor_\wedge^* = \lfloor \lceil \lfloor \lceil A \rceil_\vee^\cap \rfloor_\vee \rceil_\wedge^{\square\cup} \rfloor_\wedge^* = \lfloor \lceil A \rceil_\vee^{\cap\square\cup} \rfloor_\wedge^* \supseteq \lfloor \lceil A \rceil_\vee^* \rfloor_\wedge^* \supseteq A$. Similarly $B \subseteq B^{\downarrow_\times \uparrow_\times}$. □

The following theorem is a direct consequence of the main theorem of concept lattices [14]. It says that concept lattice of the formal fuzzy context corresponding to isotone Galois connection $\langle {}^{\uparrow_\times}, {}^{\downarrow_\times} \rangle$ forms a complete lattice and each complete lattice satisfying some particular technical condition is isomorphic to the concept lattice of a formal context $\langle U, V, I_{\langle \uparrow_\times, \downarrow_\times \rangle} \rangle$ which is given by the antitone the Galois connection defined by (40) and by (20).

Theorem 15

1. *$\mathcal{B}(U^{\Uparrow}, V^{\Downarrow}, I_{\langle \uparrow_\times, \downarrow_\times \rangle})$ equipped with $\leqslant$, defined by $\langle A_1, B_1 \rangle \leqslant \langle A_2, B_2 \rangle$ iff $A_1 \subseteq A_2$, is a complete lattice where the infima and suprema are given by*

$$\bigwedge_{j\in J} \langle A_j, B_j \rangle = \left\langle \bigcap_{j\in J} A_j, \left(\bigcup_{j\in J} B_j \right)^{\downarrow_\times \uparrow_\times} \right\rangle, \tag{41}$$

$$\bigvee_{j\in J} \langle A_j, B_j \rangle = \left\langle \left(\bigcup_{j\in J} A_j \right)^{\uparrow_\times \downarrow_\times}, \bigcap_{j\in J} B_j \right\rangle. \tag{42}$$

2. *Moreover, an arbitrary complete lattice $\mathbf{K} = \langle K, \leqslant \rangle$ is isomorphic to $\mathcal{B}(U, V, I_{\langle \uparrow_\times, \downarrow_\times \rangle})$ iff there are mappings $\mu: U \to K$, $\nu: V \to K$ such that*

   (a) *$\mu(U)$ is $\bigvee$-dense in K, $\nu(V)$ is $\bigwedge$-dense in K;*
   (b) *$\mu(u) \leqslant \nu(v)$ iff $\langle u, v \rangle \in I_{\langle \uparrow_\times, \downarrow_\times \rangle}$.*

Lemma 16. *The (crisp) relation $I^\times = I_{\langle \uparrow_\times, \downarrow_\times \rangle}$ between $X \times \mathrm{fix}(^*)$ and $Y \times \mathrm{fix}\,(^\square)$ corresponding to Galois connection $\langle {}^{\uparrow_\times}, {}^{\downarrow_\times} \rangle$ defined by (40) is given by*

$$\langle \langle x, a \rangle, \langle y, b \rangle \rangle \in I^\times \quad \textit{iff} \;\; I(x, y) \leqslant a \to b. \tag{43}$$

Proof. We have $\langle \langle x,a \rangle, \langle y,b \rangle \rangle \in I^\times$ iff $\langle y, b \rangle \in \{ \langle x, a \rangle \}^{\uparrow_\times}$. By definition of ${}^{\uparrow_\times}$, this is equivalent to $\langle y, b \rangle \in \lfloor \lceil \{ \langle x \cdot a \rangle \} \rceil_\vee^\cap \rfloor_\vee^\square$. Since $\lfloor \lceil \{ \langle x, a \rangle \} \rceil_\vee^\cap \rfloor_\vee^\square = \lfloor \{ {}^a/x \}^\cap \rfloor_\vee^\square$ and since the smallest c such that $\langle y, c \rangle \in \lfloor \{ {}^a/x \}^\cap \rfloor_\vee{}^\square$ is $c = (\{ {}^a/x \}^\cap(y))^\square$, the last assertion is equivalent to $(\{ {}^a/x \}^\cap(y))^\square \leqslant b$. Since $b = b^\square$, this is equivalent to $(\{ {}^a/x \}^\cap(y)) \leqslant b$. Now, $\{ {}^a/x \}^\cap(y) = a* \otimes I(x,y) = a \otimes I(x,y)$, whence $\{ {}^a/x \}^\cap(y) \leqslant b$ is equivalent to $I(x,y) \leqslant a \to b$ by adjointness. □

Theorem 17. $\mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$ (concept lattice with hedges) is isomorphic to $\mathcal{B}(X \times \mathrm{fix}(*)^{\uparrow_\times}, Y \times \mathrm{fix}(^\square)^{\downarrow_\times}, I^\times)$ (ordinary concept lattice). The isomorphism

$$h : \mathcal{B}(X^{*\cap}, Y^{\square\cup}, I) \to \mathcal{B}(X \times \mathrm{fix}(^*)^{\uparrow_\times}, Y \times \mathrm{fix}(^\square)^{\downarrow_\times}, I^\times)$$

and its inverse

$$g : \mathcal{B}(X \times \mathrm{fix}(^*)^{\uparrow_\times}, Y \times \mathrm{fix}(^\square)^{\downarrow_\times}, I^\times) \to \mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$$

are given by

$$h(\langle A, B \rangle) = \langle \lfloor A \rfloor_\vee^*, \lfloor B \rfloor_\wedge^\square \rangle, \tag{44}$$

$$g(\langle A', B' \rangle) = \langle \lceil A' \rceil_\vee^{\cap\cup}, \lceil B' \rceil_\wedge^{\cup\cap} \rangle. \tag{45}$$

Proof. We need to show, that (a) h and g are defined correctly, (b) h is order-preserving, (c) $g(h(\langle A,B \rangle)) = \langle A,B \rangle$ and $h(g(\langle A', B' \rangle)) = \langle A', B' \rangle$.

(a) For $\langle A,B \rangle$ in $\mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$ we have $\lfloor A \rfloor_\vee^{*\uparrow_\times} = \lfloor A^\cap \rfloor_\wedge^\square$ and $\lfloor B \rfloor_\wedge^{\square\downarrow_\times} = \lfloor B^\cup \rfloor_\vee^*$ directly from definitions of operators ${}^{\uparrow_\times}$ and ${}^{\downarrow_\times}$ (40). For $\langle A', B' \rangle$ in $\mathcal{B}(X \times \mathrm{fix}(^*)^{\uparrow_\times}, Y \times \mathrm{fix}(^\square)^{\downarrow_\times}, I^\times)$, $\lceil A' \rceil_\vee^{\cap\cup\cap} = \lceil \lfloor \lceil A' \rceil_\vee^\cap \rfloor_\wedge \rceil_\wedge^{\cup\cap} = \lceil A'^{\uparrow_\times} \rceil_\wedge{}^{\cup\cap} = \lceil \lfloor \lceil A'^{\uparrow_\times} \rceil_\wedge^\cup \rfloor_\vee \rceil_\vee^\cap = \lceil A'^{\uparrow_\times \downarrow_\times} \rceil_\vee^\cap = \lceil A' \rceil_\vee^\cap$. Similarly $(\lceil B' \rceil_\wedge^{\cup\cap})^\cup = \lceil B' \rceil_\wedge^\cup$.
(b) For $\langle A_1, B_1 \rangle, \langle A_2, B_2 \rangle \in \mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$ we have $A_1 \subseteq A_2$ iff $\lfloor A_1 \rfloor_\vee \subseteq \lfloor A_2 \rfloor_\vee$ iff $\lfloor A_1 \rfloor_\vee * \subseteq \lfloor A_2 \rfloor_\vee *$.
(c) For $\langle A,B \rangle$ in $\mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$ we have

$$\lceil \lfloor A \rfloor_\vee^* \rceil_\vee^{\cap\cup} = \lceil \lfloor A^* \rfloor_\vee \rceil_\vee^{\cap\cup} = A^{*\cap\cup} = A.$$

For $\langle A', B'\rangle$ in $\mathcal{B}(X \times \text{fix}(*)^{\uparrow_\times}, Y \times \text{fix}(^{\square})^{\downarrow_\times}, I^{\times})$ we have $\lfloor\lceil A'\rceil^{\cap\cup}_{\vee}\rfloor^{*}_{\vee} = \lfloor\lceil\lfloor\lceil A'\rceil^{\cap}_{\vee}\rfloor^{\square}_{\wedge}\rceil^{\cup}_{\wedge}\rfloor^{*}_{\vee} = \lfloor\lceil A^{\uparrow_\times}\rceil^{\cup}_{\wedge}\rfloor^{*}_{\vee} = A'^{\uparrow_\times\downarrow_\times} = A'$ □

Theorem 18

1. *$\mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$ equipped with $\leqslant$, defined by $\langle A_1, B_1\rangle \leqslant \langle A_2, B_2\rangle$ iff $A_1 \subseteq A_2$, is a complete lattice where the infima and suprema are given by*

$$\bigwedge_{j\in J}\langle A_j, B_j\rangle = \left\langle \left(\bigcap_{j\in J} A_j\right)^{\cap\cup}, \left(\bigcap_{j\in J} B_j^{\square}\right)^{\cup\cap} \right\rangle, \tag{46}$$

$$\bigvee_{j\in J}\langle A_j, B_j\rangle = \left\langle \left(\bigcup_{j\in J} A_j^{*}\right)^{\cap\cup}, \left(\bigcup_{j\in J} B_j\right)^{\cup\cap} \right\rangle. \tag{47}$$

2. *Moreover, an arbitrary complete lattice $\mathbf{K} = \langle K, \leqslant\rangle$ is isomorphic to $\mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$ iff there are mappings μ: fix$(^*) \times X \to K$, ν: fix$(^{\square}) \times Y \to K$ such that*

   (a) *μ(fix $(^*) \times X)$ is $\bigvee$-dense in K, ν(fix $(^{\square}) \times Y)$ is $\bigwedge$-dense in K.*
   (b) *$\mu(a,x) \leqslant \nu(b,y)$ iff $I(x,y) \leqslant a \to b$.*

Proof. From Theorems 15 and 17. □

4. Reducing the size of concept lattices

The main idea of generalizing concept-forming operators $\langle^{\cap},^{\cup}\rangle$ by a truth-stressing hedge and a truth-depressing hedge is to gain control on the size of the resulting concept lattice. In the case of the original isotone concept-forming operators $\langle^{\Cap},^{\Cup}\rangle$, the number of formal fuzzy concepts can be inconveniently big. For instance in the example below, we obtain 207 formal fuzzy concepts from formal context with 6 objects and 4 attributes. Proper selection of the truth-stressing hedge and truth-depressing hedge decreases the number of formal fuzzy concepts in the resulting concept lattice as demonstrated in this section. We also provide a theoretical result about sizes of concept lattice.

Example 1. We demonstrate the influence of hedges by the following example. Consider the formal fuzzy context represented by Table 2. The table describes six books and their graded attributes. For the five-valued Łukasiewicz chain

$$\mathbf{L} = \langle\{0, 0.25, 0.5, 0.75, 1\}, \min, \max, \otimes, \to, 0, 1\rangle$$

as our structure of truth degrees, there are 40 combinations of truth-stressing hedge $*$ and truth-depressing hedge $^{\square}$ (5 possible choices of $*$ and 8 possible (independent) choices of $^{\square}$, see Figs. 1 and 2). For each combination of $*$ and $^{\square}$ we compute the corresponding concept lattice $\mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$. The concept lattices are depicted in Fig. 5. Note that the concept lattices $\mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$ are displayed in a standard manner by means of their line diagrams (Hasse diagrams).

One can notice that in Fig. 5 we get interesting alternating of big and small sizes of the concept lattices. For instance in the first column we have sizes 5, 20, 12, 25, 7, 25, 12, and 25. In the rest of this section we explain why this effect occurs.

Let $\mathbf{L}$ be a complete residuated lattice and TD($\mathbf{L}$) the set of all truth-depressing hedges and TS($\mathbf{L}$) set of all truth-stressing hedges.

Define partial order $\leqslant$ in TS($\mathbf{L}$) by

$$^{*_1} \leqslant {}^{*_2} \quad \text{iff } \text{fix}(^{*_1}) \subseteq \text{fix}(^{*_2}). \tag{48}$$

And define partial order $\leqslant$ in TD($\mathbf{L}$) by

$$^{\square_1} \leqslant {}^{\square_2} \quad \text{iff } \text{fix}(^{\square_1}) \subseteq \text{fix}(^{\square_2}). \tag{49}$$

Table 2
Context of books and their graded properties.

	High rating	Large no. of pages	Low price	Top sales rank
1	0.75	0.00	1.00	0.00
2	0.50	1.00	0.25	0.50
3	1.00	1.00	0.25	0.50
4	0.75	0.50	0.25	1.00
5	0.75	0.25	0.75	0.00
6	1.00	0.00	0.75	0.25

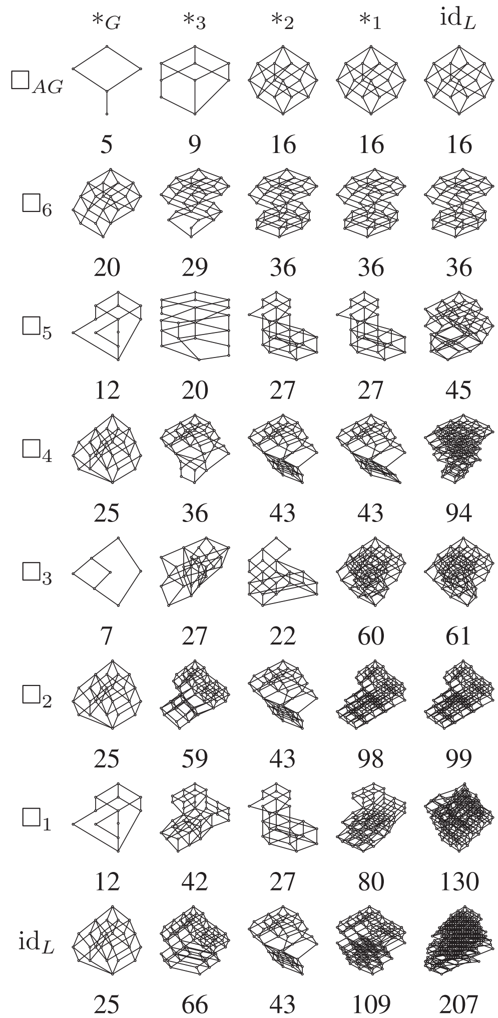


Fig. 5. Concept lattices $\mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$ induced by the context from Table 2 and numbers of their formal concepts. The picture shows concept lattices resulting by all combinations of truth-stressing hedge $*$ and truth-depressing hedge $^{\square}$ from Figs. 1 and 2.

Note that the truth-depressing hedges from Fig. 4 form a partially ordered set depicted in Fig. 4(left), and that the truth-stressing hedges from Fig. 1 form partially ordered set depicted in Fig. 4(right).

Theorem 19. *For a formal fuzzy context $\langle X, Y, I\rangle$, truth-depressing hedges $^{\square_a}, {}^{\square_b} \in \mathrm{TD}(\mathbf{L})$ s.t. $^{\square_a} \leqslant {}^{\square_b}$, and truth-stressing hedges $^{*_a}, {}^{*_b} \in TS(\mathbf{L})$ s.t. $^{*_a} \leqslant {}^{*_b}$ we have*

$$|\mathcal{B}(X^{*_a\cap}, Y^{\square_a\cup}, I)| \leqslant |\mathcal{B}(X^{*_b\cap}, Y^{\square_b\cup}, I)|. \tag{50}$$

Moreover,

$$\mathrm{Ext}(X^{\mathrm{id}_L\cap}, Y^{\square_a\cup}, I) \subseteq \mathrm{Ext}(X^{\mathrm{id}_L\cap}, Y^{\square_b\cup}, I), \tag{51}$$

$$\mathrm{Int}(X^{*_a\cap}, Y^{\mathrm{id}_L\cup}, I) \subseteq \mathrm{Int}(X^{*_b\cap}, Y^{\mathrm{id}_L\cup}, I). \tag{52}$$

Proof. Denote $\langle X'_1, Y'_1, I'_1\rangle := \langle X \times \mathrm{fix}({}^{*_a}), Y \times \mathrm{fix}({}^{\square_a}), I_1^{\times}\rangle$ and $\langle X'_2, Y'_2, I'_2\rangle := \langle X \times \mathrm{fix}({}^{*_b}), Y \times \mathrm{fix}({}^{\square_b}), I_2^{\times}\rangle$. Note that $\langle X'_1, Y'_1, I'_1\rangle$ is a subcontext of $\langle X'_2, Y'_2, I'_2\rangle$; i.e. $X'_1 \subseteq X'_2, Y'_1 \subseteq Y'_2$ and I'_1 is a restriction of I'_2 to X'_1, Y'_1: $I'_1 = I'_2 \cap X'_1 \times Y'_1$. The theorem follows from Theorem 17 and properties of subcontexts (see chapter 3 in [14]). □

Remark 4. Note that the second part of Theorem 19 does not generally hold for a truth-stresser $*$ different from identity. For instance, for the formal fuzzy context $\langle X, Y, I\rangle$ shown in Table 3, truth-stressing hedges ^{*_1}, truth-depressing hedges $^{\square_5} \leqslant {}^{\square_1}$ (see Fig. 4),we have: $\{^{0.75}/x_1, {}^{1.00}/x_2\} \in \mathrm{Ext}(X^{*_1\cap}, Y^{\square_5\cup}, I)$ but $\{^{0.75}/x_1, {}^{1.00}/x_2\} \notin \mathrm{Ext}(X^{*_1\cap}, Y^{\square_1\cup}, I)$.

Table 3
Formal fuzzy context from Remark 4.

	y_1	y_2
x_1	0.75	0.00
x_2	0.00	0.25

5. Conclusions and further issues

We have developed foundations of isotone Galois connections with a truth-stressing hedge and a truth-depressing hedge. We have explored basic calculus of such connections, i.e. on the properties analogous to those which are essential for the other type of Galois connections studied in the literature. We studied structure of $\mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$ and proved an analogy of the main theorem of concept lattices for our setting. Moreover, we compared our generalization with the approach studied in [1]. On an example, we showed how parameterization by hedges influences size of resulting concept lattice.

Further research will include the following topics:

- Optimal decomposition of matrices. We proved in our earlier work that fixed points of Galois connections can be used to find optimal decompositions of matrices with degrees [10]. Fixed points of isotone Galois connections serve for triangular decompositions of matrices with degrees. Usage of hedges introduces additional constraints.
- Study of attribute dependencies related to isotone Galois connections with hedges. This topic has been studied recently (see [11]).

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B A Calculus for Containment of Fuzzy Attributes

[16] Radim Belohlavek and Jan Konecny. A calculus for containment of fuzzy attributes. *Soft Computing*, pages 1–12, 2017.

In this paper, we examine the second main output of FCA in the graded setting with denial—attribute implications using semantics based on the graded denials.

Each expression of the form

$$A \Rightarrow B,$$

in which A and B are fuzzy sets of attributes (i.e. $A, B \in L^Y$) is called a *fuzzy attribute implication* (FAI) over Y. While FAIs are identical with the formulas in Section 2.9 as far as syntax is concerned, their semantics is different. While those in Section 2.9 are linked to graded affirmations the present ones are linked to graded denials. Their intended meaning is:

> "if all attributes of an object are contained in A then they are contained in B"

or, in terms of the graded denials,

> "if an object has attributes at most to degrees given by A then it has attributes at most to degrees given by B."

Similarly, as in Section 2.9 two natural options for the formalization of the semantics are possible—assuming the containment as bivalent or as graded. We provide general semantics which covers both these options as particular cases.

Let x denote an object and $M \in L^Y$ an $\mathbf{L}$-set representing the attributes of x, i.e. for each $y \in Y$ the degree to which object x has attribute y is M. We define the truth degree, denoted $\|A \Rightarrow B\|_M$, of $A \Rightarrow B$ in M, i.e. the truth degree to which $A \Rightarrow B$ is true for object x. For the bivalent containment, the fact that $A \Rightarrow B$ is fully true in M (in symbols $\|A \Rightarrow B\|_M = 1$) means

$$\text{if } M \subseteq A \text{ then } M \subseteq B. \tag{40}$$

For the graded containment, the fact that $A \Rightarrow B$ is fully true in M means

$$S(M, A) \leqslant S(M, B), \tag{41}$$

i.e. a degree of inclusion of M in A is less than or equal to the degree of inclusion of M in B, cf. (5). Analogously, as in Section 2.9, both the options can be obtained as particular cases of the following definition, in which the truth-stressing hedge * acts as a parameter. We define the *degree* $\|A \Rightarrow B\|_M \in L$ *to which* $A \Rightarrow B$ *is valid in* M as

$$\|A \Rightarrow B\|_M = S(M, A)^* \to S(M, B). \tag{42}$$

Among the main results established in the paper are: results regarding validity of dependencies, their models, and entailment; connections to existing dependencies for fuzzy as well as Boolean attributes, connections to interior- and closure-like structures, definition and properties of semantic entailment including an efficient check of entailment, various model-theoretical properties, a logical calculus of the dependencies inspired by the well-known Armstrong rules with its ordinary-style as well as graded-style syntactico-semantical completeness, basic results on bases, i.e. minimal fully informative sets of dependencies that are true in a given data.

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FOUNDATIONS

A calculus for containment of fuzzy attributes

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Abstract
Dependencies in data describing objects and their attributes represent a key topic in understanding relational data. In this paper, we examine certain dependencies of data described by fuzzy attributes such as *green* or *high performance*, i.e. attributes which apply to objects to certain degrees. Such attributes subsume Boolean attributes as a particular case. We utilize the framework of residuated structures of truth degrees as developed in modern fuzzy logic and examine several fundamental problems for our dependencies. These include connections to existing dependencies for fuzzy as well as Boolean attributes, connections to interior- and closure-like structures, definition and properties of semantic entailment including an efficient check of entailment, various model-theoretical properties, a logical calculus of the dependencies inspired by the well-known Armstrong rules with its ordinary-style as well as graded-style syntactico-semantical completeness, fully informative sets of all dependencies that are valid in given data including a constructive description of minimal such sets, as well as various other problems.

Keywords Fuzzy logic · Dependencies of fuzzy attributes · Fuzzy closure structures · Formal concept analysis

1 Introduction

1.1 Problem setting

We assume that the dependencies we study pertain to data in the form of a table with rows and columns corresponding to objects x in a set X and their logical attributes y in a set Y, respectively. While in the classic, Boolean case, every attribute y either applies or does not apply to any given object x, we assume a more general setting in which the attributes are fuzzy. That is, with every attribute y and object x, there is associated a degree to which y applies to x. We furthermore assume that these truth degrees form a partially ordered set L bounded by 0 and 1 (representing falsity and truth, respectively) which is, moreover, equipped with truth functions of logical connectives such as conjunction and implications, as detailed below. The classical case then becomes a particular case in which the only members of L are 0 and 1 and in which the truth functions are the truth functions of Boolean logical connectives.

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In our paper, we examine certain dependencies which concern containment of attributes. In particular, we introduce basic syntactic and semantic notions which are inspired by two basic meanings of containment of fuzzy attributes, namely binary and graded containment (Sect. 2), explore connections to interior-like structures and outline ramifications of these connections (Sect. 3), develop a logic for our dependencies with two kinds of completeness (Sect. 4), and provide results regarding minimal fully informative sets of if–then rules (Sect. 5).

1.2 Preliminaries

As the above-mentioned scales of truth degrees, we use complete residuated lattices. Since these are well known (Belohlavek 2012; Goguen 1969; Hájek 1998), we restrict to recalling basic facts. A *complete residuated lattice* with a truth-stressing hedge (shortly, a hedge) is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, {}^*, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid; $\otimes$ and $\rightarrow$ satisfy so-called adjointness property:

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \tag{1}$$

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for each $a, b, c \in L$; hedge * satisfies

$$1^* = 1, \quad a^* \leq a, \quad (a \to b)^* \leq a^* \to b^*, \quad a^{**} = a^*, \tag{2}$$

for each $a, b \in L$, $a_i \in L$ $(i \in I)$. Elements a of L are called truth degrees. $\otimes$ and $\to$ are (truth functions of) many-valued conjunction and implication. Hedge * may be seen as a (truth function of) logical connective "very true." Properties (2) have natural interpretations, e.g. second one can be read: "if a is very true, then a is true," the third one as: "if $a \to b$ is very true and if a is very true, then b is very true." Note that other properties of hedges are sometimes imposed, see, e.g. (Hájek 1998).

A common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\to$; or with L being a finite chain with appropriate operations. Two boundary cases of (truth-stressing) hedges are (i) identity, i.e. $a^* = a$ $(a \in L)$, and (ii) globalization: $a^* = 1$ for $a = 1$ and $a^* = 0$ for $a < 1$. An important special case of a complete residuated lattice with hedge is the two-element Boolean algebra $\langle \{0, 1\}, \wedge, \vee, \otimes, \to, ^*, 0, 1\rangle$, denoted by $\mathbf{2}$, which is the structure of truth degrees of the classical logic. That is, the operations $\wedge, \vee, \otimes, \to$ of $\mathbf{2}$ are the truth functions (interpretations) of the corresponding logical connectives of the classical logic and $0^* = 0$, $1^* = 1$.

We exploit the usual notions and notation: an $\mathbf{L}$-set (fuzzy set) A in universe U is a mapping $A : U \to L$, $A(u)$ being interpreted as "the degree to which u belongs to A." The collection of all $\mathbf{L}$-sets in U is denoted by $\mathbf{L}^U$. The operations with $\mathbf{L}$-sets are defined componentwise. Binary $\mathbf{L}$-relations between X and Y can be thought of as $\mathbf{L}$-sets in the universe $X \times Y$. That is, a binary $\mathbf{L}$-relation $I \in \mathbf{L}^{X \times Y}$ is a mapping assigning to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$ (a degree to which x and y are related by I). An $\mathbf{L}$-set $A \in \mathbf{L}^X$ is called crisp if $A(x) \in \{0, 1\}$ for each $x \in X$. An $\mathbf{L}$-set $A \in \mathbf{L}^X$ is called empty (denoted by $\emptyset$) if $A(x) = 0$ for each $x \in X$. For $a \in L$ and $A \in \mathbf{L}^X$, $a \otimes A \in \mathbf{L}^X$ and $a \to A \in \mathbf{L}^X$ are defined by

$$(a \otimes A)(x) = a \otimes A(x) \text{ and } (a \to A)(x) = a \to A(x).$$

Given $A, B \in \mathbf{L}^U$, we define a subsethood degree

$$S(A, B) = \textstyle\bigwedge_{u \in U} \big(A(u) \to B(u)\big), \tag{3}$$

which generalizes the ordinary subsethood relation $\subseteq$. $S(A, B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A, B) = 1$. As a consequence, $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$. Throughout the rest of the paper, $\mathbf{L}$ denotes an arbitrary complete residuated lattice with a hedge.

In what follows, we utilize the following inequality which is easy to prove using the definitions.

Lemma 1 *For $A, B \in \mathbf{L}^U$, we have*

$$S(A, B)^* \leq S(A^*, B^*). \tag{4}$$

1.3 Previous work

Explorations of various kinds of dependencies among Boolean attributes, developments of various logical calculi describing such dependencies as well as various computational problems related to these dependencies represent fundamental issues in data management and have been thoroughly studied in the past.

Most important among such dependencies are various kinds of if–then rules. These rules, which basically describe that when certain attributes are present then certain other attributes are present as well, are thoroughly examined along with their connections to functional dependencies in the classic book (Maier 1983). They have further been explored for data analytic purposes, and their algorithmic properties are examined in the influential paper (Guigues and Duquenne 1986) and in (Ganter and Wille 1999), which works pay a particular attention to extraction of a smallest fully informative set of dependencies from Boolean data. Taking into account almost valid if–then rules leads to association rules and considerations regarding confidence and support of such rules. Related explorations represent a major research stream in data mining, and we refer to (Zhang and Zhang 2002) for an overview and to (Rauch 2005) for related logical calculi of these rules as well as calculi of much more general dependencies in Boolean data, namely those of (Hájek and Havránek 2012) which subsume association rules as a very particular case.

Directly connected to the topic of our paper are recent explorations of dependencies in data with fuzzy rather than Boolean attributes, i.e. attributes such as for graded attributes, such as *green* or *high performance*. These have been codeveloped by one of the present authors in a series of papers, see (Belohlavek and Vychodil 2016, 2017) for a comprehensive treatment and (Belohlavek and Vychodil 2006) for early explorations. In this paper, we present a logic of if–then rules $A \Rightarrow B$ for graded attributes whose basic meaning is: if all attributes of an object are contained A then they are contained in B. These rules have the same syntactic form as those in (Belohlavek and Vychodil 2016, 2017), but have a different semantics: they represent restrictions on what attributes may be possessed by objects. The technical difference from the rules in (Belohlavek and Vychodil 2016, 2017) consists in the fact that the new rules are closely related to interior-like structures, while the former rules are related to closure-like structures. In the Boolean case, the two kinds of dependencies are mutually reducible: The reducibility derives from the fact that interior- and closure-like structures are mutually reducible in the Boolean case. In the setting of fuzzy attributes, such reducibility is not available, as is well known, due to the lack of the law of double negation. Consequently, our new kind of dependencies needs to be carefully explored anew. In a broader perspective, the new rules manifest the variety of structures naturally associated with object-attribute data, which has also been examined, e.g. in (Belohlavek and Konecny 2012; Ciucci et al. 2014; Georgescu and Popescu 2004; Konecny 2011), and further contribute to understanding these structures.

2 Syntax and semantics

Suppose $\mathbf{L}$ is a complete residuated lattice with a hedge (i.e. a scale of grades equipped with logical operations) and Y be a set of (symbols of) fuzzy attributes. Each expression of the form

$$A \Rightarrow B,$$

in which A and B are fuzzy sets of attributes (i.e. $A, B \in \mathbf{L}^Y$) is called a *fuzzy attribute implication* (FAI) over Y; FAIs are our basic formulas. While they are identical with the formulas in (Belohlavek and Vychodil 2016, 2017) as far syntax is concerned, their semantics is different. Put verbally, the intended meaning is:

> If all attributes of an object are contained in A then they are contained in B.

Since in a fuzzy setting, whether an object has an attribute is a matter of degree, validity of our formulas is a matter of degree as well. In the semantics described, one needs to be careful about the meaning of containment since there are two natural options possible—taking containment as bivalent or graded. We provide a general semantics which covers both these options as particular cases.

Let x denote an object and $M \in \mathbf{L}^Y$ a fuzzy set representing the attributes of x, i.e. for each $y \in Y$ the degree to which object x has attribute y is M. Our aim is to define the truth degree, denoted $||A \Rightarrow B||_M$, of $A \Rightarrow B$ in M, i.e. the truth degree to which $A \Rightarrow B$ is true for object x. As mentioned above, we provide a general definition which subsumes two particular cases, one for bivalent and one for graded containment. For bivalent containment, the fact that $A \Rightarrow B$ is fully true in M (in symbols $||A \Rightarrow B||_M = 1$) means:

$$\text{if} \quad M \subseteq A \quad \text{then} \quad M \subseteq B, \tag{5}$$

where $M \subseteq A$ denotes full containment, i.e. $M(y) \leq A(y)$ for all $y \in Y$. For a graded containment, the fact that $A \Rightarrow B$ is fully true in M means:

$$S(M, A) \leq S(M, B), \tag{6}$$

i.e. a degree of inclusion of M in A is less than or equal to the degree of inclusion of M in B, cf. (3). Now, both approaches can be obtained as particular cases of the following definition, in which the hedge * acts as a parameter (see below):

Definition 1 For a fuzzy attribute implication $A \Rightarrow B$ and a fuzzy set M of attributes (of some object x), we define the *degree* $||A \Rightarrow B||_M \in L$ *to which* $A \Rightarrow B$ *is valid in* M as follows:

$$||A \Rightarrow B||_M = S(M, A)^* \to S(M, B). \tag{7}$$

One easily verifies that if * is globalization and identity, respectively, (7) meets the above cases corresponding to bivalent and graded inclusion, (5) and (6), respectively. Let us emphasize, however, that the degree of validity $||A \Rightarrow B||_M$ is a general truth degree in L, i.e. it need not be equal to 0 or 1.

Clearly, from the perspective of the current literature on structures related to fuzzy attributes, our formulas $A \Rightarrow B$ may be interpreted in tables with fuzzy attributes. Note that each such table may be identified with a triplet $\langle X, Y, I\rangle$, in which X and Y are sets of objects (table rows) and attributes (table columns), and I is a fuzzy relation between X and Y for which $I(x, y)$ is interpreted as the degree to which the object x has the attribute y. The corresponding definitions needed are as follows

Definition 2 For a collection $\mathcal{M}$ of fuzzy sets M of attributes in Y, we define the degree to which $A \Rightarrow B$ is valid in $\mathcal{M}$ as follows:

$$||A \Rightarrow B||_{\mathcal{M}} = \textstyle\bigwedge_{M \in \mathcal{M}} ||A \Rightarrow B||_M. \tag{8}$$

For a table $\langle X, Y, I\rangle$ with fuzzy attributes, we define the degree to which $A \Rightarrow B$ is valid in $\langle X, Y, I\rangle$ by

$$||A \Rightarrow B||_{\langle X,Y,I\rangle} = ||A \Rightarrow B||_{\{I_x | x \in X\}}, \tag{9}$$

where I_x denotes the fuzzy set representing the row corresponding to object x, i.e. $I_x(y) = I(x, y)$ for each $y \in Y$.

Remark 1 (*alternative to failure dependencies*) The dependencies just introduced represent a natural alternative to failure dependencies in the theory of knowledge spaces (Doignon and Falmagne 2012) in the setting in which mastering pieces of knowledge is a matter of degree (Bartl and Belohlavek 2011). Recall that in ordinary knowledge spaces, one deals with subsets of items in a set Q (representing questions in some area, for instance). If M is a subset of Q representing a given person's state of knowledge (a set of questions the person is capable of answering correctly), a failure dependency $A \Rightarrow B$, where $A, B \subseteq Q$, holds for M if

$$A \cap M = \emptyset \text{ implies } B \cap M = \emptyset,$$

i.e. failure in answering all questions in A implies failure in answering all questions in B. This notion has been extended in (Bartl and Belohlavek 2011) to a graded setting in which M, A, and B are allowed to be fuzzy sets in Q in which the membership degrees model situations in which mastering of or failure on a particular question is a matter of degree. In particular, the degree to which $A \Rightarrow B$ is true in M has been defined in (Bartl and Belohlavek 2011) by

$$S(A \otimes M, \emptyset)^* \to S(B \otimes M, \emptyset);$$

here, $(A \otimes M)(q) = A(q) \otimes M(q)$ is the $\otimes$-based intersection of fuzzy sets A and M. This definition corresponds directly to the above definition from the ordinary case, but it has the disadvantage that it involves the many-valued negation $\neg a = a \to 0$ associated with the underlying structure $\mathbf{L}$ of truth degrees. This property is disadvantageous because such negation lacks certain properties of ordinary negation (e.g. the law of double negation) due to which fact only certain properties from the ordinary setting carry over to the fuzzy setting.

The dependencies we study in the present paper, nevertheless, offer another way to capture failure dependencies. Namely, it is easily seen that in the ordinary setting a failure dependency $C \Rightarrow D$ is true in a state $M \subseteq Q$ iff the following holds true: if $M \subseteq \overline{C}$ then $M \subseteq \overline{D}$. Denoting $A = \overline{C}$ and $B = \overline{D}$, this may be rewritten as: if $M \subseteq A$ then $M \subseteq B$. While C and D represent failures on questions in failure dependencies, A and B in the new kind of dependencies (which is obviously the kind studied in this paper) represent mastering of questions. Namely, the meaning of this new kind of dependency is described as: if all questions the individual has mastered are in A, then all questions he has mastered are in B as well. Hence, the new type of dependencies may aptly be called *mastering dependencies*.

A direct generalization of mastering dependencies to a fuzzy setting clearly yields the dependencies whose semantics is defined by Definition 1. While failure and mastering dependencies are equivalent in the ordinary setting (due to the law of double negation, a mastering dependency $A \Rightarrow B$ is equivalent to the failure dependency $\overline{A} \Rightarrow \overline{B}$), they are no longer equivalent in a fuzzy setting (clearly, they are equivalent if the fuzzy logic connective of negation involved satisfies the law of double negation, such as the Łukasiewicz negation for instance). In a fuzzy setting, they both describe the same type of dependency, but technically, mastering dependencies, as formalized by Definition 1, are more convenient because they do not involve the possibly problematic logical connective of negation. □

Our semantics of fuzzy attribute implications is closely connected to particular Galois-like connections and their fixpoints. These Galois-like connections have been introduced in (Georgescu and Popescu 2004), see also (Konecny 2011): For $\langle X, Y, I\rangle$ as above, consider the operators $^{\cap}: \mathbf{L}^X \to \mathbf{L}^Y$ and $^{\cup}: \mathbf{L}^Y \to \mathbf{L}^X$ defined by

$$A^{\cap}(y) = \bigvee_{x\in X}(A(x)^* \otimes I(x, y)), \tag{10}$$

$$B^{\cup}(x) = \bigwedge_{y\in Y}(I(x, y) \to B(y)), \tag{11}$$

for any fuzzy set A in X and B in Y. Let furthermore $\mathcal{B}(X^{\cap}, Y^{\cup}, I) = \{\langle A, B\rangle \mid A^{\cap} = B, B^{\cup} = A\}$ denote the lattice of the fixpoints of $^{\cap}$ and $^{\cup}$ and $\mathrm{Int}(X^{\cap}, Y^{\cup}, I) = \{B \mid \langle A, B\rangle \in \mathcal{B}(X^{\cap}, Y^{\cup}, I)$ for some $A\}$ the corresponding system of intents (i.e. of the second components of the fixpoints). We also use just $\mathrm{Int}(I)$ instead of $\mathrm{Int}(X^{\cap}, Y^{\cup}, I)$. The following theorem reveals an important relationship: the validity of our attribute dependencies $A \Rightarrow B$ in a table $\langle X, Y, I\rangle$ coincides with the validity in the intents of $\langle X, Y, I\rangle$ and also with the degree to which $A^{\cup\cap}$ is contained in B. This theorem is utilized below in a characterization of complete sets of fuzzy attribute implications.

Theorem 1 *Given a table $\langle X, Y, I\rangle$ with fuzzy attributes and a fuzzy attribute implication $A \Rightarrow B$ over Y, we have*

$$||A \Rightarrow B||_{\langle X,Y,I\rangle} = ||A \Rightarrow B||_{\mathrm{Int}(I)} \quad = S(A^{\cup\cap}, B). \tag{12}$$

Proof First, we check $||A \Rightarrow B||_{\langle X,Y,I\rangle} = ||A \Rightarrow B||_{\mathrm{Int}(I)}$. Observe, that $||A \Rightarrow B||_{\langle X,Y,I\rangle} \leq ||A \Rightarrow B||_{\mathrm{Int}(I)}$ iff for each $M \in \mathrm{Int}(I)$ we have

$$||A \Rightarrow B||_{\langle X,Y,I\rangle} \leq ||A \Rightarrow B||_M,$$

i.e.

$$\bigwedge_{x\in X}(S(I_x, A)^* \to S(I_x, B)) \leq S(M, A)^* \to S(M, B).$$

As $I_x(y) = I(x, y)$, we have

$$S(I_x, A) = \bigwedge_{y\in Y} I_x(y) \to A(y) = A^{\cup}(x).$$

Therefore, the last inequality is equivalent to

$$\bigwedge_{x\in X}(A^{\cup}(x)^* \to B^{\cup}(x)) \leq S(M, A)^* \to S(M, B),$$

i.e. to

$$S(A^{\cup}, B^{\cup}) = \bigwedge_{x\in X}(A^{\cup}(x)^* \to B^{\cup}(x)) \\ \leq S(M, A)^* \to S(M, B),$$

which is equivalent to

$$S(M, A)^* \otimes S(A^{\Cup}, B^{\Cup}) \leq S(M, B) \tag{13}$$

due to adjointness of $\otimes$ and $\rightarrow$. Thus, it suffices to prove (13) for each $M \in \text{Int}(I)$. For this purpose, consider the operator $^{\Cap}$, the "unhedged" version of $^{\cap}$ defined by

$$A^{\Cap}(y) = \bigvee_{x \in X}(A(x) \otimes I(x, y)).$$

The pair $\langle ^{\Cap}, ^{\Cup} \rangle$ forms an isotone **L**-Galois connection and hence satisfies $S(C_1, C_2) \leq S(C_1^{\Cap}, C_2^{\Cap})$, $S(D_1, D_2) \leq S(D_1^{\Cup}, D_2^{\Cup})$, and $D^{\Cup\Cap} \subseteq D$. Due to the fact that $M = M^{\Cup\cap}$ and since $S(C, D) \otimes S(D, E) \leq S(C, E)$, we obtain

$$\begin{aligned} S(M, A)^* \otimes S(A^{\Cup}, B^{\Cup}) &\leq S(M^{\Cup}, A^{\Cup})^* \otimes S(A^{\Cup}, B^{\Cup}) \\ &\leq S(M^{\Cup *}, A^{\Cup *}) \otimes S(A^{\Cup}, B^{\Cup}) \\ &\leq S(M^{\Cup *}, B^{\Cup}) \\ &\leq S(M^{\Cup * \Cap}, B^{\Cup\Cap}) \\ &= S(M^{\Cup\cap}, B^{\Cup\Cap}) = S(M, B^{\Cup\Cap}) \leq S(M, B), \end{aligned}$$

verifying (13) and thus $||A \Rightarrow B||_{\langle X,Y,I \rangle} \leq ||A \Rightarrow B||_{\text{Int}(I)}$. To check $||A \Rightarrow B||_{\langle X,Y,I \rangle} \geq ||A \Rightarrow B||_{\text{Int}(I)}$, it is sufficient to observe that for each $x \in X$, $I_x \in \text{Int}(I)$.

Second, we check $||A \Rightarrow B||_{\langle X,Y,I \rangle} = S(A^{\Cup\cap}, B)$. We have

$$\begin{aligned} ||A \Rightarrow B||_{\langle X,Y,I \rangle} &= ||A \Rightarrow B||_{\{I_x | x \in X\}} \\ &= \bigwedge_{x \in X} (S(I_x, A)^* \rightarrow S(I_x, B)) \\ &= \bigwedge_{x \in X} (A^{\Cup}(x)^* \rightarrow B^{\Cup}(x)) \\ &= \bigwedge_{x \in X} (A^{\Cup *}(x) \rightarrow B^{\Cup}(x)) \\ &= S(A^{\Cup *}, B^{\Cup}) \\ &= S(A^{\Cup * \cap}, B) \\ &= S(A^{\Cup\cap}, B), \end{aligned}$$

proving the claim. ◻

Having defined validity, we now consider theories of our fuzzy attribute implications and models of these theories. Recall that according to a seminal work of Pavelka (Pavelka 1979a, b, c), a theory in a fuzzy setting naturally consists of formulas to which degrees of truth are attached, i.e. a theory is a fuzzy set of formulas; see also (Gerla 2001; Hájek 1998). Therefore, we define a *theory* to be a fuzzy set T of fuzzy attribute implications. We furthermore say that a theory is *crisp* if T is crisp as a fuzzy set, in which case we write $A \Rightarrow B \in T$ if $T(A \Rightarrow B) = 1$ and $A \Rightarrow B \notin T$ if $T(A \Rightarrow B) = 0$.

Note that the degree to which an implication $A \Rightarrow B$ is a member of T, i.e. the degree $T(A \Rightarrow B)$, may naturally be interpreted as the degree to which the validity of $A \Rightarrow B$ is assumed. In addition, T may alternatively be regarded as a fuzzy set of dependencies extracted from data, in which case $T(A \Rightarrow B)$ is interpreted as the degree to which $A \Rightarrow B$ is valid in the data.

The set $\text{Mod}(T)$ of all *models* of a given theory T is then defined as

$$\begin{aligned} \text{Mod}(T) = \{M \in \mathbf{L}^Y \mid \text{for each } A, B \in \mathbf{L}^Y : \\ T(A \Rightarrow B) \leq ||A \Rightarrow B||_M\}. \end{aligned}$$

Observe that according to this definition, M is a model of T if for every implication $A \Rightarrow B$ it holds that the degree to which $A \Rightarrow B$ is valid in M is greater than or equal than the degree $T(A \Rightarrow B)$ that the theory "prescribes" for $A \Rightarrow B$. Clearly, if T is crisp then $\text{Mod}(T) = \{M \in \mathbf{L}^Y \mid \text{ for each } A \Rightarrow B \in T : ||A \Rightarrow B||_M = 1\}$.

The degree to which a given implication $A \Rightarrow B$ *semantically follows* from a theory T of implications is then naturally defined by

$$||A \Rightarrow B||_T = \bigwedge_{M \in \text{Mod}(T)} ||A \Rightarrow B||_M.$$

Interestingly, the general concept of degree of validity of fuzzy attribute implications may be reduced to the seemingly narrower, particular concept of full validity (i.e. validity to degree 1):

Lemma 2 *For a fuzzy attribute implication $A \Rightarrow B$, a fuzzy set M of attributes, and a truth degree $c \in L$ we have*

$$c \leq ||A \Rightarrow B||_M \textit{ iff } ||A \Rightarrow c \rightarrow B||_M = 1.$$

Proof Using $\alpha \rightarrow (\beta \rightarrow \gamma) = \beta \rightarrow (\alpha \rightarrow \gamma)$ and $\alpha \rightarrow \bigwedge_k \beta_k = \bigwedge_k(\alpha \rightarrow \beta_k)$, one easily obtains $||A \Rightarrow (c \rightarrow B)||_M = c \rightarrow ||A \Rightarrow B||_M$. The claim then follows from the fact that $\alpha \rightarrow \beta = 1$ iff $\alpha \leq \beta$. ◻

We may now, in a sense, reduce the concepts of model and entailment for general theories (i.e. theories which involve truth degrees) to the concepts of model and entailment for crisp theories:

Lemma 3 *For a theory T of fuzzy attribute implications, denote by $\text{cr}(T)$ a crisp theory as follows:*

$$\begin{aligned} \text{cr}(T) = \{A \Rightarrow T(A \Rightarrow B) \rightarrow B \mid A, B \in \mathbf{L}^Y \\ \textit{and } T(A \Rightarrow B) \rightarrow B \neq Y\}. \end{aligned} \tag{14}$$

Then,

$$\text{Mod}(T) = \text{Mod}(\text{cr}(T)), \textit{ and} \tag{15}$$

$$||A \Rightarrow B||_T = ||A \Rightarrow B||_{\mathrm{cr}(T)} \tag{16}$$

for any fuzzy attribute implication $A \Rightarrow B$.

Proof The equality in (15) is a direct consequence of Lemma 2. The equality in (16) follows by definition from (15). □

Interestingly, one may now reduce the concept of general degree of entailment from a theory to that of bivalent (i.e. to degree 1) entailment from a crisp theory:

Lemma 4 *For a fuzzy attribute implication* $A \Rightarrow B$ *and a theory* T *of implications, we have*

$$||A \Rightarrow B||_T = \bigvee\{c \in L \mid ||A \Rightarrow c \to B||_T = 1\},$$
$$||A \Rightarrow B||_T = \bigvee\{c \in L \mid ||A \Rightarrow c \to B||_{\mathrm{cr}(T)} = 1\}.$$

Proof Using Lemma 2, we have

$$\begin{aligned}||A \Rightarrow B||_T &= \textstyle\bigwedge_{M \in \mathrm{Mod}(T)} ||A \Rightarrow B||_M \\ &= \bigvee\{c \in L \mid c \leq ||A \Rightarrow B||_M \text{ for each } M \in \mathrm{Mod}(T)\} \\ &= \bigvee\{c \in L \mid ||A \Rightarrow c \to B||_T = 1\},\end{aligned}$$

establishing the first equality. The second one is a direct consequence of the first and of (16). □

Lemma 4 is conveniently used when later proving graded completeness theorem for our logic.

3 Models and their connection to interior-like structures

We now explore the structure of models of theories and establish important connections to interior-like structures. Let us recall from (Belohlavek et al. 2005) that an $\mathbf{L}^*$-interior operator on a set Y is a mapping $I : \mathbf{L}^Y \to \mathbf{L}^Y$ which satisfies

$$I(A) \subseteq A, \tag{17}$$
$$S(A_1, A_2)^* \leq S(I(A_1), I(A_2)), \tag{18}$$
$$I(A) = I(I(A)), \tag{19}$$

for every $A, A_1, A_2 \in \mathbf{L}^Y$. Note that when $L = \{0, 1\}$ (two-valued Boolean case) then the concept of an $\mathbf{L}^*$-interior operator coincides (modulo identifying crisp fuzzy sets with ordinary sets) with the ordinary concept of interior operator.

Let us further recall (Belohlavek et al. 2005) that an $\mathbf{L}^*$-interior system on Y is a set $\mathcal{S} \subseteq \mathbf{L}^Y$ of fuzzy sets in Y which is closed under unions of fuzzy sets and so-called a^*-multiplications of fuzzy sets; that is to say, $\mathcal{S}$ is required to satisfy:

$$\text{if } A_j \in \mathcal{S} \text{ for every } j \in J \text{ then } \textstyle\bigcup_{j \in J} A_j \in \mathcal{S}, \tag{20}$$
$$\text{if } A \in \mathcal{S} \text{ and } a \in L \text{ then } a^* \otimes A \in \mathcal{S}. \tag{21}$$

Now again, for $L = \{0, 1\}$, $\mathbf{L}^*$-interior systems coincide with ordinary interior systems.

Let now for an $\mathbf{L}^*$-interior system $\mathcal{S}$ on Y and an $\mathbf{L}^*$-interior operator I on Y define the mapping $I_{\mathcal{S}}$ and the system $\mathcal{S}_I$ by

$$I_{\mathcal{S}}(B) = \bigcup_{i \in I} (S(A_i, B)^* \otimes A_i)$$

and

$$\mathcal{S}_I = \{A \in \mathbf{L}^Y \mid A = I(A)\}.$$

Then, as proved in (Belohlavek et al. 2005), $I_{\mathcal{S}}$ is an $\mathbf{L}^*$-interior operator on Y and $\mathcal{S}_I$ is an $\mathbf{L}^*$-interior system on Y; furthermore, the mappings sending $\mathcal{S}$ to $I_{\mathcal{S}}$ and I to $\mathcal{S}_I$ are bijective and are mutually inverse.

Interestingly, as the following two theorems show, models of theories of our formulas are just the $\mathbf{L}^*$-interior systems on Y.

Theorem 2 *Let* T *be a theory of fuzzy attribute implications over* Y*. Then, the system* $\mathrm{Mod}(T)$ *of all models of* T *is an* $\mathbf{L}^*$*-interior system on* Y.

Proof According to the definition, we need to verify that $\mathrm{Mod}(T)$ satisfies conditions (20) and (21). By virtue of Lemma 3, we may safely suppose that T is crisp.

For (20): If $M_j \in \mathrm{Mod}(T)$ for $j \in J$, then $||A \Rightarrow B||_{M_j} = 1$, i.e. $S(M_j, A)^* \leq S(M_j, B)$, for any $A \Rightarrow B \in T$. Now, since $(\bigwedge_{j \in J} a_j)^* \leq \bigwedge_{j \in J} a_j^*$, we get

$$\begin{aligned}S(\textstyle\bigcup_{j \in J} M_j, A)^* &= (\textstyle\bigwedge_{j \in J} S(M_j, A))^* \\ &\leq \textstyle\bigwedge_{j \in J} S(M_j, A)^* \\ &\leq \textstyle\bigwedge_{j \in J} S(M_j, B) = S(\textstyle\bigcup_{j \in J} M_j, B),\end{aligned}$$

proving $||A \Rightarrow B||_{\bigcup_{j \in J} M_j} = 1$, hence $\bigcup_{j \in J} M_j \in \mathrm{Mod}(T)$.

For (21): If $M \in \mathrm{Mod}(T)$, then for each $A \Rightarrow B \in T$ we have $||A \Rightarrow B||_M = 1$, i.e. $S(M_j, A)^* \leq S(M_j, B)$. For each $a \in L$, we thus have

$$\begin{aligned}S(a^* \otimes M, A)^* &= (a^* \to S(M, A))^* \leq a^{**} \to S(M, A)^* \\ &= a^* \to S(M, A)^* \leq a^* \to S(M, B) = S(a^* \otimes M, B),\end{aligned}$$

establishing $||A \Rightarrow B||_{a^* \otimes M} = 1$, whence $a^* \otimes M \in \mathrm{Mod}(T)$. □

Theorem 3 *For every* $\mathbf{L}^*$*-interior system* $\mathcal{S}$ *on* Y*, there is a theory* T *of fuzzy attribute implications over* Y *whose models are just the elements of* $\mathcal{S}$*, i.e. for which* $\mathcal{S} = \mathrm{Mod}(T)$.

Proof We verify that the theory $T = \{A \Rightarrow I_{\mathcal{S}}(A) \mid A \in \mathbf{L}^Y\}$ has the required property. Take any $M \in \mathcal{S}$. Then, since $I_{\mathcal{S}}$ is the corresponding operator, $M = I_{\mathcal{S}}(M)$. According to (18), we obtain

$$S(M, A)^* \leq S(I_{\mathcal{S}}(M), I_{\mathcal{S}}(A)) = S(M, I_{\mathcal{S}}(A)).$$

This way, we proved that $||A \Rightarrow I_{\mathcal{S}}(A)||_M = 1$, hence $M \in \mathrm{Mod}(T)$. This first inclusion, $\mathcal{S} \subseteq \mathrm{Mod}(T)$, is therefore established.

We establish the second inclusion by showing that if $M \notin \mathcal{S}$ then $M \notin \mathrm{Mod}(T)$. Take any $M \notin \mathcal{S}$. Then, clearly, $M \neq I_{\mathcal{S}}(M)$. Since $I_{\mathcal{S}}$ is an interior operator, we have $I_{\mathcal{S}}(M) \subset M$, cf. (17). Consequently, $S(M, I_{\mathcal{S}}(M)) \neq 1$ due to the definition of graded inclusion S. Since

$$||M \Rightarrow I_{\mathcal{S}}(M)||_M = S(M, M)^* \to S(M, I_{\mathcal{S}}(M))$$

and since $S(M, M)^* = 1^* = 1$, we obtain

$$\begin{aligned} ||M \Rightarrow I_{\mathcal{S}}(M)||_M &= 1 \to S(M, I_{\mathcal{S}}(M)) \\ &= S(M, I_{\mathcal{S}}(M)) \neq 1. \end{aligned}$$

This means that M is not a model of T, i.e. $M \notin \mathrm{Mod}(T)$, finishing the proof. □

Since $\mathrm{Mod}(T)$ is an $\mathbf{L}^*$-interior system on Y, we may consider for every $A \in \mathbf{L}^Y$ the largest model in $\mathrm{Mod}(T)$ covered by A. This largest model, which is clearly $I_{\mathrm{Mod}(T)}(A)$, has a very important property. Namely, as shown in the next theorem, the degree $||A \Rightarrow B||_T$ of entailment of any $A \Rightarrow B$ from T equals the degree to which $A \Rightarrow B$ is valid in this single model $I_{\mathrm{Mod}(T)}(A)$, as well as to the degree to which this model is included in B:

Theorem 4 *Let $A \Rightarrow B$ be an arbitrary fuzzy attribute implication, and let T be any theory of implications. Then,*

the degree of entailment $||A \Rightarrow B||_T$ equals

the degree of validity $||A \Rightarrow B||_{I_{\mathrm{Mod}(T)}(A)}$ equals

the degree of inclusion $S(I_{\mathrm{Mod}(T)}(A), B)$.

Proof Since $I_{\mathrm{Mod}(T)}(A)$ is a model of T, the definition of semantic entailment yields $||A \Rightarrow B||_T \leq ||A \Rightarrow B||_{I_{\mathrm{Mod}(T)}(A)}$. Due to property (17), which is obeyed by $I_{\mathrm{Mod}(T)}$, and since $1^* = 1$, we get $S(I_{\mathrm{Mod}(T)}(A), A)^* = 1$. Now, since by definition,

$$\begin{aligned} &||A \Rightarrow B||_{I_{\mathrm{Mod}(T)}(A)} \\ &\quad = S(I_{\mathrm{Mod}(T)}(A), A)^* \to S(I_{\mathrm{Mod}(T)}(A), B), \end{aligned}$$

and since $1 \to a = a$, we easily obtain

$$||A \Rightarrow B||_{I_{\mathrm{Mod}(T)}(A)} = S(I_{\mathrm{Mod}(T)}(A), B).$$

Consider now any model M of T. As $I_{\mathrm{Mod}(T)}$ is the interior operator corresponding to T, we have $M = I_{\mathrm{Mod}(T)}(M)$. Applying (18) and $M = I_{\mathrm{Mod}(T)}(M)$, we obtain

$$\begin{aligned} &S(M, A)^* \otimes S(I_{\mathrm{Mod}(T)}(A), B) \\ &\quad \leq S(I_{\mathrm{Mod}(T)}(M), I_{\mathrm{Mod}(T)}(A)) \otimes S(I_{\mathrm{Mod}(T)}(A), B) \\ &\quad \leq S(I_{\mathrm{Mod}(T)}(M), B) = S(M, B). \end{aligned}$$

The adjointness property applied to the previous inequality finally yields

$$S(I_{\mathrm{Mod}(T)}(A), B) \leq S(M, A)^* \to S(M, B),$$

for each $M \in \mathrm{Mod}(T)$, which is the required inequality because $S(M, A)^* \to S(M, B) = ||A \Rightarrow B||_M$. We proved $S(I_{\mathrm{Mod}(T)}(A), B) \leq ||A \Rightarrow B||_T$. □

4 Syntactico-semantical completeness

In this section, we introduce an axiomatic system for our logic, which is inspired by the classic Armstrong system (Armstrong 1974). We then proceed to establish two kinds of completeness theorem for our system. First is the ordinary-style completeness according to which an arbitrary implication $A \Rightarrow B$ is provable from a crisp theory T of implications if and only if $A \Rightarrow B$ semantically follows from T to degree 1. Second is the graded-style completeness according to which it holds that the degree of provability from a theory T of an arbitrary implication $A \Rightarrow B$ equals the degree to which $A \Rightarrow B$ semantically follows from T.

We start by presenting our basic deduction rules:

(Ax) (from any premises) infer conclusion $A \Rightarrow A \cup B$,
(DCut) from premises $A \Rightarrow B$ and $B \cap C \Rightarrow D$ infer conclusion $A \cap C \Rightarrow D$,
(Sh) from premise $A \Rightarrow B$ infer conclusion $c^* \to A \Rightarrow c^* \to B$

for each $A, B, C, D \in \mathbf{L}^Y$, and $c \in L$. Note that the fuzzy set $c^* \to A$ is defined by $(c^* \to A)(y) = c^* \to A(y)$. Note furthermore that the rule (Ax) is essentially an axiom, and that according to this rule, any formula of the form $A \Rightarrow A \cup B$ can be derived in a single inference step.

While (Ax) and (DCut) are inspired by the ordinary rules of axiom and cut (in fact, (DCut) is dual to the usual rule of cut), rule (Sh) is a new rule in our setting. It is easy to see that if the hedge * is the globalization, rule (Sh) may be dropped. This is because if c equals 1 (Sh) clearly says "from $A \Rightarrow B$ infer $A \Rightarrow B$" and is thus a trivial rule. If $c < 1$, then because $c^* = 0$, rule (Sh) allows us to infer from $A \Rightarrow B$ the trivial conclusion $Y \Rightarrow Y$, can be inferred by (Ax) and thus can be omitted.

For ordinary provability, we use the usual notions. Thus, for an ordinary (i.e. crisp) theory T we denote by $T \vdash_{\mathcal{R}} A \Rightarrow B$ the fact that $A \Rightarrow B$ is provable from T, which means that it may be derived from the implications in T using a set $\mathcal{R}$ of deduction rules. Furthermore, we call a deduction rule of the form "from $\varphi_1, \ldots, \varphi_n$ infer φ" derivable from a set $\mathcal{R}$ of other rules if $\{\varphi_1, \ldots, \varphi_n\} \vdash_{\mathcal{R}} \varphi$. If the subscript $\mathcal{R}$ is omitted, it means that $\mathcal{R}$ consists of (Ax)–(Sh).

It is a matter of routine verification that the following rules are derivable from (Ax) and (DCut):

(Ref) (from any premises) infer conclusion $A \Rightarrow A$,
(Wea) from premise $A \Rightarrow B$ infer conclusion $A \cap C \Rightarrow B$,
(Add) from premises $A \Rightarrow B$ and $A \Rightarrow C$ infer conclusion $A \Rightarrow B \cap C$,
(Pro) from premise $A \Rightarrow B \cap C$ infer conclusion $A \Rightarrow B$,
(Tra) from premises $A \Rightarrow B$ and $B \Rightarrow C$ infer conclusion $A \Rightarrow C$,

for each $A, B, C, D \in \mathbf{L}^Y$.

4.1 Ordinary-style completeness

Before proving the completeness theorem, we need some auxiliary results. We call a deduction rule *sound* if every model of its premises is also a model of its conclusions, which for a rule "from $\varphi_1, \ldots, \varphi_n$ infer φ" means that if $M \in \mathrm{Mod}(\{\varphi_1, \ldots, \varphi_n\})$ then $M \in \mathrm{Mod}(\{\varphi\})$. Soundness of a rule thus means that whenever the premises are true, the conclusion is true as well. Our rules are sound:

Lemma 5 *The deduction rules* (Ax), (DCut), *and* (Sh) *are sound.*

Proof (Ax): Clearly, for any M we have $S(M, A)^* \leq S(M, A \cup B)$, i.e. $||A \Rightarrow A \cup B||_M = 1$, proving $M \in \mathrm{Mod}(\{A \Rightarrow A \cup B\})$.

(DCut): We need to check that $M \in \mathrm{Mod}(\{A \Rightarrow B, B \cap C \Rightarrow D\})$ implies $M \in \mathrm{Mod}(\{A \cap C \Rightarrow D\})$. We prove a stronger claim, namely

$$(||A \Rightarrow B||_M)^* \otimes (||B \cap C \Rightarrow D||_M)^* \leq ||A \cap C \Rightarrow D||_M.$$

As one easily observes, this claim is equivalent to

$$S(M, A \cap C)^* \otimes \left[S(M, A)^* \to S(M, B)\right]^* \otimes \left[S(M, B \cap C)^* \to S(M, D)\right]^* \leq S(M, D).$$

The latter inequality holds true since

$$\begin{aligned} &S(M, A \cap C)^* \otimes (S(M, A)^* \to S(M, B))^* \otimes \left[S(M, B \cap C)^* \to S(M, D)\right]^* \\ &\leq (S(M, A)^* \wedge S(M, C)^*) \otimes (S(M, A)^* \to S(M, B))^* \otimes \left[S(M, B \cap C)^* \to S(M, D)\right]^* \\ &\leq S(M, A)^* \otimes \left[S(M, A)^* \to S(M, B)\right]^* \otimes S(M, C)^* \otimes \left[S(M, B \cap C)^* \to S(M, D)\right] \\ &\leq S(M, B)^* \otimes S(M, C)^* \otimes \left[S(M, B \cap C)^* \to S(M, D)\right] \\ &\leq S(M, D). \end{aligned}$$

(Sh): Let $M \in \mathrm{Mod}(\{A \Rightarrow B\})$. We have to show that $M \in \mathrm{Mod}(\{c^* \to A \Rightarrow c^* \to B\})$. Observe first that $M \in \mathrm{Mod}(\{A \Rightarrow B\})$ iff $||A \Rightarrow B||_M = 1$ iff $S(M, A)^* \leq S(M, B)$ iff

$$\text{for each } y \in Y: \; M(y) \otimes S(M, A)^* \leq B(y). \tag{22}$$

Now, $M \in \mathrm{Mod}(\{c^* \to A \Rightarrow c^* \to B\})$ iff $S(M, c^* \to A)^* \leq S(M, c^* \to B)$ iff for each $y \in Y$ we have $M(y) \otimes S(M, c^* \to A)^* \leq c^* \to B(y)$, i.e. $M(y) \otimes c^* \otimes S(M, c^* \to A)^* \leq B(y)$, which is true:

$$\begin{aligned} &M(y) \otimes c^* \otimes S(M, c^* \to A)^* \\ &\leq M(y) \otimes c^* \otimes (c^{**} \to S(M, A)^*) \\ &= M(y) \otimes c^* \otimes (c^* \to S(M, A)^*) \\ &\leq M(y) \otimes S(M, A)^* \leq B(y), \end{aligned}$$

the last inequality being true due to (22), proving the soundness of (Sh). □

Call an ordinary theory T of fuzzy attribute implications *semantically closed* if all its semantics consequences are already contained in T, i.e. if for every $A \Rightarrow B$ we have $A \Rightarrow B \in T$ if and only if $||A \Rightarrow B||_T = 1$. Analogously, T is called *syntactically closed* if the same is true for its syntactic consequences (formulas provable from T), i.e. if for every $A \Rightarrow B$ we have $A \Rightarrow B \in T$ if and only if $T \vdash A \Rightarrow B$.

Lemma 6 *If an ordinary theory T of fuzzy attribute implications is semantically closed, it is also syntactically closed.*

Proof One may easily observe that an ordinary theory T of FAIs is syntactically closed iff we have:

$A \Rightarrow A \cup B \in T$,
if $A \Rightarrow B \in T$ and $B \cap C \Rightarrow D \in T$ then $A \cap C \Rightarrow D \in T$,
if $A \Rightarrow B \in T$ then $c^* \to A \Rightarrow c^* \to B \in T$

for each $A, B, C, D \in \mathbf{L}^Y$, and $c \in L$. Therefore, we have to show that for each deduction rule "from $\varphi_1, \ldots, \varphi_n$ infer φ," i.e. one of (Ax)–(Sh), $\varphi_1, \ldots \varphi_n \in T$ implies $\varphi \in T$. Let thus $\varphi_1, \ldots \varphi_n \in T$. Since $\{\varphi_1, \ldots \varphi_n\} \subseteq T$, for any model $M \in \mathrm{Mod}(T)$ we have

$$M \in \mathrm{Mod}(\{\varphi_1, \ldots \varphi_n\}).$$

Since each of the rules (Ax)–(Sh) is sound by Lemma 5, we conclude $M \in \text{Mod}(\{\varphi\})$. Since M is an arbitrary model of T, this shows that φ semantically follows from T. Since T is semantically closed, we get $\varphi \in T$. □

The following lemma is crucial for our proof of completeness. Note that similarly as in case of Armstrong rules, the assumption of finiteness may be dropped if we employ infinitary rules (we omit this technical issue).

Lemma 7 *If an ordinary theory T of fuzzy attribute implications over a finite set Y and a finite set L of truth degrees is syntactically closed, it is also semantically closed.*

Proof To verify that a syntactically closed theory T is semantically closed, it is sufficient to verify that $\{A \Rightarrow B \mid ||A \Rightarrow B||_T = 1\} \subseteq T$. To check this inclusion, we show that $A \Rightarrow B \notin T$ implies $A \Rightarrow B \notin \{A \Rightarrow B \mid ||A \Rightarrow B||_T = 1\}$. Notice for this purpose that since T is closed syntactically, it is also closed under any of the derived rules (Ref)–(Tra) listed above.

Assume $A \Rightarrow B \notin T$. We demonstrate that a model $M \in \text{Mod}(T)$ exists that is not a model of $A \Rightarrow B$, and this clearly implies the required inclusion $A \Rightarrow B \notin \{A \Rightarrow B \mid ||A \Rightarrow B||_T = 1\}$. Consider the fuzzy set $M = A^-$ of attributes defined as follows: A^- is the smallest fuzzy set for which $A \Rightarrow A^-$ is in T.

Observe first that such A^- exists: Since $A \Rightarrow A \in T$ due to (Ref), the set $S = \{C \mid A \Rightarrow C \in T\}$ is non-empty; moreover, since Y and L are finite, S is finite; finally, S is closed under intersections because if $A \Rightarrow C_1, \ldots, A \Rightarrow C_n \in T$ then $A \Rightarrow \bigcap_{i=1}^{n} C_i \in T$ by a repeated use of (Add).

Next, we show that (a) A^- is not a model of $A \Rightarrow B$, i.e. $||A \Rightarrow B||_{A^-} \neq 1$, and that (b) A^- is a model of T, i.e. $||C \Rightarrow D||_{A^-} = 1$ for every $C \Rightarrow D \in T$.

(a): Assume, by way of contradiction, that $||A \Rightarrow B||_{A^-} = 1$. From $A^- \subseteq A$ it follows $A^- \subseteq B$ because $1 = ||A \Rightarrow B||_{A^-} = S(A^-, A)^* \to S(A^-, B) = 1 \to S(A^-, B) = S(A^-, B)$. An application of rule (Pro) to $A \Rightarrow A^- \in T$ now yields $A \Rightarrow B \in T$, which is a contradiction with our assumption.

(b): For any $C \Rightarrow D \in T$, we need to check that $||C \Rightarrow D||_{A^-} = 1$. That is, to check $S(A^-, C)^* \to S(A^-, D) = 1$. The latter equality holds iff

$$S(A^-, C)^* \otimes A^- \subseteq D, \text{ i.e. iff } A^- \subseteq S(A^-, C)^* \to D.$$

To verify the last inclusion, we prove

$$A \Rightarrow S(A^-, C)^* \to D \in T.$$

Observe that this is indeed sufficient, as A^- is the smallest fuzzy set with $A \Rightarrow A^- \in T$. The following three assertions are furthermore available:

(i) $A \Rightarrow A^- \in T$ (directly from the definition of A^-),
(ii) $A^- \Rightarrow S(A^-, C)^* \to C \in T$
(namely, $A^- \Rightarrow S(A^-, C)^* \to C$ is an instance of (Ax) because $A^- \subseteq S(A^-, C)^* \to C$),
(iii) $S(A^-, C)^* \to C \Rightarrow S(A^-, C)^* \to D \in T$ (just apply (Sh) to $C \Rightarrow D \in T$).

Applying now (Tra) twice to (i), (ii), and (iii), $A \Rightarrow S(A^-, C)^* \to D \in T$ is readily obtained. □

We thus obtain the ordinary-style completeness for our logic:

Theorem 5 *For finite Y and L, and an ordinary theory T of fuzzy attribute implications, $T \vdash A \Rightarrow B$ if and only if $||A \Rightarrow B||_T = 1$, i.e. $A \Rightarrow B$ is provable from T iff $A \Rightarrow B$ semantically follows from T.*

Proof Denote by $\text{syn}(T)$ and $\text{sem}(T)$ the least syntactically and semantically closed ordinary theory of FAIs that contains T, respectively. It is easily shown that both syn and sem are closure operators and that $\text{syn}(T) = \{A \Rightarrow B \mid T \vdash A \Rightarrow B\}$ and $\text{sem}(T) = \{A \Rightarrow B \mid ||A \Rightarrow B||_T = 1\}$. To prove the claim, it is thus sufficient to establish $\text{syn}(T) = \text{sem}(T)$. As $\text{syn}(T)$ is syntactically closed, it is also semantically closed by Lemma 7, which means $\text{sem}(\text{syn}(T)) \subseteq \text{syn}(T)$. Therefore, since $T \subseteq \text{syn}(T)$, monotony of sem yields

$$\text{sem}(T) \subseteq \text{sem}(\text{syn}(T)) \subseteq \text{syn}(T).$$

In a similar manner, we get $\text{syn}(T) \subseteq \text{sem}(T)$, showing $\text{syn}(T) = \text{sem}(T)$, completing the proof. □

4.2 Graded-style completeness

Even though Theorem 5 connects provability and entailment, one may naturally ask if general degrees of entailment—different from 1 to which Theorem 5 restricts—may be characterized by a kind of generalized provability concept. For this purpose, we employ a concept of degree of provability of $A \Rightarrow B$ from T, denoted by $|A \Rightarrow B|_T$. With this concept in hand, we establish that $|A \Rightarrow B|_T = ||A \Rightarrow B||_T$, which equality may be looked at as expressing a completeness-to-degrees of our logic. This concept is inspired by the framework of Pavelka-style logic (Pavelka 1979a, b, c), see also, e.g. (Gerla 2001; Hájek 1998).

Let therefore T be a theory of fuzzy attribute implications, $A \Rightarrow B$ be an implication, and define the *degree to which $A \Rightarrow B$ is provable from T* by

$$|A \Rightarrow B|_T = \bigvee \{c \in L \mid \text{cr}(T) \vdash A \Rightarrow c \otimes B\}. \quad (23)$$

Note that in this definition, $\text{cr}(T)$ is defined as in Lemma 3. Observe that we made use in this definition of Lemma 4 which reduces entailment to an arbitrary degree to entailment to degree 1. One then obtains our graded-style completeness theorem:

Theorem 6 *For a theory T over finite Y and finite $\mathbf{L}$, it holds*

$$|A \Rightarrow B|_T = ||A \Rightarrow B||_T.$$

Proof Directly by using Lemma 4 and Theorem 5 and the above considerations. □

5 Non-redundant bases

In this section, we describe certain non-redundant fully informative sets of implications true in a given table $\langle X, Y, I\rangle$.

Definition 3 We call an ordinary theory (i.e. a set) T of fuzzy attribute implications *complete in* a table $\langle X, Y, I\rangle$ if for each implication $A \Rightarrow B$ we have $||A \Rightarrow B||_T = ||A \Rightarrow B||_{\langle X,Y,I\rangle}$. If, moreover, no proper subset of T is complete in $\langle X, Y, I\rangle$, we call T a *base of* $\langle X, Y, I\rangle$.

Clearly, if T is complete then each $A \Rightarrow B \in T$ is valid in $\langle X, Y, I\rangle$ to degree 1 and, furthermore, for any other implication $C \Rightarrow D$, the degree $||C \Rightarrow D||_{\langle X,Y,I\rangle}$ to which $C \Rightarrow D$ is valid in $\langle X, Y, I\rangle$ is equal to the degree $||C \Rightarrow D||_T$ to which $C \Rightarrow D$ semantically follows from T. In this sense, bases are non-redundant ordinary theories with complete information about validity in data. The following theorem characterizes complete ordinary theories:

Theorem 7 *An ordinary theory T is complete if and only if* $\mathrm{Mod}(T) = \mathrm{Int}(I)$.

Proof Take a complete T and assume that $M \in \mathrm{Mod}(T)$, i.e. M is a model of T. Due to (12), we have $||M \Rightarrow M^{\cup\cap}||_{\mathrm{Int}(I)} = S(M^{\cup\cap}, M^{\cup\cap}) = 1$. Hence, $||M \Rightarrow M^{\cup\cap}||_T = 1$ because T is complete and because we have (12). From $M \in \mathrm{Mod}(T)$, we get $||M \Rightarrow M^{\cup\cap}||_M = 1$, hence $1 = S(M, M)^* \leq S(M, M^{\cup\cap})$ and, therefore, $M \subseteq M^{\cup\cap}$. Now, one always has $M^{\cup\cap} \subseteq M$, whence $M \in \mathrm{Int}(I)$, finishing the proof of $\mathrm{Mod}(T) \subseteq \mathrm{Int}(I)$.

Let furthermore $M \in \mathrm{Int}(I)$. Due to (12), $||A \Rightarrow B||_M \geq ||A \Rightarrow B||_{\mathrm{Int}(I)} = ||A \Rightarrow B||_{\mathrm{Mod}(T)} = 1$ for every $A \Rightarrow B \in T$. This means that M is a model of T, establishing $\mathrm{Int}(I) \subseteq \mathrm{Mod}(T)$.

On the other hand, from $\mathrm{Mod}(T) = \mathrm{Int}(I)$ we obtain using (12) that $||A \Rightarrow B||_T = ||A \Rightarrow B||_{\mathrm{Int}(I)} = ||A \Rightarrow B||_{\langle X,Y,I\rangle}$, i.e. T is complete. □

We now describe bases of $\langle X, Y, I\rangle$ that may be obtained from systems of so-called pseudo-intents of $\langle X, Y, I\rangle$. In case of attribute implications over Boolean attributes, the notion of a pseudo-intent goes back to (Guigues and Duquenne 1986) [see also (Ganter and Wille 1999)]; our notion for graded attributes is inspired by (Belohlavek and Vychodil 2016, 2017).

Definition 4 We call a system $\mathcal{P}$ of fuzzy sets in Y a *system of pseudo-intents* of a table $\langle X, Y, I\rangle$ if for every fuzzy set $P \in \mathbf{L}^Y$ the following holds: $P \in \mathcal{P}$ if and only if $P \neq P^{\cup\cap}$ and for each $Q \in \mathcal{P}$ with $Q \neq P$ we have $||Q \Rightarrow Q^{\cup\cap}||_P = 1$.

From now on, let $\mathcal{P}$ denote a system of pseudo-intents of $\langle X, Y, I\rangle$. We need the following auxiliary results.

Lemma 8 *For a system of pseudo-intents $\mathcal{P}$ of $\langle X, Y, I\rangle$, consider the set $T = \{P \Rightarrow P^{\cup\cap} \mid P \in \mathcal{P}\}$ of implications. Then,* $\mathrm{Mod}(T) \subseteq \mathrm{Int}(I)$*, i.e. every model of T in an intent of I.*

Proof We proceed by way of contradiction. To show that $\mathrm{Mod}(T) \subseteq \mathrm{Int}(I)$, we assume $M \notin \mathrm{Int}(I)$ which means $M \neq M^{\cup\cap}$. As M is a model of T, it follows that for each $Q \in \mathcal{P}$ one has $||Q \Rightarrow Q^{\cup\cap}||_M = 1$. By definition of a system of pseudo-intents, $M \in \mathcal{P}$ which implies that $M \Rightarrow M^{\cup\cap} \in T$. Now,

$$\begin{aligned} ||M \Rightarrow M^{\cup\cap}||_M &= S(M, M)^* \to S(M, M^{\cup\cap}) \\ &= S(M, M^{\cup\cap}) \neq 1, \end{aligned}$$

a contradiction to $M \in \mathrm{Mod}(T)$. □

Lemma 9 *For any $A, M \in \mathbf{L}^Y$, we have $||A \Rightarrow A^{\cup\cap}||_M = 1$ for every $A \in \mathbf{L}^Y$ and* $M \in \mathrm{Int}(I)$.

Proof Let $M \in \mathrm{Int}(I)$, i.e. $M = M^{\cup\cap}$. We have

$$\begin{aligned} S(M, A)^* &\leq S(M^{\cup}, A^{\cup})^* \\ &\leq S(M^{\cup *}, A^{\cup *}) \\ &\leq S(M^{\cup\cap}, A^{\cup\cap}) \\ &= S(M, A^{\cup\cap}). \end{aligned}$$

Thus, $S(M, A)^* \to S(M, A^{\cup\cap}) = 1$, i.e. $||A \Rightarrow A^{\cup\cap}||_M = 1$. □

Lemma 10 *For any system $\mathcal{P}$ of pseudo-intents of a table $\langle X, Y, I\rangle$, the set $T = \{P \Rightarrow P^{\cup\cap} \mid P \in \mathcal{P}\}$ is complete in $\langle X, Y, I\rangle$.*

Proof Due to (12), it suffices to verify that for any implication $A \Rightarrow B$ we have

$$||A \Rightarrow B||_T = ||A \Rightarrow B||_{\mathrm{Int}(I)}$$

On the one hand, $||A \Rightarrow B||_T \leq ||A \Rightarrow B||_{\mathrm{Int}(I)}$ follows from the fact that every intent in $\mathrm{Int}(I)$ is a model of T, which was established in Lemma 9. Conversely, $||A \Rightarrow B||_T \geq ||A \Rightarrow B||_{\mathrm{Int}(I)}$ follows from Lemma 8. □

We thus obtain the main result in the present section.

Theorem 8 $T = \{P \Rightarrow P^{\cup\cap} \mid P \in \mathcal{P}\}$ *is a base of* $\langle X, Y, I\rangle$.

Proof By Lemma 10, T is complete. It remains to check minimality of T. Let $T' \subset T$. Take some $P \in \mathcal{P}$ such that $P \Rightarrow P^{\cup\cap}$ does not belong to T'. Definition 4 implies that $||Q \Rightarrow Q^{\cup\cap}||_P = 1$ for every $Q \in \mathcal{P}$ and $Q \neq P$, which means $P \in \mathrm{Mod}(T')$. Since $||P \Rightarrow P^{\cup\cap}||_P = S(P, P^{\cup\cap}) \neq 1$, we obtain $||P \Rightarrow P^{\cup\cap}||_{T'} \neq 1$. On the other hand, Lemma 9 and (12) yield $||P \Rightarrow P^{\cup\cap}||_{\langle X,Y,I\rangle} = 1$, hence T' is not complete in $\langle X, Y, I\rangle$. □

In the remainder, we show that if L is finite and * is the globalization, the base T in Theorem 8 is in fact the smallest one. For this purpose, we need the following auxiliary result.

Lemma 11 *Suppose that fuzzy sets P and Q in Y are pseudo-intents or intents of* $\langle X, Y, I\rangle$, *i.e.* $P, Q \in \mathcal{P} \cup \mathrm{Int}(I)$, *satisfying*

$$S(Q, P)^* \leq S(P \cup Q, P^{\cup\cap})$$

and

$$S(P, Q)^* \leq S(P \cup Q, Q^{\cup\cap}).$$

Then, $P \cup Q$ *is an intent of* $\langle X, Y, I\rangle$, *i.e.* $P \cup Q \in \mathrm{Int}(I)$.

Proof Let T be the set of fuzzy attribute implications in Theorem 8 and consider its subset $T' = T - \{P \Rightarrow P^{\cup\cap}, Q \Rightarrow Q^{\cup\cap}\}$. On account of Definition 4 and Lemma 9, we obtain than both P and Q are models of T'. It follows that for every implication $A \Rightarrow B$ in T' one has $S(P, A)^* \leq S(P, B)$ and $S(Q, A)^* \leq S(Q, B)$. Therefore, $S(P \cup Q, A)^* = (S(P, A) \wedge S(Q, A))^* \leq S(P, A)^* \wedge S(Q, A)^* \leq S(P, B) \wedge S(Q, B) = S(P \cup Q, B)$. This inequality implies that $P \cup Q \in \mathrm{Mod}(T')$. On account of Lemma 8, it remains to prove that $P \cup Q \in \mathrm{Mod}(\{P \Rightarrow P^{\cup\cap}, Q \Rightarrow Q^{\cup\cap}\})$. Due to the two inequalities assumed in the present lemma, $S(P \cup Q, P)^* = S(Q, P)^* \leq S(P \cup Q, P^{\cup\cap})$ and $S(P \cup Q, Q)^* = S(P, Q)^* \leq S(P \cup Q, Q^{\cup\cap})$, which means by definition that both the required conditions, $||P \Rightarrow P^{\cup\cap}||_{P \cup Q} = 1$ and $||Q \Rightarrow Q^{\cup\cap}||_{P \cup Q} = 1$, are met and hence $P \cup Q \in \mathrm{Mod}(\{P \Rightarrow P^{\cup\cap}, Q \Rightarrow Q^{\cup\cap}\})$ is indeed the case. □

Now we obtain:

Theorem 9 *Let* $\mathbf{L}$ *be a finite residuated lattice with* * *being the globalization, let Y be finite, let* $T = \{P \Rightarrow P^{\cup\cap} \mid P \in \mathcal{P}\}$. *If* T' *is complete in* $\langle X, Y, I\rangle$ *then* $|T| \leq |T'|$.

Proof We prove the claim by showing that for each $P \in \mathcal{P}$, T' contains an implication $A \Rightarrow B$ with $P \subseteq A$ and $A^{\cup\cap} = P^{\cup\cap}$ and that for mutually different $P_1, P_2 \in \mathcal{P}$, their corresponding implications in T' are also different.

Consider any pseudo-intent $P \in \mathcal{P}$. Because, by definition of a pseudo-intent, $P \neq P^{\cup\cap}$, and because T' is complete, we obtain by virtue of Theorem 7 that an implication $A \Rightarrow B$ exists in T' for which $||A \Rightarrow B||_P \neq 1$. As * is the globalization, we have $P \subseteq A$ and $P \nsubseteq B$. Thus, $P^{\cup\cap} \subseteq A^{\cup\cap}$. Now, since T' is complete and due to (12), we conclude $S(A^{\cup\cap}, B) = 1$, whence $A^{\cup\cap} \subseteq B$. As $P \nsubseteq B$, we finally get $P \nsubseteq A^{\cup\cap}$.

We now easily obtain that $A^{\cup\cap} \cup P \notin \mathrm{Int}(I)$: $P \nsubseteq A^{\cup\cap}$ implies $A^{\cup\cap} \subset A^{\cup\cap} \cup P$; $P \subseteq A$ and $A^{\cup\cap} \subseteq A$ yield $A^{\cup\cap} \cup P \subseteq A$, hence $(A^{\cup\cap} \cup P)^{\cup\cap} \subseteq A^{\cup\cap}$ by monotony of $^{\cup\cap}$. To sum up, $(A^{\cup\cap} \cup P)^{\cup\cap} \subset A^{\cup\cap} \cup P$, i.e. $A^{\cup\cap} \cup P$ is not an intent.

We now verify $A^{\cup\cap} = P^{\cup\cap}$. First, the above-observed fact $P \subseteq A$ implies $P^{\cup\cap} \subseteq A^{\cup\cap}$. It remains to prove $A^{\cup\cap} \subseteq P^{\cup\cap}$. This inclusion readily follows from $A^{\cup\cap} \subseteq P$ which we now verify. By contradiction, if $A^{\cup\cap} \nsubseteq P$ then the fact $P \nsubseteq A^{\cup\cap}$ observed above and Lemma 11 yield $A^{\cup\cap} \cup P \in \mathrm{Int}(I)$, contradicting the above observation.

It remains to check that if $P_1, P_2 \in \mathcal{P}$ are different, then no single $A \Rightarrow B \in T'$ can satisfy $P_1, P_2 \subseteq A$ and $P_1^{\cup\cap} = A^{\cup\cap} = P_2^{\cup\cap}$. By contradiction, assume that $A \Rightarrow B \in T'$ has this property. Observe first that $P_1 \subset P_2$ cannot be the case: Due to the definition of a pseudo-intent, $P_1 \subset P_2$ implies $P_1 \subseteq P_2^{\cup\cap}$, hence $A^{\cup\cap} = P_1^{\cup\cap} \subset P_1 \subseteq P_2^{\cup\cap} = A^{\cup\cap}$, a contradiction. Similarly one observes that $P_2 \subset P_1$ cannot be the case, hence we have $S(P_1, P_2) < 1$ and $S(P_2, P_1) < 1$, and thus, $S(P_1, P_2)^* = 0$ and $S(P_2, P_1)^* = 0$. Lemma 11 now yields $P_1 \cup P_2 \in \mathrm{Int}(I)$. As $P_1, P_2 \subseteq A$, we have $P_1 \cup P_2 \subseteq A$ and thus also $(P_1 \cup P_2)^{\cup\cap} \subseteq A^{\cup\cap}$. Since P_1 is a pseudo-intent, we have $P_1^{\cup\cap} \subset P_1$, hence also $P_1^{\cup\cap} \subset P_1 \cup P_2 = (P_1 \cup P_2)^{\cup\cap} \subseteq A^{\cup\cap}$, a contradiction to the assumption $P_1^{\cup\cap} = A^{\cup\cap}$. □

6 Conclusions and further issues

We examined a logic for dependencies describing containment of fuzzy attributes and established several results for this logic. The grades are assumed to be members of a complete residuated lattice and our semantics is based on the operations in this lattice. The dependencies involved may be seen as dual to those of (Belohlavek and Vychodil 2016, 2017). However, the lack of certain laws in residuated structures of truth degrees, such as the law of double negation, prevents a reduction of the present dependencies to those from (Belohlavek and Vychodil 2016, 2017) and requires a separate inquiry.

Among the main results established in the paper are: results regarding validity of dependencies, their models, and entailment; connections to related structures, particularly to isotone Galois connections and the lattices of their fixpoints; an axiomatic system for reasoning with the dependencies including two versions of completeness theorem;

basic results on bases, i.e. minimal fully informative sets of dependencies that are true in a given data. To keep the paper concise, we did not examine related computational problems. These problems include the problem of computing a degree of entailment of a given dependency from a given set of dependencies, the problem of computing various kinds of bases, including the significant base we described, and various other problems ordinarily studied for data dependencies. Such problems remain a future research topic.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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C Concept Lattices of Isotone vs. Antitone Galois Connections in Graded Setting: Mutual Reducibility Revisited

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In this paper, we explore relationship between standard concept lattices and attribute-oriented concept lattices in the graded setting. It is fundamental work for the framework for handling denial since it shows that the attribute-oriented concept-forming operators contain an implicit negation which makes it appropriate to use for handling denial.

It is well known that in the basic setting the standard and attribute-oriented concept lattices of a formal context and its complement are isomorphic, via a natural isomorphism which maps the extents to themselves and intents to their complements. It is also known that in the graded setting, this and similar kinds of reductions fail to hold. We show that when the usual notion of a complement, based on a residuum w.r.t. 0, is replaced by a new one, based on residua w.r.t. arbitrary truth degrees, the above-mentioned reduction remains valid.

On the one hand, it is well known that with $\mathbf{L}$ satisfying the double negation law ($\neg\neg a = a$) the attribute-oriented case is easily reducible to the standard case, and *vice versa*, via a set complement. Specifically, the attribute-oriented concept lattice $\mathcal{B}^{\cap\cup}(X, Y, I)$ is isomorphic to the standard concept lattice $\mathcal{B}^{\uparrow\downarrow}(X, Y, \neg I)$. The isomorphism $i\colon \mathcal{B}^{\cap\cup}(X, Y, I) \to \mathcal{B}^{\uparrow\downarrow}(X, Y, \neg I)$, as well as its inverse $i^{-1}\colon \mathcal{B}^{\uparrow\downarrow}(X, Y, \neg I) \to \mathcal{B}^{\cap\cup}(X, Y, I)$, is given by

$$i, i^{-1}\colon \langle A, B\rangle \mapsto \langle A, \neg B\rangle. \tag{43}$$

Clearly, we also have

$$\mathrm{Ext}^{\cap\cup}(X, Y, I) = \mathrm{Ext}^{\uparrow\downarrow}(X, Y, \neg I). \tag{44}$$

On the other hand, this is no longer the case in the graded setting as the double negation law does not hold generally. We propose a new notion of complement of an $\mathbf{L}$-relation: $\mathbf{L}$-complement w.r.t. $K \subseteq L$ of an $\mathbf{L}$-relation $I \in L^{X\times Y}$ is $\mathbf{L}$-relation $\neg_K I \in L^{X\times(Y\times K)}$ given by

$$\neg_K I(x, \langle y, a\rangle) = I(x, y) \to a \tag{45}$$

for all $x \in X, y \in Y, a \in K$.

Utilizing this notion of complement, we can state one-way reducibility of the standard case to the attribute-oriented case:

Theorem 8. *Let $\langle X, Y, I\rangle$ be an* $\mathbf{L}$*-context. Then* $\mathcal{B}^{\cap\cup}(X, Y, I)$ *is isomorphic to* $\mathcal{B}^{\uparrow\downarrow}(X, Y \times (L\backslash\{0\}), \neg_{L\backslash\{0\}} I)$ *with* $i : \langle A, A^{\cap}\rangle \mapsto \langle A, A^{\uparrow}\rangle$ *being the isomorphism from* $\mathcal{B}^{\cap\cup}(X, Y, I)$ *to* $\mathcal{B}^{\uparrow\downarrow}(X, Y \times L, \neg_{L\backslash\{0\}} I)$*. Particularly,*

$$\mathrm{Ext}^{\cap\cup}(X, Y, I) = \mathrm{Ext}^{\uparrow\downarrow}(X, Y \times L, \neg_{L\backslash\{0\}} I).$$

The result reveals a new, deeper root of the reduction: it is not the availability of the law of double negation, but rather the fact that negations are implicitly present in the construction of attribute-oriented concept lattices.

A converse statement to Theorem 8 does not hold. That is, there is no notion of a complement $\sim$ such that for any fuzzy relation I, the set $\mathrm{Ext}^{\uparrow\downarrow}(X, Y, I)$ is equal to $\mathrm{Ext}^{\cap\cup}(X, Z, \sim I)$ for any suitable Z. This is because for some fuzzy relations I, $\mathrm{Ext}^{\uparrow\downarrow}(X, Y, I)$ is not a system of extents of any fuzzy relation J w.r.t. the operators $\langle^{\cap}, ^{\cup}\rangle$. This was demonstrated in [14].

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Concept lattices of isotone vs. antitone Galois connections in graded setting: Mutual reducibility revisited

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ABSTRACT

It is well known that concept lattices of isotone and antitone Galois connections induced by an ordinary binary relation and its complement are isomorphic, via a natural isomorphism mapping extents to themselves and intents to their complements. It is also known that in a fuzzy setting, this and similar kinds of reduction fail to hold. In this note, we show that when the usual notion of a complement, based on a residuum w.r.t. 0, is replaced by a new one, based on residua w.r.t. arbitrary truth degrees, the above-mentioned reduction remains valid. For ordinary relations, the new and the usual complement coincide. The result we present reveals a new, deeper root of the reduction: It is not the availability of the law of double negation but rather the fact that negations are implicitly present in the construction of concept lattices of isotone Galois connections.

1. Problem setting

As is well-known, a given ordinary binary relation $I \in \{0,1\}^{X\times Y}$ (representing, e.g. a yes/no relationship between objects $x \in X$ and attributes $y \in Y$) induces two important pairs of operators between $\{0,1\}^X$ and $\{0,1\}^Y$. Namely, a pair $\langle {}^{\uparrow_I}, {}^{\downarrow_I}\rangle$ defined by

$$A^{\uparrow_I} = \{y \in Y \mid \text{ for each } x \in A : \langle x,y\rangle \in I\},$$
$$B^{\downarrow_I} = \{x \in X \mid \text{ for each } y \in B : \langle x,y\rangle \in I\}, \qquad (1)$$

and a pair $\langle {}^{\cap_I}, {}^{\cup_I}\rangle$ defined by

$$A^{\cap_I} = \{y \in Y \mid \text{there exists } x \in A \text{ such that } \langle x,y\rangle \in I\},$$
$$B^{\cup_I} = \{x \in X \mid \text{for each } y \in Y : \langle x,y\rangle \in I \text{ implies } y \in B\}, \qquad (2)$$

for all subsets A of X and B of Y. These operators are employed in several areas including data analysis, such as formal concept analysis in particular [8] or association rules, logic and reasoning about data [7], or ordered sets and their applications [6]. It is well known that the two pairs of operators are mutually definable [7]. An important consequence is that with $\neg$ denoting the set complement, the sets of fixpoints, i.e. the concept lattices

$$\mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I) \quad \text{and} \quad \mathcal{B}(X^{\uparrow_{\neg I}}, Y^{\downarrow_{\neg I}}, \neg I) \text{ are isomorphic as lattices,} \qquad (3)$$

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(or, equivalently, $\mathcal{B}(X^{\cap_{\neg I}}, Y^{\cup_{\neg I}}, \neg I)$ and $\mathcal{B}(X^{\uparrow_I}, Y^{\downarrow_I}, I)$ are isomorphic), with $\langle A, B\rangle \mapsto \langle A, \neg B\rangle$ being an isomorphism. Hence, in particular,

$$\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I) = \mathrm{Ext}(X^{\uparrow_{\neg I}}, Y^{\downarrow_{\neg I}}, \neg I), \tag{4}$$

i.e. the corresponding sets of extents are equal. Here, the concept lattices and the sets of extents of a binary relation $I \in \{0,1\}^{X\times Y}$ are defined by

$$\mathcal{B}(X^{\uparrow_I}, Y^{\downarrow_I}, I) = \{\langle C, D\rangle \in \{0,1\}^X \times \{0,1\}^Y | C^{\uparrow_I} = D, D^{\downarrow_I} = C\}, \tag{5}$$

$$\mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I) = \{\langle C, D\rangle \in \{0,1\}^X \times \{0,1\}^Y | C^{\cap_I} = D, D^{\cup_I} = C\}, \tag{6}$$

$$\mathrm{Ext}(X^{\uparrow_I}, Y^{\downarrow_I}, I) = \{C \in \{0,1\}^X | \langle C, D\rangle \in \mathcal{B}(X^{\uparrow_I}, Y^{\downarrow_I}, I) \text{ for some } D\}, \tag{7}$$

$$\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I) = \{C \in \{0,1\}^X | \langle C, D\rangle \in \mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I) \text{ for some } D\}. \tag{8}$$

The above reducibility results mean that, in a sense, one need not investigate the properties of the concept lattices of $\langle {}^{\uparrow_I}, {}^{\downarrow_I}\rangle$ and $\langle {}^{\cap_I}, {}^{\cup_I}\rangle$ separately because the properties of one are derivable from those of the other.

However, as shown in [9], when fuzzy relations instead of ordinary relations I are considered (i.e. graded attributes rather than yes/no attributes are considered), the above mutual reducibility results are no longer true. In this note, we show that when the notion of a complement of a fuzzy relation is defined in a new way, (3) and (4) remain valid even in the setting of fuzzy relations. We also show that in the other direction, the reducibility results cannot be saved even with the new notion of complement. Since in the case of ordinary relations the new notion of complement coincides with the usual one, our result puts the known reducibility results in a different perspective that we discuss.

2. Result and remarks

We assume that the set L of truth degrees along with the truth functions $\otimes$ of conjunction and $\to$ of implication forms a complete residuated lattice, i.e. a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ satisfying: $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice; $\langle L, \otimes, 1\rangle$ is a commutative monoid; $\otimes$ and $\to$ satisfy adjointness, i.e. $a \otimes b \leqslant c$ iff $a \leqslant b \to c$. We assume that the reader is familiar with examples and properties of residuated lattices [2,10,11,13].

A fuzzy relation $I \in L^{X\times Y}$ induces two pairs of operators between L^X and L^Y, i.e. the sets of all fuzzy sets in X and Y, defined by

$$A^{\uparrow_I}(y) = \bigwedge_{x\in X}(A(x) \to I(x,y)), \quad B^{\downarrow_I}(x) = \bigwedge_{y\in Y}(B(y) \to I(x,y)), \tag{9}$$

$$A^{\cap_I}(y) = \bigvee_{x\in X}(A(x) \otimes I(x,y)), \quad B^{\cup_I}(x) = \bigwedge_{y\in Y}(I(x,y) \to B(y)), \tag{10}$$

for all fuzzy sets $A \in L^X$ and $B \in L^Y$. Clearly, (9) and (10) generalize the above operators defined by (1) and (2) (just put $L = \{0,1\}$). $\mathcal{B}(X^{\uparrow_I}, Y^{\downarrow_I}, I)$, $\mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I)$, $\mathrm{Ext}(X^{\uparrow_I}, Y^{\downarrow_I}, I)$, and $\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I)$ are defined by the same formulas as in (5)–(8) with $\{0,1\}$ replaced by L. For more information we refer, e.g. to [2–4,9].

As was mentioned above, when I is a fuzzy relation (3) and (4) fail to hold. This fact was for the first time observed in [9] and is well known. In this observation, however, it is crucial that the complement $\neg I$ of a fuzzy relation I between X and Y is conceived as a fuzzy relation between X and Y defined by

$$\neg I(x,y) = I(x,y) \to 0. \tag{11}$$

That is, one uses the truth function $\neg$ of negation defined by

$$\neg a = a \to 0, \tag{12}$$

for each $a \in L$ and the standard way of defining a complement of a fuzzy set by means of $\neg$.

As we show in what follows, there is another notion of complement of I. Both $\neg I$, as defined above, and the new notion of complement coincide with the ordinary notion of complement in the ordinary case, i.e. when $L = \{0,1\}$. However, the new notion of complement has the advantage that a part of the reducibility results, namely (3) and (4), remain true even when I is a fuzzy relation (see Remark 2b for a reducibility result that does not hold with any notion of complement).

The classical notion of complement $\neg I$ of a fuzzy relation may be looked at the following way. Each attribute $y \in Y$ in the data table representing I is replaced by its complement. That it, each fuzzy set $I_y \in L^X$, representing attribute y, defined by $I_y(x) = I(x,y)$ is replaced in the table by its complement $\neg I_y$ defined by

$$(\neg I_y)(x) = \neg(I_y(x)), \quad \text{i.e.} \quad (\neg I_y)(x) = I_y(x) \to 0.$$

The complement (12) is in fact the residuum of a w.r.t. 0. However, one may also consider a residuum of $a \in L$ w.r.t. to an arbitrary element $b \in L$, i.e. one may consider

$$\neg_b a = a \to b, \tag{13}$$

of which $\neg a$ is a particular case because $\neg a = \neg_0 a$. In addition to $\neg I_y$, the "negation relative to 0" one may therefore also consider $\neg_b I_y$, the "negation relative to b", for other degrees b, defined by

$$(\neg_b I_y)(x) = \neg_b(I_y(x)), \quad \text{i.e.} \quad (\neg_b I_y)(x) = I_y(x) \to b.$$

For every original attribute y, I_y may therefore be replaced not just by the complement $\neg_0 I_y$ w.r.t. 0 but by several complements $\neg_b I_y$ w.r.t. $b \in K$ with $K \subseteq L$ being a set of selected values, bringing us the following definition.

Definition 1. For a set $K \subseteq L$, the K-complement of a fuzzy relation I between X and Y is a fuzzy relation $\neg_K I$ between X and $Y \times K$ defined by

$$(\neg_K I)(x, \langle y, b\rangle) = \neg_b I(x, y), \tag{14}$$

for every $x \in X$, $y \in Y$, and $b \in K$.

Remark 1.

(a) Going from I to $\neg_K I$ may be seen as replacing every attribute $y \in Y$, represented by I_y in I, by a collection of new attributes $\langle y, b\rangle \in Y \times K$, represented by $\neg_b I_y$ in $\neg_K I$ for $b \in K$.
(b) Clearly, for $K = \{0\}$, $\neg_K I$ may be identified with $\neg I$, because $Y \times \{0\}$ may be identified with Y and $\neg_K I(x, \langle y, \{0\}\rangle) = \neg I(x, y)$.
(c) Observe that for $L = \{0,1\}$ (the ordinary case), $\neg_{L-\{1\}} I = \neg_{\{0\}} I$, i.e. in view of (b) of this Remark, $\neg_{L-\{1\}} I$ may be identified with the classical complement $\neg I$ of I.

In view of Remark 1c, there are two ways to generalize the notion of a complement of an ordinary relation I between X and Y to a fuzzy setting:

(i) First, a complement of I may be defined as a fuzzy relation between X and Y by (11).
(ii) Second, a complement of I may be defined as a fuzzy relation between X and $Y \times K$ by (14) with $K = L - \{1\}$.

While (3) and (4) fail to hold in a fuzzy setting for (i), they do hold in a fuzzy setting with the complement understood according to (ii):

Theorem 1. *For a fuzzy relation I between X and Y, let $\lrcorner I$ denote $\neg_{L-\{1\}} I$. Then $\mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I)$ and $\mathcal{B}(X^{\uparrow_{\lrcorner I}}, Y \times (L - \{1\})^{\downarrow_{\lrcorner I}}, \lrcorner I)$ are isomorphic as lattices, with the mappings $\langle A, B\rangle \mapsto \langle A, D\rangle$, where*

$$D(y, b) = \neg_b B(y), \tag{15}$$

for $y \in Y$, $b \in L - \{1\}$, and $\langle A, D\rangle \mapsto \langle A, B\rangle$, where

$$B(y) = \bigwedge_{b \in L-\{1\}} \neg_b D(y, b), \tag{16}$$

for $y \in Y$, being the isomorphism and its inverse. Hence, in particular,

$$\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I) = \mathrm{Ext}(X^{\uparrow_{\lrcorner I}}, Y \times (L - \{1\})^{\downarrow_{\lrcorner I}}, \lrcorner I). \tag{17}$$

Proof. We first prove (17). Since ${}^{\uparrow_{\lrcorner I}\downarrow_{\lrcorner I}}$ is an **L**-closure operator in X [2], it follows that $\mathrm{Ext}(X^{\uparrow_{\lrcorner I}}, Y \times (L - \{1\})^{\downarrow_{\lrcorner I}}, \lrcorner I)$ is an **L**-closure system in X, i.e. it is closed under arbitrary $\bigwedge$-intersections and left $\to$-multiplications. This means that for all $A_j \in \mathrm{Ext}(X^{\uparrow_{\lrcorner I}}, Y \times (L - \{1\})^{\downarrow_{\lrcorner I}}, \lrcorner I)$, $j \in J$, we have $\bigwedge_{j \in J} A_j \in \mathrm{Ext}(X^{\uparrow_{\lrcorner I}}, Y \times (L - \{1\})^{\downarrow_{\lrcorner I}}, \lrcorner I)$ and for each $a \in L$ and $A \in \mathrm{Ext}(X^{\uparrow_{\lrcorner I}}, Y \times (L - \{1\})^{\downarrow_{\lrcorner I}}, \lrcorner I)$ we have $a \to A \in \mathrm{Ext}(X^{\uparrow_{\lrcorner I}}, Y \times (L - \{1\})^{\downarrow_{\lrcorner I}}, \lrcorner I)$ with $a \to A \in L^X$ defined by $(a \to A)(x) = a \to A(x)$ for each $x \in X$. Moreover, [4, Theorem 2 (10)] implies that $\mathrm{Ext}(X^{\uparrow_{\lrcorner I}}, Y \times (L - \{1\})^{\downarrow_{\lrcorner I}}, \lrcorner I)$ is the least **L**-closure system in X containing every column of $\lrcorner I$, i.e. every $\neg_b I_y$ for each $b \in L - \{1\}$.

To prove (17), it is therefore sufficient to show that $\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I)$ is the least **L**-closure system in X containing every column of $\lrcorner I$. This assertion follows from the fact that $\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I)$ is always an **L**-closure system and from the following claim. □

Claim 1. $\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I)$ *consists of all possible $\bigwedge$-intersections of fuzzy sets $\neg_b I_y (y \in Y, b \in L - \{1\})$.*

*Namely, if $\mathcal{S}$ is an **L**-closure system that contains every column of $\lrcorner I$, it contains all intersections of the columns of $\lrcorner I$ and, due to Claim, it contains $\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I)$. Therefore, to prove (17), it remains to prove Claim.*

Proof of Claim 1. Since ${}^{\cap_I}$ and ${}^{\cup_I}$ form an isotone Galois connection, we have

$$\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I) = \{B^{\cup_I} | B \in L^Y\}. \tag{18}$$

On one hand, every $B^{\cup_I}$ is an intersection of fuzzy sets of the form $\neg_b I_y$ because

$$B^{\cup_I}(x) = \bigwedge_{y\in Y}(I(x,y) \to B(y)) = \bigwedge_{y\in Y}\neg_{B(y)} I_y. \tag{19}$$

On the other hand, consider an arbitrary intersection A of $\neg_b I_y$s, i.e. $A = \bigwedge_{\langle y,b\rangle\in P}\neg_b I_y$ for some $P \subseteq Y\times (L-\{1\})$. Define $B(y) = \bigwedge_{\langle y,b\rangle\in P} b$. Then

$$A(x) = \bigwedge_{y\in Y}\bigwedge_{\langle y,b\rangle\in P}(I(x,y)\to b) = \bigwedge_{y\in Y} I(x,y) \to \bigwedge_{\langle y,b\rangle\in P} b = \bigwedge_{y\in Y} I(x,y) \to B(y) = B^{\cup_I}(x),$$

hence $A \in \mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I)$, finishing the proof of Claim and hence also the proof of (17).

Now, since $\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I)$ and $\mathrm{Ext}(X^{\uparrow_{\lrcorner I}}, Y\times(L-\{1\})^{\downarrow_{\lrcorner I}}, \lrcorner I)$ are isomorphic as lattices to $\mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I)$ and $\mathcal{B}(X^{\uparrow_{\lrcorner I}}, Y\times(L-\{1\})^{\downarrow_{\lrcorner I}}, \lrcorner I)$, respectively, it follows that $\mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I)$ and $\mathcal{B}(X^{\uparrow_{\lrcorner I}}, Y\times(L-\{1\})^{\downarrow_{\lrcorner I}}, \lrcorner I)$ are isomorphic as lattices.

Take any $\langle A,B\rangle \in \mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I)$ and the corresponding $\langle A,D\rangle \in \mathcal{B}(X^{\uparrow_{\lrcorner I}}, Y\times(L-\{1\})^{\downarrow_{\lrcorner I}}, \lrcorner I)$. Then

$$\begin{aligned} D(y,b) = A^{\uparrow_{\lrcorner I}}(y,b) &= \bigwedge_{x\in X} A(x) \to \lrcorner I(x,\langle y,b\rangle) = \bigwedge_{x\in X} A(x) \to (I(x,y)\to b) = \bigwedge_{x\in X}((A(x)\otimes I(x,y))\to b) \\ &= \left[\bigvee_{x\in X}(A(x)\otimes I(x,y))\right] \to b = A^{\cap_I}(y)\to b = B(y)\to b = \neg_b B(y), \end{aligned}$$

verifying (15). To check (16), consider any $A\in L^X$ and the corresponding $B = A^{\cap_I}$ and $D = A^{\uparrow_{\lrcorner I}}$. Observe first that

$$B(y) \leqslant \neg_b D(y,b), \tag{20}$$

for each $b \in L-\{1\}$. Indeed, taking into account $a \leqslant (a\to b)\to b = \neg_b\neg_b a$ for any $a\in L$ and (15), we have $B(y) \leqslant \neg_b\neg_b B(y) = \neg_b D(y,b)$. This verifies the "$\leqslant$" part of (16). Let now $c = B(y)$. If $c < 1$, then c is one of the degrees from $L-\{1\}$ over which the infimum in (16) is taken and since $\neg_c D(y,c) = \neg_c\neg_c B(y) = \neg_c\neg_c c = c = B(y)$ in this case, the infimum in (16) is indeed equal to $B(y)$. If $c = 1$ then due to (20), $\neg_b D(y,b) = 1$ for each $b \in L-\{1\}$, hence also the infimum in (16) is equal to 1, i.e. equal to $B(y)$. □

Remark 2

(a) One easily checks that since $\neg_1 I_y(x) = 1$ for each $x\in X$, one may replace $L-\{1\}$ by L in Theorem 1.

(b) A converse statement to Theorem 1 does not hold. That is, there is no notion of a complement $\sim$ such that for any fuzzy relation I, $\mathrm{Ext}(X^{\uparrow_I}, Y^{\downarrow_I}, I)$ is equal to $\mathrm{Ext}(X^{\cap_{\sim I}}, Z^{\cup_{\sim I}}, \sim I)$ for any suitable Z. This is because for some fuzzy relations I, $\mathrm{Ext}(X^{\uparrow_I}, Y^{\downarrow_I}, I)$ is not a system of extents of any fuzzy relation J w.r.t. the operators ${}^{\cap_J}$ and ${}^{\cup_J}$ [5].

(c) In view of Remark 1c, Theorem 1 generalizes (3) and (4) and its proof does not use the law of double negation.

3. Conclusions

We proposed a new notion of complement of a fuzzy relation. We showed that this notion helps to save certain results that are known not to hold with the ordinary notion of a complement. A further exploration of the new notion of complement remains a subject for future research.

It is an interesting question to explore to what extent the new notion may be used in various other areas of fuzzy set theory to replace the usual notion of complement in such a way that the resulting concepts behave as in the classical, bivalent case. In particular, in the context of closure structures associated to fuzzy relations, it seems reasonable to use the new notion of complement to define a new semantics of failure dependencies in knowledge spaces with graded knowledge states [1]. Another topic worth further investigation is provided by [12]. One of the main results in [12] is a description of a scaling of a fuzzy relation $I\in L^{X\times Y}$ to an ordinary relation $I_c \subseteq (X\times L)\times(Y\times L)$ such that $\mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I)$ and $\mathcal{B}(X\times L^{\uparrow_{I_c}}, Y\times L^{\downarrow_{I_c}}, I_c)$ are isomorphic as lattices. This result is a consequence of Theorem 1. Furthermore, [12] considers general isotone Galois connections that employ linguistic hedges to parameterize the concept of an isotone Galois connection and to reduce the size of the resulting concept lattice. An analogous reduction may be obtained by using $\neg_K I$ with $K \subseteq L-\{1\}$. These issues will be subject to a future work.

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D L-concept Analysis with Positive and Negative Attributes

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We describe an extension of FCA in the graded setting, allowing a user to choose which incidences are viewed as affirmations and which are viewed as denials. The two sets are then handled using a combination of the standard and attribute-oriented concept-forming operators. Specifically, we extend the notion of formal **L**-context to contain two **L**-relations, ${}^{+}I$ and ${}^{-}I$, between objects and attributes. The membership degrees in ${}^{+}I$ present graded affirmations while the membership degrees in ${}^{-}I$ present graded denials. It is natural to assume that ${}^{+}I \subseteq {}^{-}I$. The intervals $[{}^{+}I(x,y), {}^{-}I(x,y)]$ are then seen as sets of truth degrees in which object x can have attribute y. As intents, we use pairs $\langle {}^{+}B, {}^{-}B \rangle \in L^Y \times L^Y$, where the **L**-sets ${}^{+}B, {}^{-}B$ respectively represent affirmations and and denials about attributes.

The concept-forming operators $\vartriangle: L^X \to L^Y \times L^Y$ and $\triangledown: L^Y \times L^Y \to L^X$ are defined as

$$A^{\vartriangle} = \langle A^{\uparrow}, A^{\cap} \rangle \text{ and } \langle {}^{+}B, {}^{-}B \rangle^{\triangledown} = {}^{+}B^{\downarrow} \cap {}^{-}B^{\cup} \tag{46}$$

for each $A \in L^X, {}^{+}B, {}^{-}B \in L^Y$; where the pair $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ is induced by $\langle X, Y, {}^{+}I \rangle$ and the pair $\langle {}^{\cap}, {}^{\cup} \rangle$ is induced by $\langle X, Y, {}^{-}I \rangle$.

Both the two main outputs of FCA are presented. In the first part, an analogy of the main theorem of concept lattices and a relationship between the new concept lattice and the previously studied concept lattices is shown.

In the second part, we describe the second main output of FCA. We present a general logic of if-then rules $\mathbf{A} \Rightarrow \mathbf{B}$ ($\mathbf{A}, \mathbf{B} \in L^Y \times L^Y$), called **L**-containment implications, for graded attributes which can be read: if all attributes of an object are contained in **A** then they are contained in **B**. Specifically, for $\mathcal{M} \subseteq L^Y \times L^Y$, the degree $\|\mathbf{A} \Rightarrow \mathbf{B}\|_{\mathcal{M}}$ in which $\mathbf{A} \Rightarrow \mathbf{B}$ is valid in $\mathcal{M}$ is defined as $\|\mathbf{A} \Rightarrow \mathbf{B}\|_{\mathcal{M}} = \bigwedge_{\mathbf{M} \in \mathcal{M}} \|\mathbf{A} \Rightarrow \mathbf{B}\|_{\mathbf{M}}$.

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L-concept analysis with positive and negative attributes☆

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We describe an extension of formal fuzzy concept analysis allowing a user to choose which attributes are viewed as positive and which are viewed as negative. The two sets are then handled using a combination of previously studied antitone concept-forming operators and isotone concept-forming operators, respectively. The two main outputs of formal concept analysis, namely concept lattices and attribute implications, in the setting of positive and negative attributes are presented. An analogy of the main theorem of concept lattices and a relationship between the new concept lattice and the previously studied concept lattices is showed. We introduce basic syntactic and semantic notions for attribute implications called fuzzy containment implications. We consider two settings, one where the sets of positive and negative attributes are crisp sets, and a generalization, where the two sets are fuzzy sets.

1. Introduction

Formal concept analysis [9,12] is a method of relational data analysis identifying interesting clusters (formal concepts) in a collection of objects and their attributes, and organizing them into a structure called concept lattice. The formal concept is obtained as a fixed point of so-called concept-forming operators and is characterized by a pair of sets – extent and intent. The extent contains all objects covered by the concept and the intent contains all attributes covered by the concept. Numerous generalizations of formal concept analysis, which allow to work with graded data, were provided; see [16] and references therein. In the present paper we stick with approach of Belohlavek [2] and Pollandt [17].

In a graded (fuzzy) setting, two main kinds of concept forming-operators – antitone and isotone one – were studied [3,13,17,18], compared [5,6] and even covered under a unifying framework [4,15]. The antitone concept-forming operators handle attributes in a positive way and concepts are based on sharing attributes (at least in some degree), while the isotone concept-forming operators handle attributes in a negative way and concepts are based on missing same attributes (or having them at most in some degree).

In order to clarify our motivation, we assume the data (**L**-context) in Fig. 1 with objects representing employees, and attributes representing skills.

We consider the following four situations:

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	α	β	γ
A	0.5	0	1
B	1	0.5	1
C	0	0.5	0.5
D	0.5	0.5	1

Fig. 1. Example of **L**-context with objects A, B, C, D and attributes α, β, γ; **L** is a chain $0 < 0.5 < 1$ with Łukasiewicz operations.

(a) One can handle the attributes in positive way and form concepts based on having the same skills at least in some degree. Such concepts are formed by antitone concept-forming operators denoted by $\langle\uparrow, \downarrow\rangle$. Extents of the concepts can be interpreted as maximal collections of employees able to fulfill a task which requires particular skill set. For example, the collection of employees able to fulfill a task which requires the skill α in full degree and the skill β at least in half degree can be found as $\{\alpha, \deg 0.5\beta\}^{\downarrow}$.

(b) Or, one can handle the attributes in negative way and form concepts based on having the same skills at most in some degree. Such concepts are created by isotone concept-forming operators which we denote by $\langle\cap, \cup\rangle$. Extents of the concepts can be interpreted as maximal collections of employees who lack the same skills and need some training to gain them. For example, the maximal collection of employees who lack the skill α and have the skill β at most in degree 0.5 can be found as $\{\deg 0.5\beta, \gamma\}^{\cup}$.

(c) Now, consider a training course for the skill β for which is essential to have the skill α at least in degree 0.5. Concept covering just employees appropriate for the training course (i.e. employees who meet the requirement but have not mastered β yet) is not formed neither by antitone nor isotone concept-forming operators. While the attribute α is positive, β is negative and they must be handled in a different way.

(d) Finally, consider another training course to master the skill β for which is essential to already have the skill β in degree 0.5. In such a case, we are interested in a concept covering just employees having the skill β exactly in degree 0.5. In such a case, β is considered positively and negatively at the same time. Again, it cannot be formed neither by antitone nor isotone concept-forming operators.

This example motivates us to extend formal concept analysis in such a way that a user is allowed to specify a set ${}^{+}Y$ of positive attributes and a set ${}^{-}Y$ of negative attributes. Attributes in ${}^{+}Y$ and ${}^{-}Y$ are then handled using antitone concept-forming operators and isotone concept-forming operators, respectively. The present approach enables us to incorporate the concepts from (a) and concepts from (b) into one concept lattice, and to form concepts from (c) and (d).

We study the two main outputs of formal concept analysis, i.e. concept lattices and attribute implications, in the setting of positive and negative attributes.

We have considered a similar setting in [1] where each attribute has both positive and negative occurrence in a formal context. Membership degrees of positive and negative attributes in intents serve as their lower and upper approximations, respectively, in fuzzy rough set setting.[1] Recently, [19] considered a framework with positive and negative attributes for crisp setting. The present work can be considered to be a generalization of both [1] and [19].

The paper is structured as follows. In Section 2 we recall basic notions important for the rest of the paper; namely residuated lattices, fuzzy sets and fuzzy relations, completely lattice **L**-ordered sets, and formal fuzzy concept analysis. Section 3 introduces the extension of formal concept analysis where a user can select sets ${}^{+}Y$ and ${}^{-}Y$ of positive and negative attributes, respectively. Section 4 then generalizes the results of Section 3 by making ${}^{+}Y$ and ${}^{-}Y$ fuzzy sets. In Section 5 we provide results on attribute implications in the present setting.

2. Preliminaries

2.1. Residuated lattices, *L*-sets and *L*-relations

We use complete residuated lattices as basic structures of truth degrees. A complete residuated lattice [2,14,21] is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, i.e. a partially ordered set in which arbitrary infima and suprema exist (the partial order of **L** is denoted by $\leq$); $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation which is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$; $\otimes$ and $\rightarrow$ satisfy adjointness, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$. Operations $\otimes$ (multiplication) and $\rightarrow$ (residuum) play the role of truth functions of “fuzzy conjunction” and “fuzzy implication.” 0 and 1 denote the least and greatest elements. Throughout this work, **L** denotes an arbitrary complete residuated lattice.

Common examples of complete residuated lattices include those defined on the unit interval (i.e. $L = [0, 1]$), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding residuum $\rightarrow$ given by $a \rightarrow b = \max\{c \mid a \otimes c \leq b\}$. The three most important pairs of adjoint operations on the unit interval are

[1] Note that fuzzy interval-valued approaches, e.g. [7,8,10,20], use lower and upper approximations in formal contexts as well but handle both approximations in a positive way.

- Łukasiewicz

$$a \otimes b = \max(a + b - 1, 0),$$
$$a \to b = \min(1 - a + b, 1),$$

- Gödel

$$a \otimes b = \min(a, b),$$
$$a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

- Goguen (product)

$$a \otimes b = a \cdot b,$$
$$a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ \frac{b}{a} & \text{otherwise.} \end{cases}$$

Instead of unit interval we can also consider a finite chain, e.g. $L = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$. All operations on this chain are then defined analogously, see [2].

The following lemma summarizes the properties of complete residuated lattices used in this paper.

Lemma 1. *For all $a, b, c, a_i, b_i \in L$ we have*

$$(a \otimes b) \to c = a \to (b \to c), \tag{1}$$

$$a \to \textstyle\bigwedge_i b_i = \bigwedge_i (a \to b_i), \tag{2}$$

$$\textstyle\bigwedge_i (a_i \to b) = (\bigvee_i a_i) \to b, \tag{3}$$

$$\textstyle\bigwedge_i a_i \leq a_i, \tag{4}$$

$$1 \to a = a, \tag{5}$$

$$a \to a = 1, \tag{6}$$

$$b \leq (a \to b) \to b, \tag{7}$$

$$c = \bigwedge_{a \in L - \{0\}} (c \to a) \to a. \tag{8}$$

Proof. (1)–(7) can be found in [2].

To prove (8) note, that for $c = 1$ the equality follows directly from (7). For $c \in L - \{1\}$, we have $c \leq (c \to a) \to a$ for all $a \in L - \{1\}$ due (7). Thus we have

$$c \leq \bigwedge_{a \in L - \{1\}} (c \to a) \to a \leq (c \to c) \to c = 1 \to c = c$$

using (4)–(6). □

An **L**-set A in a universe set X is a mapping assigning to each $x \in X$ some truth degree $A(x) \in L$. The set of all **L**-sets in a universe X is denoted L^X.

The operations with **L**-sets are defined componentwise. For instance, for $a \in L$ and $A \in L^X$ we define **L**-sets $a \to A$ a $a \otimes A$ in X by $(a \to A)(x) = a \to A(x)$ and $(a \otimes A)(x) = a \otimes A(x)$ for all $x \in X$, respectively. The intersection of **L**-sets $A, B \in L^X$ is an **L**-set $A \cap B$ in X such that $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in X$. Similarly for union of the two **L**-sets.

Intersection and union of two **L**-sets can be generalized to any number of **L**-sets and even to an **L**-set of **L**-sets. For an **L**-set $\mathcal{M} \in L^{L^X}$, the intersection $\bigcap \mathcal{M}$ and union $\bigcup \mathcal{M}$ are **L**-sets in X, defined by

$$\textstyle\bigcap \mathcal{M} = \bigcap_{A \in L^X} (\mathcal{M}(A) \to A), \quad \bigcup \mathcal{M} = \bigcup_{A \in L^X} (\mathcal{M}(A) \otimes A). \tag{9}$$

An **L**-set $A \in L^X$ is also denoted $\{^{A(x)}/x \mid x \in X\}$. If for all $y \in X$ distinct from $x_1, \ldots, x_n$ we have $A(y) = 0$, we also write $\{^{A(x_1)}/x_1, \ldots, {}^{A(x_n)}/x_n\}$. An **L**-set $A \in L^X$ is called crisp if $A(x) \in \{0, 1\}$ for each $x \in X$. Crisp **L**-sets can be identified with ordinary sets. For a crisp set A, we also write $x \in A$ for $A(x) = 1$ and $x \notin A$ for $A(x) = 0$.

For $A, B \in L^X$ we define the degree of inclusion of A in B by

$$S(A, B) = \textstyle\bigwedge_{x \in X} (A(x) \to B(x)). \tag{10}$$

Graded inclusion generalizes the classical inclusion relation. Described verbally, $S(A, B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A, B) = 1$. As a consequence, we have $A \subseteq B$ iff $A(x) \leq B(x)$ for each $x \in X$. Further, we set

$$E(A, B) = S(A, B) \wedge S(B, A). \tag{11}$$

The value $E(A, B)$ is interpreted as the degree to which the sets A and B are similar.

A binary **L**-relation R between sets X and Y is an **L**-set in $X \times Y$, i.e. $R \in L^{X \times Y}$. A binary **L**-relation R on a set X is an **L**-set in $X \times X$, i.e. $R \in L^{X \times X}$.

A binary **L**-relation R on a set X is called reflexive if $R(x, x) = 1$ for any $x \in X$, symmetric if $R(x, y) = R(y, x)$ for any x, $y \in X$, and transitive if $R(x, y) \otimes R(y, z) \leq R(x, z)$ for any x, y, $z \in X$. R is called an **L**-tolerance if it is reflexive and symmetric, and an **L**-equivalence if it is reflexive, symmetric and transitive. If R is an **L**-equivalence such that for any x, $y \in X$ from $R(x, y) = 1$ it follows $x = y$, then R is called an **L**-equality on X.

Let $\sim$ be an **L**-equivalence on X. We say that a binary **L**-relation R on X is compatible with $\sim$, if for each x, x', y, $y' \in X$,

$$R(x, y) \otimes (x \sim x') \otimes (y \sim y') \leq R(x', y').$$

2.2. Completely lattice **L**-ordered sets

An **L**-order on a set X with an **L**-equality $\approx$ on X is a binary **L**-relation $\preceq$ on X which is compatible with $\approx$, reflexive, transitive and satisfies $(x \preceq y) \wedge (y \preceq x) \leq x \approx y$ for any x, $y \in X$ (antisymmetry). An immediate consequence of the definition is that for any x, $y \in X$ it holds

$$x \approx y = (x \preceq y) \wedge (y \preceq x). \tag{12}$$

The tuple $\langle\langle X, \approx\rangle, \preceq\rangle$ is called an **L**-ordered set. If there is no danger of confusion, we denote **L**-ordered sets shortly by $\langle X, \preceq\rangle$.

Let $\langle X, \preceq\rangle$ be an **L**-ordered set. For $A \in L^X$ we define **L**-sets $\mathcal{L}(A), \mathcal{U}(A) \in L^X$ by

$$\mathcal{L}(A)(y) = \textstyle\bigwedge_{x \in X}(A(x) \to (y \preceq x)), \quad \mathcal{U}(A)(y) = \textstyle\bigwedge_{x \in X}(A(x) \to (x \preceq y))$$

for all $y \in X$. The right-hand side of the first equation is the degree of "for each $x \in X$, if x is in A, then y is less than or equal to x, " and similarly for the second equation. Thus, $\mathcal{L}(A)(y)$ (resp. $\mathcal{U}(A)(y)$) can be seen as the degree to which y is less (resp. greater) than or equal to each element of A. $\mathcal{L}(A)$ (resp. $\mathcal{U}(A)$) is called *the lower cone* (resp. *the upper cone*) of A.

For any **L**-set $A \in L^X$ there exists at most one element $x \in X$ such that $\mathcal{L}(A)(x) \wedge \mathcal{U}(\mathcal{L}(A))(x) = 1$ (resp. $\mathcal{U}(A)(x) \wedge \mathcal{L}(\mathcal{U}(A))(x) = 1$), see [2,3]. If there is such an element, we call it *the infimum of* A (resp. *the supremum of* A) and denote it by $\inf A$ (resp. $\sup A$); otherwise we say that the infimum (resp. the supremum) does not exist. For $x \in X$ and $a \in L$ we define

$$a \to x = \inf(\{^a/x\}), \quad a \otimes x = \sup(\{^a/x\}).$$

An **L**-ordered set $\langle X, \preceq\rangle$ is called *completely lattice* **L**-*ordered*, if for each $A \in L^X$, both $\inf A$ and $\sup A$ exist. An important example of a completely lattice **L**-ordered set is the tuple $\mathbf{L}^X = \langle\langle L^X, E\rangle, S\rangle$, where X is an arbitrary set and E and S are given by (11) and (10), respectively. Infima and suprema in $\mathbf{L}^X$ are given by intersections and unions: for any $\mathcal{M} \in L^{L^X}$ we have

$$\inf \mathcal{M} = \bigcap \mathcal{M}, \quad \sup \mathcal{M} = \bigcup \mathcal{M}.$$

Let $\mathbf{X} = \langle X, \preceq\rangle$ be a completely lattice **L**-ordered set. A subset $K \subseteq X$ is {0, 1}-infimally dense (resp. {0, 1}-supremally dense) in $\mathbf{X}$ if for each $x \in X$ there is some $K' \subseteq K$ such that $x = \inf K'$ (resp. $x = \sup K'$).

2.3. Galois connections, **L**-closure operators and **L**-closure systems

An antitone Galois connection between the **L**-ordered sets $\langle X, \preceq_X\rangle$ and $\langle Y, \preceq_Y\rangle$ is a pair $\langle f, g\rangle$ of mappings f: $X \to Y$, g: $Y \to X$, satisfying

$$(x \preceq_X g(y)) = (y \preceq_Y f(x)) \tag{13}$$

for every $x \in X$, $y \in Y$.

An isotone Galois connection between the **L**-ordered sets $\langle X, \preceq_X\rangle$ and $\langle Y, \preceq_Y\rangle$ is a pair $\langle f, g\rangle$ of mappings f: $X \to Y$, g: $Y \to X$, satisfying

$$(x \preceq_X g(y)) = (f(x) \preceq_Y y) \tag{14}$$

for every $x \in X$, $y \in Y$.

An **L**-closure operator on an **L**-ordered set $\langle X, \preceq\rangle$ is a mapping c: $X \to X$ satisfying

$$(x \preceq c(x)) = 1,$$
$$(x_1 \preceq x_2) \leq (c(x_1) \preceq c(x_2)),$$
$$c(x) = c(c(x))$$

for all x, x_1, $x_2 \in X$.

An **L**-interior operator on an **L**-ordered set $\langle X, \preceq\rangle$ is a mapping i: $X \to X$ satisfying

$$(i(x) \preceq x) = 1,$$
$$(x_1 \preceq x_2) \leq (i(x_1) \preceq i(x_2)),$$
$$i(x) = i(i(x))$$

for all x, x_1, $x_2 \in X$.

A system $\mathcal{S} = \{x_i \in X \mid i \in I\}$ is called an **L**-closure system on an **L**-ordered set $\langle X, \preceq\rangle$ if it is closed under general $\preceq$-intersections, i.e. for each $x \in X$ it holds true that

$$\inf_{i \in I}((x \preceq x_i) \to x_i) \in \mathcal{S}.$$

A system $\mathcal{S} = \{x_i \in X \mid i \in I\}$ is called an **L**-interior system on an **L**-ordered set $\langle X, \preceq\rangle$ if it is closed under general $\preceq$-unions, i.e. for each $x \in X$ it holds true that

$$\sup_{i \in I}((x_i \preceq x) \otimes x_i) \in \mathcal{S}.$$

There is a close relationship between antitone/isotone Galois connections, **L**-closure/**L**-interior operators, and **L**-closure/**L**-interior systems, for more details see [2]. Particularly, for an antitone Galois connection $\langle f, g\rangle$ between $\langle X, \preceq_X\rangle$ and $\langle Y, \preceq_Y\rangle$, the composite mapping fg is an **L**-closure operator on $\langle X, \preceq_X\rangle$, and the composite mapping gf is an **L**-closure operator on $\langle Y, \preceq_Y\rangle$. Similarly, for an isotone Galois connection $\langle f, g\rangle$ between $\langle X, \preceq_X\rangle$ and $\langle Y, \preceq_Y\rangle$, the composite mapping fg is an **L**-closure operator on $\langle X, \preceq_X\rangle$, and the composite mapping gf is an **L**-interior operator on $\langle Y, \preceq_Y\rangle$. Moreover, for **L**-closure operator c on $\langle X, \preceq_X\rangle$, the system $\{x \in X \mid x = c(x)\}$ is an **L**-closure system on $\langle X, \preceq_X\rangle$. Similarly, for **L**-interior operator i on $\langle X, \preceq_X\rangle$, the system $\{x \in X \mid x = i(x)\}$ is an **L**-interior system on $\langle X, \preceq_X\rangle$.

2.4. Formal concept analysis of data with fuzzy attributes

An **L**-context is a triplet $\langle X, Y, I\rangle$ where X and Y are (ordinary) sets and $I \in L^{X \times Y}$ is an **L**-relation between X and Y. Elements of X are called objects, elements of Y are called attributes, I is called an incidence relation. $I(x, y) = a$ is read: "The object x has the attribute y to degree a." An **L**-context may be described as a table with the objects corresponding to the rows of the table, the attributes corresponding to the columns of the table and $I(x, y)$ written in cells of the table (for an example see Fig. 1).

Consider the following pairs of operators induced by an **L**-context $\langle X, Y, I\rangle$. First, the pair $\langle\uparrow, \downarrow\rangle$ of antitone concept-forming operators $^\uparrow$: $L^X \to L^Y$ and $^\downarrow$: $L^Y \to L^X$ is defined by

$$A^{\uparrow}(y) = \textstyle\bigwedge_{x \in X}(A(x) \to I(x, y)), \quad B^{\downarrow}(x) = \textstyle\bigwedge_{y \in Y}(B(y) \to I(x, y)). \tag{15}$$

Second, the pair $\langle\cap, \cup\rangle$ of isotone concept-forming operators operators $^\cap$: $L^X \to L^Y$ and $^\cup$: $L^Y \to L^X$ is defined by

$$A^{\cap}(y) = \textstyle\bigvee_{x \in X}(A(x) \otimes I(x, y)), \quad B^{\cup}(x) = \textstyle\bigwedge_{y \in Y}(I(x, y) \to B(y)). \tag{16}$$

To emphasize that the operators are induced by I, we also denote the operators by $\langle\uparrow_I, \downarrow_I\rangle$ and $\langle\cap_I, \cup_I\rangle$.

The pairs $\langle A, B\rangle \in L^X \times L^Y$, such that $A^\uparrow = B$ and $B^\downarrow = A$ are called standard **L**-concepts. Analogously, the pairs $\langle A, B\rangle \in L^X \times L^Y$, such that $A^\cap = B$ and $B^\cup = A$ are called attribute-oriented **L**-concepts. Components A and B in standard or attribute-oriented **L**-concept $\langle A, B\rangle$ are called extent and intent, respectively. For an **L**-concept lattice $\mathcal{B}(X, Y, I)$, where $\mathcal{B}$ is either $\mathcal{B}^{\uparrow\downarrow}$ or $\mathcal{B}^{\cap\cup}$, we denote the corresponding sets of extents and intents by Ext(X, Y, I) and Int(X, Y, I), respectively, i.e.

$$\begin{aligned} \mathrm{Ext}(X, Y, I) &= \{A \in L^X \mid \langle A, B\rangle \in \mathcal{B}(X, Y, I) \text{ for some } B\}, \\ \mathrm{Int}(X, Y, I) &= \{B \in L^Y \mid \langle A, B\rangle \in \mathcal{B}(X, Y, I) \text{ for some } A\}. \end{aligned} \tag{17}$$

Remark 1. For **L**-set $A \in L^X$, the truth degrees in which objects (fully) in A have attribute y are all in the upper cone of $A^\uparrow(y)$ in **L** (Fig. 2 (left)). In the case $A^\uparrow(y) = 0$, objects (fully) in A may have the attribute y in any degree (Fig. 2 (middle)). In the case $A^\uparrow(y) = 1$, objects (fully) in A have the attribute y in full degree (Fig. 2 (right)). As positive information (having an attribute) is absolute in this setting, we say that the pair of concept-forming operators $\langle\downarrow, \uparrow\rangle$ considers attributes in a *positive* way. On the contrary, the truth degrees in which objects (fully) in A have attribute y are all in the lower cone of $A^\cap(y)$ in **L** (Fig. 3 (left)). In the case $A^\cap(y) = 0$, objects (fully) in A do not have the attribute y; i.e. they have it in degree 0. (Fig. 3 (middle)). In the case $A^\cap(y) = 1$, objects (fully) in A may have the attribute y in any degree (Fig. 3 (right)). As negative information (not having an attribute) is absolute in this setting, we say that the pair of concept-forming operators $\langle\cup, \cap\rangle$ considers attributes in a *negative* way.

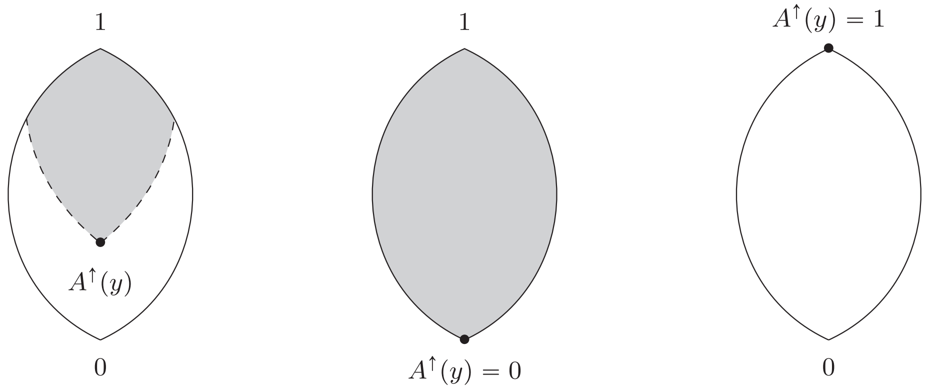


Fig. 2. The truth degrees in which objects (fully) in A may have attribute y (gray area); general case (left), extreme cases $A^{\uparrow}(y) = 0$ and $A^{\uparrow}(y) = 1$ (middle and right, respectively).

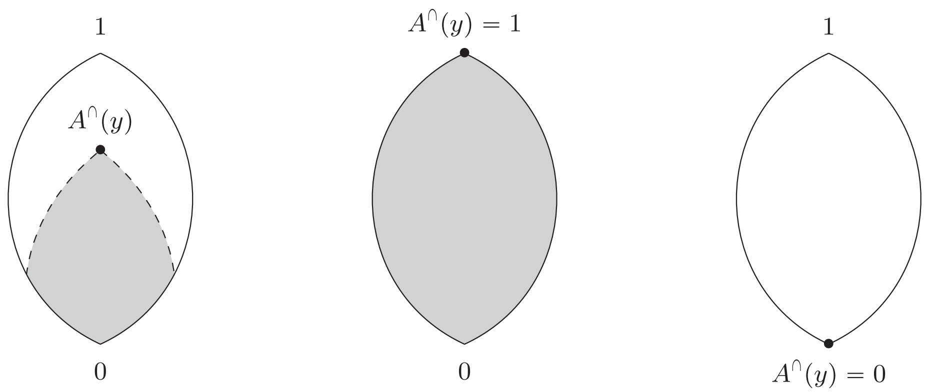


Fig. 3. The truth degrees in which objects (fully) in A may have attribute y (gray area); general case (left), extreme cases $A^{\cap}(y) = 0$ and $A^{\cap}(y) = 1$ (middle and right, respectively).

The set of all standard **L**-concepts of $\langle X, Y, I\rangle$, together with the **L**-order $\preceq$ defined by

$$\langle A_1, B_1\rangle \preceq \langle A_1, B_2\rangle = S(A_1, A_2) \tag{18}$$

for all standard **L**-concepts $\langle A_1, B_1\rangle$, $\langle A_1, B_2\rangle$, forms a completely lattice **L**-ordered set called a standard **L**-concept lattice; see example in Fig. 4. We denote it by $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$. To state this result, we need to introduce following notation. For an **L**-set $\mathcal{M}$ in $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$, we put $\bigcap_X \mathcal{M} = \bigcap \mathrm{proj}_X(\mathcal{M})$, $\bigcup_X \mathcal{M} = \bigcup \mathrm{proj}_X(\mathcal{M})$, $\bigcap_Y \mathcal{M} = \bigcap \mathrm{proj}_Y(\mathcal{M})$, $\bigcup_Y \mathcal{M} = \bigcup \mathrm{proj}_Y(\mathcal{M})$, where $\mathrm{proj}_X(\mathcal{M})$ is the **L**-set in $\mathrm{Ext}^{\uparrow\downarrow}(X, Y, I)$ defined by $(\mathrm{proj}_X(\mathcal{M}))(A) = \mathcal{M}(\langle A, A^{\uparrow}\rangle)$ for every $A \in \mathrm{Ext}^{\uparrow\downarrow}(X, Y, I)$, and similarly, $\mathrm{proj}_Y(\mathcal{M})$ is the **L**-set in $\mathrm{Int}^{\uparrow\downarrow}(X, Y, I)$ defined by $(\mathrm{proj}_Y(\mathcal{M}))(B) = \mathcal{M}(\langle B^{\downarrow}, B\rangle)$ for every $B \in \mathrm{Int}^{\uparrow\downarrow}(X, Y, I)$.

Theorem 1. *Let $\mathbb{K} = \langle X, Y, I\rangle$ be an **L**-context.*

*(a) $\langle \mathcal{B}^{\uparrow\downarrow}(X, Y, I), \preceq\rangle$ is a completely lattice **L**-ordered set with suprema and infima defined as follows for **L**-set $\mathcal{M} \in L^{\mathcal{B}^{\uparrow\downarrow}(X,Y,I)}$.*

$$\begin{aligned}\inf(\mathcal{M}) &= \langle \textstyle\bigcap_X \mathcal{M}, (\bigcup_Y \mathcal{M})^{\downarrow\uparrow}\rangle,\\ \sup(\mathcal{M}) &= \langle (\textstyle\bigcup_X \mathcal{M})^{\uparrow\downarrow}, \bigcap_Y \mathcal{M}\rangle.\end{aligned}$$

*(b) Moreover, a completely lattice **L**-ordered set $\boldsymbol{V} = \langle V, \sqsubseteq\rangle$ is isomorphic to $\langle \mathcal{B}^{\uparrow\downarrow}(X, Y, I), \preceq\rangle$ iff there are mappings*

$$\tilde{\gamma} : X \times L \to V \quad \text{and} \quad \tilde{\mu} : Y \times L \to V,$$

such that $\tilde{\gamma}(X \times L)$ is {0, 1}-supremally dense in $\boldsymbol{V}$, $\tilde{\mu}(Y \times L)$ is {0, 1}-infimally dense in $\boldsymbol{V}$, and

$$((a \otimes b) \to I(x, y)) = (\tilde{\gamma}(x, a) \sqsubseteq \tilde{\mu}(y, b)) \tag{19}$$

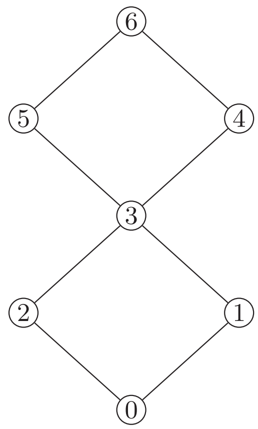
for all $x \in X$, $y \in Y$, $a, b \in L$.

Example 1. Consider the standard **L**-concept lattice $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$ in Fig. 4 and

$$\mathcal{M} = \{{}^{0.5}/③, ④, ⑤, {}^{0.5}/⑥\}.$$

One can compute $\inf(\mathcal{M})$ as infimum in completely lattice **L**-ordered set. We have

$$\mathcal{L}(\mathcal{M}) = \{⓪, ①, ②, ③, {}^{0.5}/④, {}^{0.5}/⑤, {}^{0.5}/⑥\}$$



	extent	intent
⓪	$\{{}^{0.5}/\mathrm{B}, {}^{0.5}/\mathrm{D}\}$	$\{\alpha, \beta, \gamma\}$
①	$\{{}^{0.5}/\mathrm{B}, {}^{0.5}/\mathrm{C}, {}^{0.5}/\mathrm{D}\}$	$\{\alpha, {}^{0.5}/\beta, \gamma\}$
②	$\{{}^{0.5}/\mathrm{A}, \mathrm{B}, {}^{0.5}/\mathrm{D}\}$	$\{{}^{0.5}/\alpha, \beta, \gamma\}$
③	$\{{}^{0.5}/\mathrm{A}, \mathrm{B}, {}^{0.5}/\mathrm{C}, \mathrm{D}\}$	$\{{}^{0.5}/\alpha, {}^{0.5}/\beta, \gamma\}$
④	$\{{}^{0.5}/\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$	$\{{}^{0.5}/\beta, {}^{0.5}/\gamma\}$
⑤	$\{\mathrm{A}, \mathrm{B}, {}^{0.5}/\mathrm{C}, \mathrm{D}\}$	$\{{}^{0.5}/\alpha, \gamma\}$
⑥	$\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$	$\{{}^{0.5}/\gamma\}$

$\preceq$	⓪	①	②	③	④	⑤	⑥
⓪	1	1	1	1	1	1	1
①	0.5	1	0.5	1	1	1	1
②	0.5	0.5	1	1	1	1	1
③	0.5	0.5	0.5	1	1	1	1
④	0	0.5	0	0.5	1	0.5	1
⑤	0	0	0.5	0.5	0.5	1	1
⑥	0	0	0	0.5	0.5	0.5	1

Fig. 4. Standard **L**-concept lattice $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$ (top left) of the **L**-context in Fig. 1, description of its **L**-concepts (top right) and the **L**-order $\preceq$ (bottom).

and

$$\mathcal{U}(\mathcal{L}(\mathcal{M})) = \{{}^{0.5}/⓪, {}^{0.5}/①, {}^{0.5}/②, ③, ④, ⑤, ⑥\}.$$

Obviously, $\mathcal{L}(\mathcal{M})(v) \wedge \mathcal{U}(\mathcal{L}(\mathcal{M}))(v) = 1$ only for $v =$ ③. Thus $\inf(\mathcal{M}) =$ ③. More conveniently, one can compute $\inf(\mathcal{M})$ using Theorem 1(a). The extent of $\inf(\mathcal{M})$ is equal to

$$\begin{aligned} \textstyle\bigcap_X \mathcal{M} &= \textstyle\bigcap \mathrm{proj}_X(\mathcal{M}) \\ &= 0 \to \mathrm{ext}⓪ \cap 0 \to \mathrm{ext}① \cap 0 \to \mathrm{ext}② \\ &\cap\ 0.5 \to \mathrm{ext}③ \cap 1 \to \mathrm{ext}④ \cap 1 \to \mathrm{ext}⑤ \cap 0.5 \to \mathrm{ext}⑥ \\ &= Y \cap Y \cap Y \cap Y \cap \mathrm{ext}④ \cap \mathrm{ext}⑤ \cap Y \\ &= \{\deg 0.5/\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\} \cap \{\mathrm{A}, \mathrm{B}, \deg 0.5/\mathrm{C}, \mathrm{D}\} \\ &= \{\deg 0.5/\mathrm{A}, \mathrm{B}, \deg 0.5/\mathrm{C}, \mathrm{D}\} = \mathrm{ext}③ \end{aligned}$$

where ext denotes the extent of a given concept. The reader can check that the intent part matches ③ as well.

Analogously, all attribute-oriented **L**-concepts of $\langle X, Y, I\rangle$, together with the **L**-order (18) forms a completely lattice **L**-ordered set called an attribute-oriented **L**-concept lattice and we denote it by $\mathcal{B}^{\cap\cup}(X, Y, I)$, see example in Fig. 5. We could state analogy of Theorem 1 for $\langle \mathcal{B}^{\cap\cup}(X, Y, I), \preceq\rangle$. Instead we present a relationship between standard and attribute-oriented **L**-concept lattices from which such result follows.

In [6] we proposed a new notion of complement of an **L**-relation: **L**-complement of an **L**-relation $I \in L^{X\times Y}$ is **L**-relation $\neg_L I \in L^{X\times(Y\times L)}$ given by

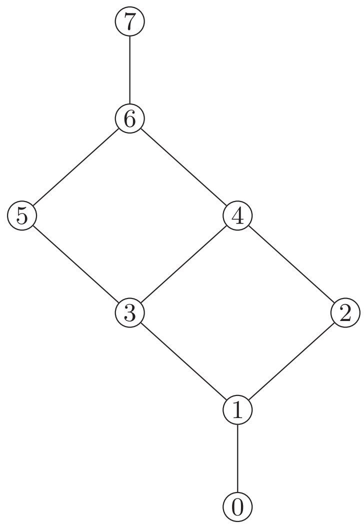
$$\neg_L I(x, \langle y, a\rangle) = I(x, y) \to a \tag{20}$$

for all $x \in X$, $y \in Y$, $a \in L$.

Theorem 2. *Let $\langle X, Y, I\rangle$ be an **L**-context. Then $\langle \mathcal{B}^{\cap\cup}(X, Y, I), \preceq\rangle$ is isomorphic to $\langle \mathcal{B}^{\uparrow\downarrow}(X, Y \times L, \neg_L I), \preceq\rangle$ with $\langle A, A^{\cap}\rangle \mapsto \langle A, A^{\uparrow}\rangle$ being the isomorphism $\mathcal{B}^{\cap\cup}(X, Y, I) \to \mathcal{B}^{\uparrow\downarrow}(X, Y \times L, \neg_L I)$. Particularly, $\mathrm{Ext}^{\cap\cup}(X, Y, I) = \mathrm{Ext}^{\uparrow\downarrow}(X, Y \times L, \neg_L I)$.*

Proof. In [6] we have shown that $\mathcal{B}^{\cap\cup}(X, Y, I)$ and $\mathcal{B}^{\uparrow\downarrow}(X, Y \times L, \neg_L I)$ isomorphic as complete lattices with $\langle A, A^{\cap}\rangle \mapsto \langle A, A^{\uparrow}\rangle$ being the isomorphism $\mathcal{B}^{\cap\cup}(X, Y, I) \to \mathcal{B}^{\uparrow\downarrow}(X, Y \times L, \neg_L I)$. The extension to completely lattice **L**-ordered sets is trivial as the order $\preceq$ is defined via graded subsethood of extents which the isomorphism preserves. □

As it is always clear, what kind of **L**-concept is referred, we call the standard **L**-concepts and the attribute-oriented **L**-concepts uniformly **L**-concepts. Similarly, we call the standard **L**-concept lattices and the attribute-oriented **L**-concept lattices just **L**-concept lattices.



	extent	intent
⓪	$\{{}^{0.5}/\mathrm{C}\}$	$\varnothing$
①	$\{{}^{0.5}/\mathrm{A}, {}^{0.5}/\mathrm{C}, {}^{0.5}/\mathrm{D}\}$	$\{{}^{0.5}/\gamma\}$
②	$\{{}^{0.5}/\mathrm{A}, \mathrm{C}, {}^{0.5}/\mathrm{D}\}$	$\{{}^{0.5}/\beta, {}^{0.5}/\gamma\}$
③	$\{{}^{0.5}/\mathrm{A}, {}^{0.5}/\mathrm{B}, {}^{0.5}/\mathrm{C}, {}^{0.5}/\mathrm{D}\}$	$\{{}^{0.5}/\alpha, {}^{0.5}/\gamma\}$
④	$\{{}^{0.5}/\mathrm{A}, {}^{0.5}/\mathrm{B}, \mathrm{C}, {}^{0.5}/\mathrm{D}\}$	$\{{}^{0.5}/\alpha, {}^{0.5}/\beta, {}^{0.5}/\gamma\}$
⑤	$\{\mathrm{A}, {}^{0.5}/\mathrm{B}, {}^{0.5}/\mathrm{C}, {}^{0.5}/\mathrm{D}\}$	$\{{}^{0.5}/\alpha, \gamma\}$
⑥	$\{\mathrm{A}, {}^{0.5}/\mathrm{B}, \mathrm{C}, \mathrm{D}\}$	$\{{}^{0.5}/\alpha, {}^{0.5}/\beta, \gamma\}$
⑦	$\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$	$\{\alpha, {}^{0.5}/\beta, \gamma\}$

$\preceq$	⓪	①	②	③	④	⑤	⑥	⑦
⓪	1	1	1	1	1	1	1	1
①	0.5	1	1	1	1	1	1	1
②	0.5	0.5	1	0.5	1	0.5	1	1
③	0.5	0.5	0.5	1	1	1	1	1
④	0.5	0.5	0.5	0.5	1	0.5	1	1
⑤	0	0.5	0.5	0.5	0.5	1	1	1
⑥	0	0.5	0.5	0.5	0.5	0.5	1	1
⑦	0	0	0	0.5	0.5	0.5	0.5	1

Fig. 5. Attribute-oriented **L**-concept lattice $\mathcal{B}^{\cap\cup}(X, Y, I)$ (top left) of the **L**-context in Fig. 1, description of its **L**-concepts (top right) and the **L**-order $\preceq$ (bottom).

*2.5. System of **L**-sets of positive and negative attributes*

In this paper, we consider **L**-concepts with intents which consist of an **L**-set of positive attributes in ${}^{+}Y$ and an **L**-set of negative attributes in ${}^{-}Y$. We denote such pairs of **L**-sets by boldface uppercase letters and their positive (negative) components by matching uppercase letters with plus (minus) written as left superscript; for instance $\mathbf{A} = \langle {}^{+}A, {}^{-}A \rangle$, $\mathbf{B} = \langle {}^{+}B, {}^{-}B \rangle$, etc. In this section, we describe how we handle such pairs from $L^{{}^{+}Y} \times L^{{}^{-}Y}$.

Denote $\preceq$ an **L**-order on $L^{{}^{+}Y} \times L^{{}^{-}Y}$ defined as

$$(\mathbf{A} \preceq \mathbf{B}) = S({}^{+}B, {}^{+}A) \wedge S({}^{-}A, {}^{-}B)$$

for each $\mathbf{A}, \mathbf{B} \in L^{{}^{+}Y} \times L^{{}^{-}Y}$. We call $\preceq$ an **L**-*containment* or *graded containment*.

For $\langle {}^{+}A, {}^{-}A \rangle, \langle {}^{+}B, {}^{-}B \rangle \in L^{{}^{+}Y} \times L^{{}^{-}Y}$ we define intersection $\sqcap$ and union $\sqcup$ as

$$\begin{aligned} \langle {}^{+}A, {}^{-}A \rangle \sqcap \langle {}^{+}B, {}^{-}B \rangle &= \langle {}^{+}A \cup {}^{+}B, {}^{-}A \cap {}^{-}B \rangle, \\ \langle {}^{+}A, {}^{-}A \rangle \sqcup \langle {}^{+}B, {}^{-}B \rangle &= \langle {}^{+}A \cap {}^{+}B, {}^{-}A \cup {}^{-}B \rangle. \end{aligned} \tag{21}$$

The intersection and union (21) can be generalized the same way as in (9): let $\mathcal{M} \in L^{L^{{}^{+}Y} \times L^{{}^{-}Y}}$, then

$$\bigsqcap \mathcal{M} = \langle \bigcup (\mathcal{M}(\mathbf{A}) \otimes {}^{+}A), \bigcap (\mathcal{M}(\mathbf{A}) \rightarrow {}^{-}A) \rangle, \tag{22}$$

$$\bigsqcup \mathcal{M} = \langle \bigcap (\mathcal{M}(\mathbf{A}) \rightarrow {}^{+}A), \bigcup (\mathcal{M}(\mathbf{A}) \otimes {}^{-}A) \rangle. \tag{23}$$

Note that $\langle L^{{}^{+}Y} \times L^{{}^{-}Y}, \preceq \rangle$ is a completely lattice **L**-ordered set and for each $\mathcal{M} \in L^{L^{{}^{+}Y} \times L^{{}^{-}Y}}$ we have

$$\inf \mathcal{M} = \bigsqcap \mathcal{M}, \quad \sup \mathcal{M} = \bigsqcup \mathcal{M}.$$

3. Crisp combinations of isotone and antitone concept-forming operators

We describe an extension of formal concept analysis where a user can select two sets ${}^{+}Y, {}^{-}Y$ of attributes from Y. The two sets do not need to be disjoint. Attributes in ${}^{+}Y$ and ${}^{-}Y$ are then handled using antitone concept-forming operators and isotone concept-forming operators, respectively.

For better understandability we first present a case where ${}^{+}Y$ and ${}^{-}Y$ are crisp sets. We omit proofs in this section, because in Section 4 we generalize the case, such that ${}^{+}Y$ and ${}^{-}Y$ are fuzzy sets, and we provide proofs of more general results therein.

Let $\langle X, Y, I\rangle$ be an **L**-context and let ${}^{+}Y, {}^{-}Y \subseteq Y$ be sets of selected positive and negative attributes, respectively. We call $\langle X, \langle {}^{+}Y, {}^{-}Y\rangle, I\rangle$ an **L**-*context with positive and negative attributes*; we also denote it by $\langle X, \mathbf{Y}, I\rangle$, considering $\mathbf{Y} = \langle {}^{+}Y, {}^{-}Y\rangle$.

Denote by ${}^{+}I = I \cap (X \times {}^{+}Y)$ and ${}^{-}I = I \cap (X \times {}^{-}Y)$. Define $\triangle : L^X \to L^{+Y} \times L^{-Y}$ and $\triangledown : L^{+Y} \times L^{-Y} \to L^X$ as

$$A^{\triangle} = \langle A^{\uparrow}, A^{\cap}\rangle,$$
$$\mathbf{B}^{\triangledown} = {}^{+}B^{\downarrow} \cap {}^{-}B^{\cup}$$

for each $A \in L^X$, $\mathbf{B} \in L^{+Y} \times L^{-Y}$, where $\langle \uparrow, \downarrow\rangle$ be a pair of antitone concept-forming operators induced by $\langle X, {}^{+}Y, {}^{+}I\rangle$ and $\langle \cap, \cup\rangle$ be a pair of isotone concept-forming operators induced by $\langle X, {}^{-}Y, {}^{-}I\rangle$.

Remark 2. Before we approach to description of properties of $\langle \triangle, \triangledown\rangle$ we need to explain why we model the negative scale using the isotone concept-forming operators.

Let us start with crisp setting. The obvious choice is to use the antitone concept-forming operators induced by a complement of I; this is how negative attributes are handled in [19]. However, isotone concept-forming operators induced by I provide us an isomorphic concept lattice

$$\mathcal{B}^{\cap\cup}(X, Y, I) \approx \mathcal{B}^{\uparrow\downarrow}(X, Y, \neg I) \tag{24}$$

with $\langle A, B\rangle \mapsto \langle A, \neg B\rangle$ being the isomorphism, see [11]. That is, both $\mathcal{B}^{\cap\cup}(X, Y, I)$ and $\mathcal{B}^{\uparrow\downarrow}(X, Y, \neg I)$ have the same extents, and the intents in $\mathcal{B}^{\uparrow\downarrow}(X, Y, \neg I)$ are complements of the corresponding intents in $\mathcal{B}^{\cap\cup}(X, Y, I)$.

In the fuzzy setting, the above considerations do not hold in general [13]. Unless **L** satisfies the double negation law, applying the complementation on data leads to its degradation; for example, when **L** is a chain with Gödel operations, complement of any **L**-relation is a crisp relation.

The new complement defined by (20) is lossless, i.e. one can obtain the original **L**-relation from its complement. With this complement, one can obtain an isomorphism similar to (24) by Theorem 2:

$$\mathcal{B}^{\cap\cup}(X, Y, I) \approx \mathcal{B}^{\uparrow\downarrow}(X, Y \times L, \neg_L I).$$

To sum up, in the fuzzy setting, we have three options:

(a) we can use the antitone concept-forming operators induced by $\neg I$ and lose some data,
(b) we can use the antitone concept-forming operators induced by $\neg_L I$ and manage pairs in $Y \times L$ instead of simple attributes, or
(c) we can use isotone Galois connections induced by I, which form the same extents as (b).

We have chosen (c) to avoid degradation of data and to avoid manage the pairs in $Y \times L$.

Theorem 3. *The pair* $\langle \triangle, \triangledown\rangle$ *forms an isotone Galois connection between* **L***-ordered sets* $\langle L^X, S\rangle$ *and* $\langle L^{+Y} \times L^{-Y}, \preceq\rangle$.

Corollary 1. *The system* $\{A \in L^X \mid A = A^{\triangle\triangledown}\}$ *is an* **L***-closure system on* $\langle L^X, S\rangle$. *The system* $\{\mathbf{B} \in L^{+Y} \times L^{-Y} \mid \mathbf{B} = \mathbf{B}^{\triangledown\triangle}\}$ *is an* **L***-interior system on* $\langle L^{+Y} \times L^{-Y}, \preceq\rangle$.

A pair $\langle A, \mathbf{B}\rangle$ is called a *formal concept* if

$$A^{\triangle} = \mathbf{B} \quad \text{and} \quad \mathbf{B}^{\triangledown} = A.$$

The set of all formal concepts in $\langle X, \mathbf{Y}, I\rangle$ is denoted by $\mathcal{B}^{\triangle\triangledown}(X, \mathbf{Y}, I)$. On $\mathcal{B}^{\triangle\triangledown}(X, \mathbf{Y}, I)$ we define **L**-order $\trianglelefteq$:

$$\langle A_1, \mathbf{B}_1\rangle \trianglelefteq \langle A_2, \mathbf{B}_2\rangle = S(A_1, A_2)$$

for $\langle A_1, \mathbf{B}_1\rangle, \langle A_2, \mathbf{B}_2\rangle \in \mathcal{B}^{\triangle\triangledown}(X, \mathbf{Y}, I)$. Since

$$S(A_1, A_2) = S(A_1, \mathbf{B}_2^{\triangledown}) = (A_1^{\triangle} \preceq \mathbf{B}_2) = (\mathbf{B}_1 \preceq \mathbf{B}_2),$$

we can define the **L**-order $\trianglelefteq$ equivalently by:

$$\langle A_1, \mathbf{B}_1\rangle \trianglelefteq \langle A_2, \mathbf{B}_2\rangle = (\mathbf{B}_1 \preceq \mathbf{B}_2).$$

Theorem 4. *Let* $\langle X, \mathbf{Y}, I\rangle$ *be an* **L***-context with positive attributes and negative attributes.*

(a) $\langle \mathcal{B}^{\triangle\triangledown}(X, \mathbf{Y}, I), \trianglelefteq\rangle$ *is a completely lattice* **L***-ordered set with suprema and infima defined as follows:*

$$\inf(\mathcal{M}) = \langle \textstyle\bigcap_X \mathcal{M}, (\bigsqcap_{\mathbf{Y}} \mathcal{M})^{\triangledown\triangle}\rangle,$$
$$\sup(\mathcal{M}) = \langle (\textstyle\bigcup_X \mathcal{M})^{\triangle\triangledown}, \bigsqcup_{\mathbf{Y}} \mathcal{M}\rangle$$

for an **L***-set* $\mathcal{M} \in L^{\mathcal{B}^{\triangle\triangledown}(X, \mathbf{Y}, I)}$.

*(b) Moreover, a completely lattice **L**-ordered set* $\boldsymbol{V} = \langle V, \sqsubseteq \rangle$ *is isomorphic to* $\langle \mathcal{B}^{\triangle\triangledown}(X, \mathbf{Y}, I), \trianglelefteq \rangle$ *iff there are mappings*[2]

$$\gamma : X \times L \to V \quad \text{and} \quad \mu : ({}^{+}Y\dot{\cup}{}^{-}Y) \times L \to V$$

such that $\gamma(X \times L)$ *is* $\{0, 1\}$*-supremally dense in* $\boldsymbol{V}$, $\mu(({}^{+}Y\dot{\cup}{}^{-}Y) \times L)$ *is* $\{0, 1\}$*-infimally dense in* $\boldsymbol{V}$*, and*

$$(\gamma(x, a) \sqsubseteq \mu(y, b)) = \begin{cases} (a \otimes b) \to {}^{+}I(x, y) & \text{if } y \in {}^{+}Y, \\ {}^{-}I(x, y) \to (a \to b) & \text{if } y \in {}^{-}Y \end{cases}$$

for all $x \in X$, $a, b \in L$.

Example 2. Consider the **L**-context from Fig. 1 with positive attributes ${}^{+}Y = \{\alpha, \gamma\}$ and negative attributes ${}^{-}Y = \{\beta, \gamma\}$. In $\mathcal{B}^{\triangle\triangledown}(X, \mathbf{Y}, I)$ the attribute α is handled using antitone concept-forming operators, the attribute β is handled using isotone concept-forming operators, and γ is handled using both pairs of concept-forming operators. The **L**-concept lattice $\mathcal{B}^{\triangle\triangledown}(X, \mathbf{Y}, I)$ is depicted in Fig. 6.

Remark 3. For ${}^{+}Y = Y$, ${}^{-}Y = \emptyset$ the concept-forming operators $\langle \triangle, \triangledown \rangle$ correspond with the antitone concept-forming operators $\langle \uparrow, \downarrow \rangle$ and for ${}^{+}Y = \emptyset$, ${}^{-}Y = Y$ the concept-forming operators $\langle \triangle, \triangledown \rangle$ correspond with the the isotone concept-forming operators $\langle \cap, \cup \rangle$.

4. Weighted combinations of isotone and antitone concept-forming operators

Let $\langle X, \mathbf{Y}, I \rangle$ be an **L**-context with positive and negative attributes such that ${}^{+}Y, {}^{-}Y \in L^{Y}$. Define weighted concept-forming operators as follows:

$$A^{\triangle} = \langle {}^{+}Y \lhd A^{\uparrow}, {}^{-}Y \circ A^{\cap} \rangle,$$
$$\mathbf{B}^{\triangledown} = ({}^{+}Y \circ {}^{+}B)^{\downarrow} \cap ({}^{-}Y \lhd {}^{-}B)^{\cup}$$

for each $A \in L^{X}$, ${}^{+}B, {}^{-}B \in L^{Y}$, where $\langle \uparrow, \downarrow \rangle$ is a pair of antitone concept-forming operators induced by $\langle X, Y, I \rangle$, $\langle \cap, \cup \rangle$ is a pair of isotone concept-forming operators induced by $\langle X, Y, I \rangle$, and composition operators $\circ$ and $\lhd$ be defined by

$$(Z \circ B)(y) = Z(y) \otimes B(y),$$
$$(Z \lhd B)(y) = Z(y) \to B(y)$$

for each $y \in Y$.

In what follows we use **L**-relations ${}^{+}I, {}^{-}I \in L^{X \times Y}$ defined by

$${}^{+}I(x, y) = {}^{+}Y(y) \to I(x, y),$$
$${}^{-}I(x, y) = {}^{-}Y(y) \otimes I(x, y)$$

for every $x \in X$ and $y \in Y$. Using this notation we can rewrite concept-forming operators $\langle \triangle, \triangledown \rangle$ as

$$A^{\triangle} = \langle A^{\uparrow_{+I}}, A^{\cap_{-I}} \rangle,$$
$$\mathbf{B}^{\triangledown} = {}^{+}B^{\downarrow_{+I}} \cap {}^{-}B^{\cup_{-I}},$$

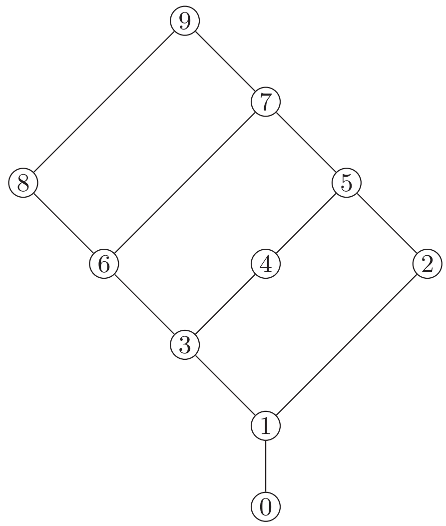
where $\langle \uparrow_{+I}, \downarrow_{+I} \rangle$ is a pair of antitone concept-forming operators induced by $\langle X, Y, {}^{+}I \rangle$, $\langle \cap_{-I}, \cup_{-I} \rangle$ is a pair of isotone concept-forming operators induced by $\langle X, Y, {}^{-}I \rangle$.

Theorem 5. *The pair* $\langle \triangle, \triangledown \rangle$ *forms an isotone Galois connection between **L**-ordered sets* $\langle L^{X}, S \rangle$ *and* $\langle L^{Y} \times L^{Y}, \preceq \rangle$.

Proof. We use the fact, that $\langle \uparrow, \downarrow \rangle$ is an antitone Galois connection between $\langle L^{X}, S \rangle$ and $\langle L^{Y}, S \rangle$, and $\langle \cap, \cup \rangle$ is an isotone Galois connection between $\langle L^{X}, S \rangle$ and $\langle L^{Y}, S^{-1} \rangle$. For each $A \in L^{X}$, $\mathbf{B} \in L^{Y} \times L^{Y}$ we have

$$\begin{aligned} S(A, \mathbf{B}^{\triangledown}) &= \bigwedge_{x \in X} (A(x) \to (({}^{+}Y \circ {}^{+}B)^{\downarrow}(x) \wedge ({}^{-}Y \lhd {}^{-}B)^{\cup}(x))) \\ &= \bigwedge_{x \in X} (A(x) \to ({}^{+}Y \circ {}^{+}B)^{\downarrow}(x)) \wedge \bigwedge_{x \in X} (A(x) \to ({}^{-}Y \lhd {}^{-}B)^{\cup}(x)) \\ &= S(A, ({}^{+}Y \circ {}^{+}B)^{\downarrow}) \wedge S(A, ({}^{-}Y \lhd {}^{-}B)^{\cup}) \\ &= S({}^{+}Y \circ {}^{+}B, A^{\uparrow}) \wedge S(A^{\cap}, {}^{-}Y \lhd {}^{-}B) \\ &= \bigwedge_{y \in Y} (({}^{+}Y \circ {}^{+}B)(y) \to A^{\uparrow}(y)) \wedge \bigwedge_{y \in Y} (A^{\cap}(y) \to ({}^{-}Y \lhd {}^{-}B)(y)) \\ &= \bigwedge_{y \in Y} (({}^{+}Y(y) \otimes {}^{+}B(y)) \to A^{\uparrow}(y)) \wedge \bigwedge_{y \in Y} (A^{\cap}(y) \to ({}^{-}Y(y) \to {}^{-}B(y))) \end{aligned}$$

[2] By ${}^{+}Y\dot{\cup}{}^{-}Y$ we denote disjoint union of ${}^{+}Y$ and ${}^{-}Y$.



	extent	positive intent	negative intent
⓪	$\emptyset$	$\{\alpha, \gamma\}$	$\emptyset$
①	$\{^{0.5}/\mathrm{A}\}$	$\{\alpha, \gamma\}$	$\{^{0.5}/\gamma\}$
②	$\{\mathrm{A}\}$	$\{^{0.5}/\alpha, \gamma\}$	$\{\gamma\}$
③	$\{^{0.5}/\mathrm{A}, ^{0.5}/\mathrm{B}, ^{0.5}/\mathrm{D}\}$	$\{\alpha, \gamma\}$	$\{^{0.5}/\beta, ^{0.5}/\gamma\}$
④	$\{^{0.5}/\mathrm{A}, \mathrm{B}, ^{0.5}/\mathrm{D}\}$	$\{\alpha, \gamma\}$	$\{^{0.5}/\beta, \gamma\}$
⑤	$\{\mathrm{A}, \mathrm{B}, \mathrm{D}\}$	$\{^{0.5}/\alpha, \gamma\}$	$\{^{0.5}/\beta, \gamma\}$
⑥	$\{^{0.5}/\mathrm{A}, ^{0.5}/\mathrm{B}, ^{0.5}/\mathrm{C}, ^{0.5}/\mathrm{D}\}$	$\{\gamma\}$	$\{^{0.5}/\beta, ^{0.5}/\gamma\}$
⑦	$\{\mathrm{A}, ^{0.5}/\mathrm{B}, ^{0.5}/\mathrm{C}, \mathrm{D}\}$	$\{\gamma\}$	$\{^{0.5}/\beta, \gamma\}$
⑧	$\{^{0.5}/\mathrm{A}, ^{0.5}/\mathrm{B}, \mathrm{C}, ^{0.5}/\mathrm{D}\}$	$\{^{0.5}/\gamma\}$	$\{^{0.5}/\beta, ^{0.5}/\gamma\}$
⑨	$\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$	$\{^{0.5}/\gamma\}$	$\{^{0.5}/\beta, \gamma\}$

$\trianglelefteq$	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨
⓪	1	1	1	1	1	1	1	1	1	1
①	0.5	1	1	1	1	1	1	1	1	1
②	0	0.5	1	0.5	0.5	1	0.5	1	0.5	1
③	0.5	0.5	0.5	1	1	1	1	1	1	1
④	0	0	0	0.5	1	1	0.5	0.5	0.5	1
⑤	0	0	0	0.5	0.5	1	0.5	0.5	0.5	1
⑥	0.5	0.5	0.5	0.5	0.5	0.5	1	1	1	1
⑦	0	0	0	0.5	0.5	0.5	0.5	1	0.5	1
⑧	0	0	0	0	0	0	0.5	0.5	1	1
⑨	0	0	0	0	0	0	0.5	0.5	0.5	1

Fig. 6. **L**-concept lattice $\mathcal{B}^{\vartriangle\triangledown}(X, \mathbf{Y}, I)$ of the **L**-context from Fig. 1 with positive attributes ${}^{+}Y = \{\alpha, \gamma\}$ and negative attributes ${}^{-}Y = \{\beta, \gamma\}$ (top); description of **L**-concepts (middle); the **L**-order $\trianglelefteq$ (bottom).

$$= \bigwedge_{y \in Y} ({}^{+}B(y) \to ({}^{+}Y(y) \to A^{\uparrow}(y))) \wedge \bigwedge_{y \in Y} (({}^{-}Y(y) \otimes A^{\cap}(y)) \to {}^{-}B(y))$$
$$= \bigwedge_{y \in Y} ({}^{+}B(y) \to ({}^{+}Y \triangleleft A^{\uparrow})(y)) \wedge \bigwedge_{y \in Y} (({}^{-}Y \circ A^{\cap})(y) \to {}^{-}B(y))$$
$$= S({}^{+}B, {}^{+}Y \triangleleft A^{\uparrow}) \wedge S({}^{-}Y \circ A^{\cap}, {}^{-}B)$$

$$= (\langle {}^{+}Y \lhd A^{\uparrow}, {}^{-}Y \circ A^{\cap} \rangle \preceq \langle {}^{+}B, {}^{-}B \rangle)$$
$$= (A^{\triangle} \preceq \mathbf{B}).$$

□

Corollary 2. *The system* $\{A \in L^X \mid A = A^{\triangle\triangledown}\}$ *is an* **L***-closure system on* $\langle L^X, S\rangle$*. The system* $\{\mathbf{B} \in L^Y \times L^Y \mid \mathbf{B} = \mathbf{B}^{\triangledown\triangle}\}$ *is an* **L***-closure system on* $\langle L^Y \times L^Y, \preceq\rangle$*.*

A pair $\langle A, \mathbf{B}\rangle$ is called a *formal concept* if

$$A^{\triangle} = \mathbf{B} \quad \text{and} \quad \mathbf{B}^{\triangledown} = A.$$

The set of all formal concepts in $\langle X, \mathbf{Y}, I\rangle$ is denoted by $\mathcal{B}^{\triangle\triangledown}(X, \mathbf{Y}, I)$. On $\mathcal{B}^{\triangle\triangledown}(X, \mathbf{Y}, I)$ we define the **L**-order $\trianglelefteq$:

$$\langle A_1, \mathbf{B}_1\rangle \trianglelefteq \langle A_2, \mathbf{B}_2\rangle = S(A_1, A_2)$$

for $\langle A_1, \mathbf{B}_1\rangle, \langle A_2, \mathbf{B}_2\rangle \in \mathcal{B}^{\triangle\triangledown}(X, \mathbf{Y}, I)$. Since

$$S(A_1, A_2) = S(A_1, \mathbf{B}_2^{\triangledown}) = (A_1^{\triangle} \preceq \mathbf{B}_2) = (\mathbf{B}_1 \preceq \mathbf{B}_2),$$

we can define the **L**-order $\trianglelefteq$ equivalently by:

$$\langle A_1, \mathbf{B}_1\rangle \trianglelefteq \langle A_2, \mathbf{B}_2\rangle = (\mathbf{B}_1 \preceq \mathbf{B}_2).$$

Theorem 6 ***(reduction to ⟨↑, ↓⟩-case).*** *Let* $\langle X, \mathbf{Y}, I\rangle$ *be an* **L***-context with positive attributes and negative attributes* ${}^{+}Y, {}^{-}Y \in L^Y$*. Define a* **L***-relation* $I' \in L^{X \times (Y \times L)}$ *as*

$$I'(x, \langle y, a\rangle) = \begin{cases} {}^{+}Y(y) \to I(x, y) & \text{if } a = 1, \\ {}^{-}Y(y) \to (I(x, y) \to a) & \text{otherwise}. \end{cases}$$

Then $\mathcal{B}^{\uparrow\downarrow}(X, Y \times L, I')$ *is isomorphic to* $\mathcal{B}^{\triangle\triangledown}(X, \mathbf{Y}, I)$ *and the corresponding isomorphism* $i : \mathcal{B}^{\uparrow\downarrow}(X, Y \times L, I') \to \mathcal{B}^{\triangle\triangledown}(X, \mathbf{Y}, I)$ *is given by*

$$i(\langle A, A^{\uparrow}\rangle) = \langle A, A^{\triangle}\rangle$$

for each $\langle A, A^{\uparrow}\rangle = \mathcal{B}^{\uparrow\downarrow}(X, Y \times L, I')$*.*

Proof. Let $\langle A, B\rangle \in \mathcal{B}^{\uparrow\downarrow}(X, Y \times L, I')$, then we have

$$\begin{aligned} A &= B^{\downarrow_{I'}} \\ &= \bigwedge_{\langle y,a\rangle \in Y \times L} (B(\langle y, a\rangle) \to I'(x, \langle y, a\rangle)) \\ &= \left(\bigwedge_{y \in Y} (B(\langle y, 1\rangle) \to ({}^{+}Y(y) \to I(x, y)))\right) \\ &\quad \wedge \left(\bigwedge_{\langle y,a\rangle \in Y \times L - \{1\}} (B(\langle y, a\rangle) \to ({}^{-}Y(y) \to (I(x, y) \to a)))\right). \end{aligned}$$

Denoting

$$B_1(y) = B(\langle y, 1\rangle) \quad \text{and} \quad B_2(y) = \bigwedge_{a \in L - \{1\}} (B(\langle y, a\rangle) \to a),$$

the first term can be written as

$$\begin{aligned} \textstyle\bigwedge_{y \in Y} (B_1(y) \to ({}^{+}Y(y) \to I(x, y))) &= \textstyle\bigwedge_{y \in Y} (({}^{+}Y(y) \otimes B_1(y)) \to I(x, y)) \\ &= \textstyle\bigwedge_{y \in Y} ((B_1 \circ {}^{+}Y)(y) \to I(x, y)) \\ &= (B_1 \circ {}^{+}Y)^{\downarrow}(x) \end{aligned}$$

and the second term can be written as

$$\begin{aligned} \bigwedge_{\langle y,a\rangle \in Y \times L - \{1\}} (B(\langle y, a\rangle) \to ({}^{-}Y(y) \to (I(x, y) \to a))) &= \bigwedge_{\langle y,a\rangle \in Y \times L - \{1\}} (I(x, y) \to ({}^{-}Y(y) \to (B(\langle y, a\rangle) \to a))) \\ &= \bigwedge_{y \in Y} \bigwedge_{a \in L - \{1\}} (I(x, y) \to ({}^{-}Y(y) \to (B(\langle y, a\rangle) \to a))) \end{aligned}$$

$$= \bigwedge_{y \in Y} (I(x,y) \to ({}^{-}Y(y) \to \bigwedge_{a \in L - \{1\}} (B(\langle y, a \rangle) \to a)))$$
$$= \bigwedge_{y \in Y} (I(x,y) \to ({}^{-}Y(y) \to B_2(y)))$$
$$= \bigwedge_{y \in Y} (I(x,y) \to ({}^{-}Y \lhd B_2)(y))$$
$$= ({}^{-}Y \lhd B_2)^{\cup}(x).$$

Putting it together, we have

$$A = (B_1 \circ {}^{+}Y)^{\downarrow} \cap ({}^{-}Y \lhd B_2)^{\cup} = \langle B_1, B_2 \rangle^{\triangledown}$$

showing that A is also an extent in $\mathcal{B}^{\vartriangle\triangledown}(X, \mathbf{Y}, I)$. One can use the same sequence of equalities backwards denoting $B(\langle x, 1 \rangle) = B_1(x)$ and $B(\langle x, a \rangle) = B_2(x) \to a$ for $a \in L - \{1\}$ (and using (8)). That shows that each extent in $\mathcal{B}^{\vartriangle\triangledown}(X, \mathbf{Y}, I)$ is an extent in $\mathcal{B}^{\uparrow\downarrow}(X, Y \times L, I')$ and proves $\mathrm{Ext}^{\uparrow\downarrow}(X, Y \times L, I') = \mathrm{Ext}^{\vartriangle\triangledown}(X, \mathbf{Y}, I)$. The statement of the theorem then easily follows from this fact. □

Theorem 7. *Let $\mathbb{K} = \langle X, \mathbf{Y}, I \rangle$ be an $\boldsymbol{L}$-context with positive attributes and negative attributes such that ${}^{+}Y, {}^{-}Y \in L^Y$.*

(a) $\langle \mathcal{B}^{\vartriangle\triangledown}(X, \mathbf{Y}, I), \trianglelefteq \rangle$ is a completely lattice $\boldsymbol{L}$-ordered set with suprema and infima defined as follows for $\boldsymbol{L}$-set $\mathcal{M} \in L^{\mathcal{B}^{\vartriangle\triangledown}(X, \mathbf{Y}, I)}$.

$$\inf(\mathcal{M}) = \langle \textstyle\bigcap_X \mathcal{M}, (\bigsqcap_{\mathbf{Y}} \mathcal{M})^{\triangledown\vartriangle} \rangle,$$
$$\sup(\mathcal{M}) = \langle (\textstyle\bigcup_X \mathcal{M})^{\vartriangle\triangledown}, \bigsqcup_{\mathbf{Y}} \mathcal{M} \rangle.$$

(b) Moreover, a completely lattice $\boldsymbol{L}$-ordered set $\boldsymbol{V} = \langle V, \sqsubseteq \rangle$ is isomorphic to $\langle \mathcal{B}^{\vartriangle\triangledown}(X, \mathbf{Y}, I), \trianglelefteq \rangle$ iff there are mappings

$$\tilde{\gamma} : X \times L \to V \quad \text{and} \quad \tilde{\mu} : Y \times L \dot{\times} L \to V,$$

where $L \dot{\times} L = (\{0\} \times L) \cup (L \times \{1\})$, such that $\tilde{\gamma}(X \times L)$ is $\{0, 1\}$-supremally dense in $\boldsymbol{V}$, $\tilde{\mu}(Y \times L \dot{\times} L)$ is $\{0, 1\}$-infimally dense in $\boldsymbol{V}$, and

$$((a \otimes b_1) \to {}^{+}I(x,y)) \wedge ({}^{-}I(x,y) \to (a \to b_2)) = (\tilde{\gamma}(x,a) \sqsubseteq \tilde{\mu}(y, b_1, b_2)) \tag{25}$$

for all $x \in X$, $y \in Y$, $a \in L$, $\langle b_1, b_2 \rangle \in L \dot{\times} L$.

Proof. (a): Follows directly from Theorem 6 and [2, Theorem 5.63].

(b, "$\Rightarrow$"): First, we suppose that $\mathbf{V}$ is isomorphic to $\mathcal{B}^{\vartriangle\triangledown}(X, \mathbf{Y}, I)$. From Theorem 6 we also have that there is isomorphism $i : \mathcal{B}^{\vartriangle\triangledown}(X, \mathbf{Y}, I) \to \mathcal{B}^{\uparrow\downarrow}(X, Y \times L, I')$ given by $i(\langle A, A^{\vartriangle} \rangle) = \langle A, A^{\uparrow} \rangle$. Therefore, it is sufficient to find mappings $\tilde{\gamma} : X \times L \to \mathcal{B}^{\uparrow\downarrow}(X, Y \times L, I')$ and $\tilde{\mu} : Y \times L \dot{\times} L \to \mathcal{B}^{\uparrow\downarrow}(X, Y \times L, I')$ of required properties. We define these mappings in the following way:

$$\tilde{\gamma} = \gamma \circ i, \text{where } \gamma : X \times L \to \mathcal{B}^{\vartriangle\triangledown}(X, \mathbf{Y}, I),$$
$$\gamma(x, a) = \langle \{{}^{a}/x\}^{\vartriangle\triangledown}, \{{}^{a}/x\}^{\vartriangle} \rangle,$$
$$\tilde{\mu} = \mu \circ i, \text{where } \mu : Y \times L \dot{\times} L \to \mathcal{B}^{\vartriangle\triangledown}(X, \mathbf{Y}, I),$$
$$\mu(y, b_1, b_2) = \langle \langle \{{}^{b_1}/y\}, \{{}^{b_2}/y\} \rangle^{\triangledown}, \langle \{{}^{b_1}/y\}, \{{}^{b_2}/y\} \rangle^{\triangledown\vartriangle} \rangle.$$

In addition, due to [2, Theorem 5.63] there exist mappings $\gamma' : X \times L \to \mathcal{B}^{\uparrow\downarrow}(X, Y \times L, I')$, $\mu' : Y \times L \times L \to \mathcal{B}^{\uparrow\downarrow}(X, Y \times L, I')$

$$\gamma'(x, a) = \langle \{{}^{a}/x\}^{\uparrow\downarrow}, \{{}^{a}/x\}^{\uparrow} \rangle,$$
$$\mu'(y, c, b) = \langle \{{}^{b}/\langle y, c \rangle\}^{\downarrow}, \{{}^{b}/\langle y, c \rangle\}^{\downarrow\uparrow} \rangle$$

such that $\gamma'(X \times L)$ is supremally dense in $\mathcal{B}^{\uparrow\downarrow}(X, Y \times L, I')$ and $\mu'(Y \times L \times L)$ is infimally dense in $\mathcal{B}^{\uparrow\downarrow}(X, Y \times L, I')$. To show that $\tilde{\gamma}(X \times L)$ is supremally dense in $\mathcal{B}^{\uparrow\downarrow}(X, Y \times L, I')$ and $\tilde{\mu}(X \times L \dot{\times} L)$ is infimally dense in $\mathcal{B}^{\uparrow\downarrow}(X, Y \times L, I')$ we just need to prove: (i) $\gamma'(X \times L) = \tilde{\gamma}(X \times L)$, and (ii) $\mu'(Y \times L \times L) = \tilde{\mu}(X \times L \dot{\times} L)$.

(i) Since the isomorphism i maps an extent to itself, it is sufficient to show that equality

$$\{\{{}^{a}/x\}^{\uparrow\downarrow} | x \in X, a \in L\} = \{\{{}^{a}/x\}^{\vartriangle\triangledown} | x \in X, a \in L\}.$$

holds true. Indeed, for every $x \in X$, $a \in L$ we have

$$\{{}^{a}/x\}^{\uparrow\downarrow}(x_0) = \bigwedge_{\langle y_0, c_0 \rangle \in Y \times L} (a \to I'(x, \langle y_0, c_0 \rangle)) \to I'(x_0, \langle y_0, c_0 \rangle)$$
$$= \left(\bigwedge_{y_0 \in Y} (a \to {}^{+}I(x, y_0)) \to {}^{+}I(x_0, y_0) \right)$$

$$\wedge\left(\bigwedge_{\langle y_0,c_0\rangle\in Y\times L\setminus\{1\}}(a\to({}^{-}I(x,y_0)\to c_0))\to({}^{-}I(x_0,y_0)\to c_0)\right)$$
$$=\{{}^{a}/x\}^{\uparrow_{+I}\downarrow_{+I}}(x_0)\wedge\{{}^{a}/x\}^{\cap_{-I}\cup_{-I}}(x_0)$$
$$=\{{}^{a}/x\}^{\triangle\triangledown}(x_0)$$

for all $x_0\in X$.

(ii) Similarly, we need to prove

$$\{\{{}^{b}/\langle y,c\rangle\}^{\downarrow}\,|\,y\in Y,b,c\in L\}=\{\langle\{{}^{b_1}/y\},\{{}^{b_2}/y\}\rangle^{\triangledown}\,|\,y\in Y,\langle b_1,b_2\rangle\in L\dot{\times}L\}.$$

For any $y\in Y$, b, $c\in L$ we have

$$\{{}^{b}/\langle y,c\rangle\}^{\downarrow}(x)=b\to I'(x,\langle y,c\rangle)=\begin{cases}b\to{}^{+}I(x,y) & \text{if } c=1,\\ {}^{-}I(x,y)\to(b\to c) & \text{otherwise.}\end{cases}\tag{26}$$

Now, we define $\langle b_1,b_2\rangle\in L\dot{\times}L$ such that $b_1=b$, $b_2=1$ if $c=1$, and $b_1=0$, $b_2=b\to c$ otherwise. Therefore, we can write

$$\{{}^{b}/\langle y,c\rangle\}^{\downarrow}(x)=\begin{cases}b_1\to{}^{+}I(x,y) & \text{if } b_2=1,\\ {}^{-}I(x,y)\to b_2 & \text{if } b_1=0,\end{cases}\tag{27}$$

so

$$\{{}^{b}/\langle y,c\rangle\}^{\downarrow}(x)=(b_1\to{}^{+}I(x,y))\wedge({}^{-}I(x,y)\to b_2)$$
$$=\{{}^{b_1}/y\}^{\downarrow_{+I}}(x)\wedge\{{}^{b_2}/x\}^{\cup_{-I}}(x)$$
$$=\langle\{{}^{b_1}/y\},\{{}^{b_2}/y\}\rangle^{\triangledown}(x)$$

for all $x\in X$.

Conversely, to get (26) from (27), we define $b=b_1$, $c=1$ if $b_2=1$; for $b_1=0$ we take arbitrary b, c such that $c\neq1$ and $b\to c=b_2$ (we can do that because for all $x\in X$ it holds $\{{}^{b}/\langle y,c\rangle\}^{\downarrow}(x)=\{{}^{b'}/\langle y,c'\rangle\}^{\downarrow}(x)$ provided that $c,c'\neq1$ and $b\to c=b'\to c'$).

Now, we prove identity (25). From [2, Theorem 5.63] we know that the mappings γ' and μ' satisfy condition $((a\otimes b)\to I'(x,\langle y,c\rangle))=(\gamma'(x,a)\sqsubseteq\mu'(y,c,b))$ for all $x\in X$, $y\in Y$, a, b, $c\in L$. By letting $b_1=b$, $b_2=1$ if $c=1$, and $b_1=0$, $b_2=b\to c$ otherwise, we obtain (by similar considerations as above)

$$(a\otimes b)\to I'(x,\langle y,c\rangle)=((a\otimes b_1)\to{}^{+}I(x,y))\wedge({}^{-}I(x,y)\to(a\to b_2)).$$

Moreover, we have seen above that for such of choice of b_1 and b_2, $\{{}^{a}/x\}^{\uparrow\downarrow}$ is an extent of $\gamma'(x,a)$ as well as of $\gamma(x,a)$, and $\{{}^{b}/\langle y,c\rangle\}^{\downarrow}$ is an extent of $\mu'(y,c,b)$ as well as of $\mu(y,b_1,b_2)$. Therefore, $\gamma'(x,a)=\tilde{\gamma}(x,a)$ and $\mu'(y,c,b)=\tilde{\mu}(y,b_1,b_2)$.

(b, "$\Leftarrow$"): Let $\tilde{\gamma}:X\times L\to V$ and $\tilde{\mu}:Y\times L\dot{\times}L\to V$ be mappings satisfying the conditions of the claim. We define mappings γ': $X\times L\to V$ and μ': $Y\times L\times L\to V$:

$$\gamma'(x,a)=\tilde{\gamma}(x,a),$$
$$\mu'(y,c,b)=\begin{cases}\tilde{\mu}(y,b,1) & \text{for } c=1,\\ \tilde{\mu}(y,0,b\to c) & \text{otherwise}\end{cases}$$

for all $x\in X$, $y\in Y$, a, b, $c\in L$. Using the properties of $\tilde{\gamma}$ and $\tilde{\mu}$ we get $\gamma'(X\times L)$ is supremally dense in $\mathbf{V}$, $\mu'(Y\times L\times L)$ is infimally dense in $\mathbf{V}$, and $((a\otimes b)\to I'(x,\langle y,c\rangle))=(\gamma'(x,a)\sqsubseteq\mu'(y,c,b))$ holds true for all $x\in X$, $y\in Y$, a, b, $c\in L$. Due to [2, Theorem 5.63] we immediately have that $\mathbf{V}$ is isomorphic to $\mathcal{B}^{\uparrow\downarrow}(X,Y\times L,I')$. Since $\mathcal{B}^{\uparrow\downarrow}(X,Y\times L,I')$ is isomorphic to $\mathcal{B}^{\triangle\triangledown}(X,\mathbf{Y},I)$ (see Theorem 6), the statement is proved. □

5. Fuzzy containment implications

In this section, we describe the second main output of formal concept analysis. We present a general logic of if-then rules $\mathbf{A}\Rightarrow\mathbf{B}$ for graded attributes which can be read: if all attributes of an object are contained in $\mathbf{A}$ then they are contained in $\mathbf{B}$. We introduce basic syntactic and semantic notions.

Let ${}^{+}Y$, ${}^{-}Y$ are sets of (symbols of) positive and negative graded attributes, respectively . An $\mathbf{L}$-containment implication in $\mathbf{Y}=\langle{}^{+}Y,{}^{-}Y\rangle$ is an expression

$$\mathbf{A}\Rightarrow\mathbf{B},$$

where $\mathbf{A},\mathbf{B}\in L^{+Y}\times L^{-Y}$.

Let $\mathbf{M}\in L^{+Y}\times L^{-Y}$. The degree $\|\mathbf{A}\Rightarrow\mathbf{B}\|_{\mathbf{M}}$ in which $\mathbf{A}\Rightarrow\mathbf{B}$ is valid in $\mathbf{M}$ is defined as

$$\|\mathbf{A}\Rightarrow\mathbf{B}\|_{\mathbf{M}}=(\mathbf{M}\preceq\mathbf{A})\to(\mathbf{M}\preceq\mathbf{B}).$$

Let $\mathcal{M} \subseteq L^{+Y} \times L^{-Y}$, the degree $\|\mathbf{A} \Rightarrow \mathbf{B}\|_{\mathcal{M}}$ in which $\mathbf{A} \Rightarrow \mathbf{B}$ is valid in $\mathcal{M}$ is defined as

$$\|\mathbf{A} \Rightarrow \mathbf{B}\|_{\mathcal{M}} = \textstyle\bigwedge_{\mathbf{M}\in\mathcal{M}} \|\mathbf{A} \Rightarrow \mathbf{B}\|_{\mathbf{M}}.$$

We are going to evaluate fuzzy containment implications in **L**-contexts with positive and negative attributes. Let $\langle X, \mathbf{Y}, I\rangle$ be an **L**-context with positive and negative attributes. The degree $\|\mathbf{A} \Rightarrow \mathbf{B}\|_{\langle X, \mathbf{Y}, I\rangle}$ in which $A \Rightarrow B$ is valid in $\langle X, \mathbf{Y}, I\rangle$ is defined as

$$\|\mathbf{A} \Rightarrow \mathbf{B}\|_{\langle X,\mathbf{Y},I\rangle} = \|\mathbf{A} \Rightarrow \mathbf{B}\|_{\{\mathbf{I}_x \,|\, x\in X\}}$$

where $\mathbf{I}_x = \langle {}^{+}I_x, {}^{-}I_x\rangle$ with ${}^{+}I_x(y) = {}^{+}I(x,y)$, ${}^{-}I_x(y) = {}^{-}I(x,y)$.

There is a close relationship between the semantics of fuzzy containment implications, the concept-forming operators $\langle\triangle, \triangledown\rangle$, and the lattices of their fixpoints which were studied in the previous chapters. The following theorem shows a basic connection between them—the validity of $\mathbf{A} \Rightarrow \mathbf{B}$ in a context coincides with the validity of $\mathbf{A} \Rightarrow \mathbf{B}$ in the system of its intents and with the degree of containment of $\mathbf{A}^{\triangledown\triangle}$ in $\mathbf{B}$.

Theorem 8. *Let $\langle X, \mathbf{Y}, I\rangle$ be an **L**-context with positive and negative attributes and let $\mathbf{A} \Rightarrow \mathbf{B}$ be a fuzzy containment implication in $\mathbf{Y}$. We have*

$$\|\mathbf{A} \Rightarrow \mathbf{B}\|_{\langle X,\mathbf{Y},I\rangle} = \|\mathbf{A} \Rightarrow \mathbf{B}\|_{\mathrm{Int}^{\triangle\triangledown}(X,\mathbf{Y},I)} \tag{28}$$

$$= S(\mathbf{A}^{\triangledown}, \mathbf{B}^{\triangledown}) \tag{29}$$

$$= (\mathbf{A}^{\triangledown\triangle} \preceq \mathbf{B}). \tag{30}$$

Proof. First, we prove (29). Note that for each $x \in X, \mathbf{A} \in L^{+Y} \times L^{-Y}, I \in X \times Y$ we have

$$(\mathbf{I}_x \preceq \mathbf{A}) = \mathbf{A}^{\triangledown}(x). \tag{31}$$

Indeed, we have

$$\begin{aligned}
(\mathbf{I}_x \preceq \mathbf{A}) &= S({}^{+}A, {}^{+}I_x) \wedge S({}^{-}I_x, {}^{-}A)\\
&= \Big(\bigwedge_{y\in{}^{+}Y} ({}^{+}A(y) \to {}^{+}I_x(y))\Big) \wedge \Big(\bigwedge_{y\in{}^{-}Y} ({}^{-}I_x(y) \to {}^{-}A(y))\Big)\\
&= \Big(\bigwedge_{y\in{}^{+}Y} ({}^{+}A(y) \to {}^{+}I(x,y))\Big) \wedge \Big(\bigwedge_{y\in{}^{-}Y} \Big({}^{-}I(x,y) \to {}^{-}A(y)\Big)\Big)\\
&= {}^{+}A^{\downarrow}(x) \wedge {}^{-}A^{\cup}(x)\\
&= ({}^{+}A^{\downarrow} \cap {}^{-}A^{\cup})(x)\\
&= \mathbf{A}^{\triangledown}(x).
\end{aligned}$$

Using (31) we can prove (29).

$$\begin{aligned}
\|\mathbf{A} \Rightarrow \mathbf{B}\|_{\langle X,\mathbf{Y},I\rangle} &= \|\mathbf{A} \Rightarrow \mathbf{B}\|_{\{\langle {}^{+}I_x, {}^{-}I_x\rangle \,|\, x\in X\}}\\
&= \bigwedge_{x\in X} \|\mathbf{A} \Rightarrow \mathbf{B}\|_{\mathbf{I}_x}\\
&= \bigwedge_{x\in X} (\mathbf{I}_x \preceq \mathbf{A}) \to (\mathbf{I}_x \preceq \mathbf{B})\\
&= \bigwedge_{x\in X} (\mathbf{A}^{\triangledown}(x) \to \mathbf{B}^{\triangledown}(x))\\
&= S(\mathbf{A}^{\triangledown}, \mathbf{B}^{\triangledown}).
\end{aligned}$$

Now we prove (28). Since $\{\mathbf{I}_x \,|\, x \in X\} \subseteq \mathrm{Int}^{\triangle\triangledown}(X, \mathbf{Y}, I)$ we have

$$\|\mathbf{A} \Rightarrow \mathbf{B}\|_{\langle X,\mathbf{Y},I\rangle} \geq \|\mathbf{A} \Rightarrow \mathbf{B}\|_{\mathrm{Int}^{\triangle\triangledown}(X,\mathbf{Y},I)}.$$

We need to show that for any $\mathbf{M} \in \mathrm{Int}^{\triangle\triangledown}(X, \mathbf{Y}, I)$ we have

$$\|\mathbf{A} \Rightarrow \mathbf{B}\|_{\langle X,\mathbf{Y},I\rangle} \leq (\mathbf{M} \preceq \mathbf{A}) \to (\mathbf{M} \preceq \mathbf{B}).$$

Since $\mathbf{M} \in \mathrm{Int}^{\triangle\triangledown}(X, \mathbf{Y}, I)$, we have $\mathbf{M} = \mathbf{M}^{\triangledown\triangle}$ and

$$(\mathbf{M} \preceq \mathbf{A}) = (\mathbf{M}^{\triangledown\triangle} \preceq \mathbf{A}) = S(\mathbf{M}^{\triangledown}, \mathbf{A}^{\triangledown}),$$

we have to show that for any $\mathbf{M} \in \mathrm{Int}^{\triangle\triangledown}(X, \mathbf{Y}, I)$

$$\|\mathbf{A} \Rightarrow \mathbf{B}\|_{\langle X,\mathbf{Y},I\rangle} \leq S(\mathbf{M}^{\triangledown}, \mathbf{A}^{\triangledown}) \to S(\mathbf{M}^{\triangledown}, \mathbf{B}^{\triangledown}).$$

By (29) it is equivalent to

$$S(\mathbf{A}^{\triangledown}, \mathbf{B}^{\triangledown}) \leq S(\mathbf{M}^{\triangledown}, \mathbf{A}^{\triangledown}) \rightarrow S(\mathbf{M}^{\triangledown}, \mathbf{B}^{\triangledown}).$$

which is equivalent to

$$S(\mathbf{M}^{\triangledown}, \mathbf{A}^{\triangledown}) \otimes S(\mathbf{A}^{\triangledown}, \mathbf{B}^{\triangledown}) \leq S(\mathbf{M}^{\triangledown}, \mathbf{B}^{\triangledown}),$$

which is known to be true. □

6. Conclusions

We described an extension of formal concept analysis where two sets of positive and negative attributes are selected by a user. These two types of attributes were handled using antitone concept-forming operators and isotone concept-forming operators, respectively. The two main outputs of formal concept analysis, i.e. concept lattices and attribute implications, in this setting were described.

Our future research in this area includes study of intercontextual information in form of bonds, reduction via block relations, and further study of fuzzy containment implications.

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E Rough Fuzzy Concept Analysis

[4] Eduard Bartl and Jan Konecny. Rough fuzzy concept analysis. *Fundamenta Informaticae*, 156(2):141–168, 2017.

We provide a new approach to the fusion of FCA with graded affirmation and denials and Rough Set Theory (RST). As a starting point we consider affirmations to represent the lower approximation, while the denials the upper approximation of a given input. Using the combination of concept-forming operators (46), we transfer the roughness of the input to the roughness of corresponding formal fuzzy concepts in the sense that a formal fuzzy concept is considered as a collection of objects accompanied with two fuzzy sets of attributes—those which are shared by all the objects and those which at least one object has. In the paper we study the properties of such formal concepts and show their relationship with concepts formed by well-known isotone and antitone operators.

We also demonstrate use RST inspired reduction of size of concept lattices based on equivalences induced by attributes and show that this reduction is natural, i.e. it preserves extents.

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Rough Fuzzy Concept Analysis

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Abstract. We provide a new approach to fusion of Fuzzy Formal Concept Analysis and Rough Set Theory. As a starting point we take into account a couple of fuzzy relations, one of them represents the lower approximation, while the other one the upper approximation of a given data table. By defining appropriate concept-forming operators we transfer the roughness of the input data table to the roughness of corresponding formal fuzzy concepts in the sense that a formal fuzzy concept is considered as a collection of objects accompanied with two fuzzy sets of attributes—those which are shared by all the objects and those which at least one object has. In the paper we study the properties of such formal concepts and show their relationship with concepts formed by well-known isotone and antitone operators.

Keywords: antitone concept-forming operator, concept lattice, formal concept analysis, Galois connection, isotone concept-forming operator, rough set, truth-depressing hedge, truth-stressing hedge

1. Introduction

We provide a new approach to fusion of Fuzzy Formal Concept Analysis (fuzzy FCA) and Rough Set Theory (RST).

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Rough Set Theory originated by Pawlak [57] is emerging powerful methodology used to study information systems characterized by inexact, uncertain or vague information with applications in the field of artificial intelligence such as pattern recognition, machine learning, and automated knowledge acquisition. A pair of unary set-theoretic operators, called approximation operators, is usually based on an equivalence relation on a set of objects called the universe. A concept, represented by a set of objects, is called a definable concept if its lower and upper approximations are equal to the set itself. An arbitrary concept is approximated from below and above by two definable concepts—lower and upper approximation, respectively. Pawlak's rough set model can be generalized to fuzzy environment to deal with quantitative data, the results are called rough fuzzy sets and fuzzy rough sets. Dubois and Prade [24, 25] were the first to generalize RST into fuzzy setting and many followed.

Formal Concept Analysis [21, 27] is a method of relational data analysis identifying interesting clusters (formal concepts) in a collection of objects and their attributes, and organizing them into a structure called concept lattice.

FCA was proved to be useful for information retrieval, software engineering, and data mining. The formal concept in FCA is obtained as a fixed point of so-called concept-forming operators and is characterized by a pair of sets—extent and intent. The extent contains all objects covered by the concept and intent contains all attributes covered by the concept. Numerous generalizations of FCA, which allow to work with graded data, were provided; see [42] and references therein. In the present paper we stick with approach of [6, 43].

In a graded (fuzzy) setting, two main kinds of concept-forming operators—antitone and isotone one—were studied [7, 28, 43, 44], compared [13, 14], and even covered under a unifying framework [9, 39]. Many researchers [38, 46, 49, 53] (see also the survey [42]) studied a fusion of (fuzzy) FCA and RST and observed that intents generated by antitone concept-forming operators behave like lower approximations and intents generated by isotone concept-forming operators behave like upper approximations in RST. Yao [55] mentions that the two kinds of operators represent two expreme cases in describing a set of objects based on their properties. To the best of our knowledge, no approach have considered a combination of the concept-forming operators to obtain both upper and lower approximation in one concept.

In the present paper we provide such combination. A concept in our setting is a triple consisting of a (fuzzy) set of objects, a (fuzzy) set of attributes they share, and a (fuzzy) set of attributes possessed by at least one of the object. The later two serve as the lower and upper approximation, respectively. We provide concept-forming operators for these concepts and describe their properties.

2. Preliminaries

In this section we summarize the basic notions used in the paper.

Residuated Lattices, L-set and L-relations

We use complete residuated lattices as basic structures of truth-degrees. A complete residuated lattice [6, 30, 50] is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, i.e. a partially ordered set in which arbitrary infima and suprema exist; $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation which is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$;

$\otimes$ and $\rightarrow$ satisfy adjointness, i.e. $a \otimes b \leqslant c$ iff $a \leqslant b \rightarrow c$. 0 and 1 denote the least and greatest elements. The partial order of $\mathbf{L}$ is denoted by $\leqslant$. Throughout this work, $\mathbf{L}$ denotes an arbitrary complete residuated lattice.

Elements of L are called truth degrees. Operations $\otimes$ (multiplication) and $\rightarrow$ (residuum) play the role of (truth functions of) "fuzzy conjunction" and "fuzzy implication". Furthermore, we define the complement of $a \in L$ as $\neg a = a \rightarrow 0$.

An $\mathbf{L}$-set (or fuzzy set) A in a universe set X is a mapping assigning to each $x \in X$ some truth degree $A(x) \in L$. The set of all $\mathbf{L}$-sets in a universe X is denoted $\mathbf{L}^X$.

The operations with $\mathbf{L}$-sets are defined componentwise. For instance, the intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^X$ is an $\mathbf{L}$-set $A \cap B$ in X such that $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in X$. An $\mathbf{L}$-set $A \in \mathbf{L}^X$ is also denoted $\{^{A(x)}/x \mid x \in X\}$. If for all $y \in X$ distinct from $x_1, \ldots, x_n$ we have $A(y) = 0$, we also write $\{^{A(x_1)}/x_1, \ldots, {}^{A(x_n)}/x_n\}$.

An $\mathbf{L}$-set $A \in \mathbf{L}^X$ is called normal if there is $x \in X$ such that $A(x) = 1$. An $\mathbf{L}$-set $A \in \mathbf{L}^X$ is called crisp if $A(x) \in \{0, 1\}$ for each $x \in X$. Crisp $\mathbf{L}$-sets can be identified with ordinary sets. For a crisp A, we also write $x \in A$ for $A(x) = 1$ and $x \notin A$ for $A(x) = 0$.

For $A, B \in \mathbf{L}^X$ we define the degree of inclusion of A in B by

$$S(A, B) = \bigwedge_{x \in X} A(x) \rightarrow B(x). \tag{1}$$

Graded inclusion generalizes the classical inclusion relation. Described verbally, $S(A, B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A, B) = 1$. As a consequence, we have $A \subseteq B$ iff $A(x) \leqslant B(x)$ for each $x \in X$.

Binary $\mathbf{L}$-relations (binary fuzzy relations) between X and Y can be thought of as $\mathbf{L}$-sets in the universe $X \times Y$. That is, a binary $\mathbf{L}$-relation $I \in \mathbf{L}^{X \times Y}$ between a set X and a set Y is a mapping assigning to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$ (a degree to which x and y are related by I). For $\mathbf{L}$-relation $I \in \mathbf{L}^{X \times Y}$ we define its inverse $I^{-1} \in \mathbf{L}^{Y \times X}$ as $I^{-1}(y, x) = I(x, y)$ for all $x \in X$, $y \in Y$.

The compositions of $\mathbf{L}$-relations $I \in \mathbf{L}^{X \times Y}$ and $J \in \mathbf{L}^{Y \times Z}$ are defined by

$$(I \circ J)(x, z) = \bigvee_{y \in Y} I(x, y) \otimes J(y, z),$$
$$(I \triangleleft J)(x, z) = \bigwedge_{y \in Y} I(x, y) \rightarrow J(y, z),$$
$$(I \triangleright J)(x, z) = \bigwedge_{y \in Y} J(y, z) \rightarrow I(x, y)$$

for every $x \in X, z \in Z$.

We will utilize following associativity properties of the compositions of $\mathbf{L}$-relations.

Lemma 2.1. ([6])
For $I \in \mathbf{L}^{X \times Y}, J \in \mathbf{L}^{Y \times Z}, K \in \mathbf{L}^{Z \times W}$

$$(I \circ J) \circ K = I \circ (J \circ K) \quad \text{and} \quad (I \triangleright J) \triangleright K = I \triangleright (J \circ K).$$

A binary $\mathbf{L}$-relation E on X is called an $\mathbf{L}$-equivalence if for any $x, y, z \in X$ it satisfies $E(x, x) = 1$ (reflexivity), $E(x, y) = E(y, x)$ (symmetry), $E(x, y) \otimes E(y, z) \leqslant E(x, z)$ (transitivity).

We say that an $\mathbf{L}$-relation $R \in \mathbf{L}^{X\times Y}$ is compatible w.r.t. $\mathbf{L}$-equivalence $E \in \mathbf{L}^{Y\times Y}$

$$R(x, y_1) \otimes E(y_1, y_2) \leqslant R(x, y_2)$$

for any $x \in X, y_1, y_2 \in Y$. Analogously, an $\mathbf{L}$-set $B \in \mathbf{L}^{Y}$ is compatible w.r.t. $E \in \mathbf{L}^{Y\times Y}$ if

$$B(y_1) \otimes E(y_1, y_2) \leqslant B(y_2)$$

for any $y_1, y_2 \in Y$.

Leibniz $\mathbf{L}$-equivalence induced by an $\mathbf{L}$-relation $R \in L^{X\times Y}$ is defined as follows

$$E_R(y_1, y_2) = \bigwedge_{x\in X} R(x, y_1) \leftrightarrow R(x, y_2) \tag{2}$$

for each $y_1, y_2 \in Y$.

Formal Concept Analysis

For a comprehensive background on FCA, we refer to [27]. The input data to FCA consists of a table with rows and columns representing objects and attributes, respectively. An entry representing object x and attribute y contains $\times$ (or 1) if x has y (y applies to x), otherwise the entry contains a blank (or 0). Formally, such a table is represented by a triplet $\langle X, Y, I\rangle$, called a *formal context*, in which I is a binary relation between X and Y and $\langle x, y\rangle \in I$ means that the object x has the attribute y. For every set $A \subseteq X$ of objects in X denote by $A^{\uparrow}$ a subset of attributes in Y defined by

$$A^{\uparrow} = \{y \in Y \mid \text{for each } x \in A : \langle x, y\rangle \in I\}.$$

Similarly, for $B \subseteq Y$ denote by $B^{\downarrow}$ a subset of X defined by

$$B^{\downarrow} = \{x \in X \mid \text{for each } y \in B : \langle x, y\rangle \in I\}.$$

A *formal concept* of $\langle X, Y, I\rangle$ is a pair $\langle A, B\rangle$ of $A \subseteq X$ and $B \subseteq Y$ satisfying $A^{\uparrow} = B$ and $B^{\downarrow} = A$. That is, a formal concept consists of a set A of objects and a set B of attributes such that A is the set of all objects in X sharing all attributes in B and, conversely, B is the collection of all attributes in Y shared by all objects from A. A and B are called the *extent* and the *intent* of $\langle A, B\rangle$, respectively. This definition formalizes the traditional approach to concepts which is due to Port-Royal logic [33]. As an example, consider the concept *dog*: its extent is the collection of all dogs, while its intent is the collection of all attributes of dogs such as “has four legs”, “barks”, etc. The set

$$\mathcal{B}(X, Y, I) = \{\langle A, B\rangle \mid A^{\uparrow} = B, B^{\downarrow} = A\}$$

of all formal concepts of $\langle X, Y, I\rangle$, called the *concept lattice of I,* can be equipped with a partial order $\leqslant$ defined by $\langle A_1, B_1\rangle \leqslant \langle A_2, B_2\rangle$ if $A_1 \subseteq A_2$ (or, equivalently, $B_2 \subseteq B_1$). Therefore, $\leqslant$ represents a *subconcept-superconcept hierarchy* due to which *dog* is a subconcept of *mammal*, etc. $\mathcal{B}(X, Y, I)$ happens to be a concept lattice whose structure is described by the so-called basic theorem of concept lattices [27].

	α	β	γ	δ
A	0.5	0	1	0
B	1	0.5	1	0.5
C	0	0.5	0.5	0.5
D	0.5	0.5	1	0.5

Figure 1. Example of $\mathbf{L}$-context with objects A,B,C,D and attributes $\alpha, \beta, \gamma, \delta$.

Formal Concept Analysis in Fuzzy Setting

There are several approaches to generalize formal concept analysis to be able to process such indeterminancy or uncertainty [7, 43, 37, 32, 20]. Many of them are based on Zadeh's theory of fuzzy sets [56]. We follow the approach of [7].

An $\mathbf{L}$-context is a triplet $\langle X, Y, I\rangle$ where X and Y are (ordinary) sets and $I \in \mathbf{L}^{X\times Y}$ is an $\mathbf{L}$-relation between X and Y. Elements of X are called objects, elements of Y are called attributes, I is called an incidence relation. $I(x, y) = a$ is read: "The object x has the attribute y to degree a." An $\mathbf{L}$-context may be displayed as a table with the objects corresponding to the rows of the table, the attributes corresponding to the columns of the table and $I(x, y)$ written in cells of the table (for an example see Fig. 1).

Consider the following pairs of operators induced by an $\mathbf{L}$-context $\langle X, Y, I\rangle$. First, the pair $\langle\uparrow,\downarrow\rangle$ of operators $^{\uparrow} : \mathbf{L}^X \to \mathbf{L}^Y$ and $^{\downarrow} : \mathbf{L}^Y \to \mathbf{L}^X$ is defined by

$$\begin{aligned} A^{\uparrow}(y) &= \bigwedge_{x\in X} A(x) \to I(x, y), \\ B^{\downarrow}(x) &= \bigwedge_{y\in Y} B(y) \to I(x, y). \end{aligned} \tag{3}$$

Second, the pair $\langle\cap,\cup\rangle$ of operators $^{\cap} : \mathbf{L}^X \to \mathbf{L}^Y$ and $^{\cup} : \mathbf{L}^Y \to \mathbf{L}^X$ is defined by

$$\begin{aligned} A^{\cap}(y) &= \bigvee_{x\in X} A(x) \otimes I(x, y), \\ B^{\cup}(x) &= \bigwedge_{y\in Y} I(x, y) \to B(y). \end{aligned} \tag{4}$$

To emphasize that the operators are induced by I, we also denote the operators by $\langle\uparrow_I,\downarrow_I\rangle$ and $\langle\cap_I,\cup_I\rangle$.

We will utilize the following properties of concept-forming operators and their compositions.

Lemma 2.2. ([12])
Let $\langle X, F, I\rangle$ and $\langle F, Y, J\rangle$ be $\mathbf{L}$-contexts, let $A \in \mathbf{L}^X$, $B \in \mathbf{L}^Y$. We have

$$A^{\cap_{I\circ J}} = A^{\cap_I\cap_J}, \qquad B^{\cup_{I\circ J}} = B^{\cup_J\cup_I}, \tag{5}$$

$$A^{\uparrow_{I\triangleleft J}} = A^{\cap_I\uparrow_J}, \qquad B^{\downarrow_{I\triangleleft J}} = B^{\downarrow_J\cup_I}. \tag{6}$$

Fixpoints of these operators are called formal concepts. The set of all formal concepts (along with set inclusion) forms a complete lattice, called **L**-concept lattice. We denote the sets of all concepts (as well as the corresponding **L**-concept lattice) by $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$ and $\mathcal{B}^{\cap\cup}(X, Y, I)$, i.e.

$$\begin{aligned} \mathcal{B}^{\uparrow\downarrow}(X, Y, I) &= \{\langle A, B\rangle \in \mathbf{L}^X \times \mathbf{L}^Y \mid A^{\uparrow} = B,\ B^{\downarrow} = A\}, \\ \mathcal{B}^{\cap\cup}(X, Y, I) &= \{\langle A, B\rangle \in \mathbf{L}^X \times \mathbf{L}^Y \mid A^{\cap} = B,\ B^{\cup} = A\} \end{aligned} \tag{7}$$

For an **L**-concept lattice $\mathcal{B}^*(X, Y, I)$, where $\mathcal{B}^*$ is either $\mathcal{B}^{\uparrow\downarrow}$ or $\mathcal{B}^{\cap\cup}$, denote the corresponding sets of extents and intents by $\mathrm{Ext}(X, Y, I)$ and $\mathrm{Int}(X, Y, I)$. That is,

$$\begin{aligned} \mathrm{Ext}^*(X, Y, I) &= \{A \in \mathbf{L}^X \mid \langle A, B\rangle \in \mathcal{B}(X, Y, I) \text{ for some } B\}, \\ \mathrm{Int}^*(X, Y, I) &= \{B \in \mathbf{L}^Y \mid \langle A, B\rangle \in \mathcal{B}(X, Y, I) \text{ for some } A\}. \end{aligned} \tag{8}$$

When displaying **L**-concept lattices, we use labeled Hasse diagrams to include all the information on extents and intents. In $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$, for any $x \in X$, $y \in Y$ and formal **L**-concept $\langle A, B\rangle$ we have $A(x) \geqslant a$ and $B(y) \geqslant b$ if and only if there is a formal concept $\langle A_1, B_1\rangle \leqslant \langle A, B\rangle$, labeled by ${}^{a}\!/x$ and a formal concept $\langle A_2, B_2\rangle \geqslant \langle A, B\rangle$, labeled by ${}^{b}\!/y$. We use labels x resp. y instead of ${}^{1}\!/x$ resp. ${}^{1}\!/y$ and omit redundant labels (i.e., if a concept has both the labels ${}^{a}\!/x$ and ${}^{b}\!/x$ then we keep only that with the greater degree; dually for attributes). The whole structure of concept lattice $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$ can be determined from the labeled diagram using the results from [7] (see also [6]).

In $\mathcal{B}^{\cap\cup}(X, Y, I)$, for any $x \in X$, $y \in Y$ and formal **L**-concept $\langle A, B\rangle$ we have $A(x) \geqslant a$ and $B(y) \leqslant b$ if and only if there is a formal concept $\langle A_1, B_1\rangle \leqslant \langle A, B\rangle$, labeled by ${}^{a}\!/x$ and a formal concept $\langle A_2, B_2\rangle \geqslant \langle A, B\rangle$, labeled by ${}^{b}\!/y$ (see examples depicted in Fig. 2).

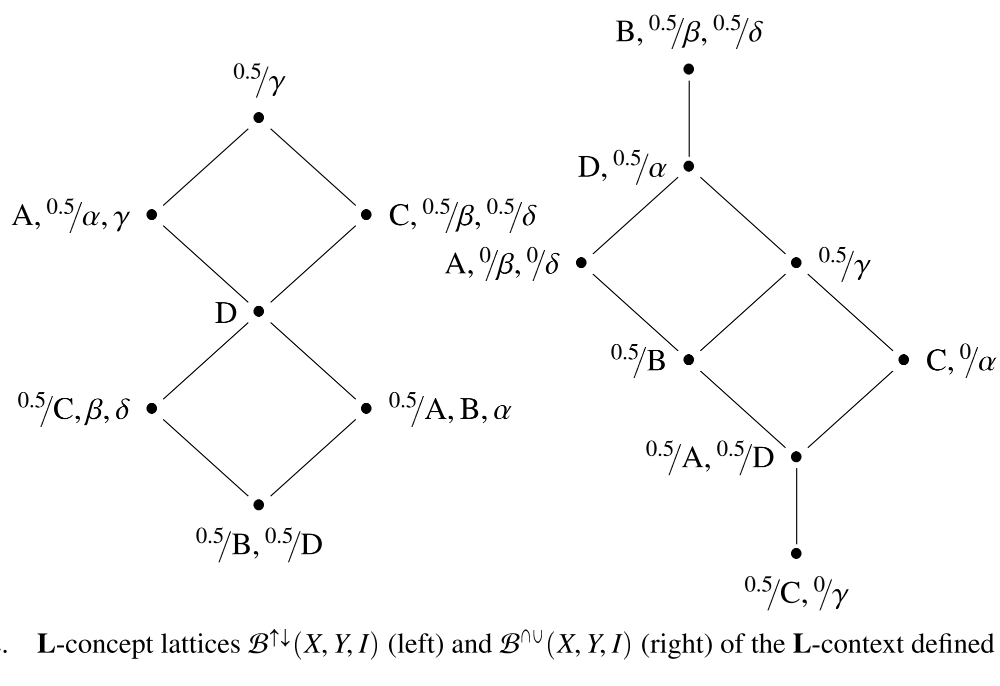

Figure 2. **L**-concept lattices $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$ (left) and $\mathcal{B}^{\cap\cup}(X, Y, I)$ (right) of the **L**-context defined in Fig. 1.

An $(\mathbf{L}_1, \mathbf{L}_2)$-Galois connection between the sets X and Y is a pair $\langle f, g\rangle$ of mappings $f : \mathbf{L}_1^X \to \mathbf{L}_2^Y$, $g : \mathbf{L}_2^Y \to \mathbf{L}_1^X$, satisfying

$$S(A, g(B)) = S(B, f(A)) \tag{9}$$

for every $A \in \mathbf{L}_1^X, B \in \mathbf{L}_2^Y$.

One can easily observe that the couple $\langle \uparrow, \downarrow\rangle$ forms an $(\mathbf{L}, \mathbf{L})$-Galois connection between X and Y, while $\langle \cap, \cup\rangle$ forms an $(\mathbf{L}, \mathbf{L}^{-1})$-Galois connection between X and Y.

Rough Set Theory

For a comprehensive background on Rough Set Theory (RST), we refer to [41]. Pawlak introduced RST where uncertain elements are approximated with respect to an equivalence relation representing indiscernibility. Formally, given *Pawlak approximation space* $\langle U, E\rangle$, where U is a non-empty set of elements (*universe*) and E is an equivalence relation on U, the *rough approximation* of a crisp set $A \subseteq U$ by E is the pair $\langle A^{\Downarrow_E}, A^{\Uparrow_E}\rangle$ of sets in U defined by

$$\begin{aligned} x \in A^{\Downarrow_E} \quad &\text{iff} \quad \text{for all } y \in U,\ \langle x, y\rangle \in E \text{ implies } y \in A,\\ x \in A^{\Uparrow_E} \quad &\text{iff} \quad \text{there exists } y \in U \text{ such that } \langle x, y\rangle \in E \text{ and } y \in A. \end{aligned}$$

The $A^{\Downarrow_E}$ and $A^{\Uparrow_E}$ are called *lower and upper approximation* of the set A by E, respectively (see Fig. 3). A pair $\langle A_1, A_2\rangle$ of sets in U is called a *rough set* in $\langle U, E\rangle$ if there is a set A in U such that $A_1 = A^{\Downarrow_E}$ and $A_2 = A^{\Uparrow_E}$ (c.f. [41]).

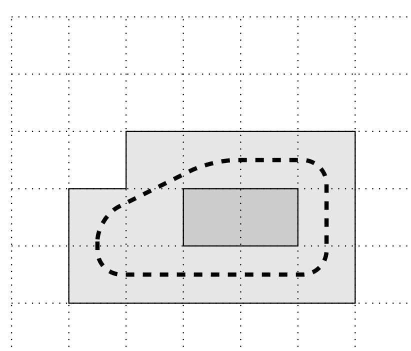

Figure 3. Schema of a rough set: grid represents the equivalence classes of E, dashed line represents an approximated set A, the dark gray area is its lower approximation $A^{\Downarrow_E}$, and union of light gray area and dark gray area is its upper approximation $A^{\Uparrow_E}$.

Pawlak approximation space $\langle U, E\rangle$ may be too restricting for some applications. Thefore, a number of various generalizations of the setting have been considered, including E being a tolerance on U or even a general relation between some sets [47, 54, 58].

Rough Set Theory in Fuzzy Setting

In the fuzzy setting, we consider E to be an $\mathbf{L}$-equivalence on U and generalize $\langle A^{\Downarrow_E}, A^{\Uparrow_E}\rangle$ as in [24, 25, 45],

$$\begin{aligned} A^{\Downarrow_E}(x) &= \bigwedge_{y\in U}(E(x,y) \to A(y)),\\ A^{\Uparrow_E}(x) &= \bigvee_{y\in U}(A(y)\otimes E(x,y)). \end{aligned} \tag{10}$$

That is, ${}^{\Downarrow_E}$ is equivalent to ${}^{\cup_E}$, and ${}^{\Uparrow_E}$ is equivalent to ${}^{\cap_E}$; i.e.

$$\langle \Downarrow_E, \Uparrow_E\rangle = \langle \cup_E, \cap_E\rangle. \tag{11}$$

Note that for $\mathbf{L}$-set A, $A^{\Downarrow_E}$ is its largest subset compatible[1] with E and $A^{\Uparrow_E}$ is its smallest superset compatible with E. Similarly, for $\mathbf{L}$-relation I, $I \triangleright E$ is the largest subrelation of I compatible with E, and $I \circ E$ is the smallest superrelation of I compatible with E. Approximations of $\mathbf{L}$-sets induced by a $\mathbf{L}$-relations were also studied in [19, 26].

For handling an generalization of rough sets, we introduce the following notation. By $\mathbf{L}^{-1}$ we denote L with dual lattice order; we do not describe its properties (they can be easily derived from properties of $\mathbf{L}$). In this paper we use $\mathbf{L}^{-1}$ in order to describe and grasp $\mathbf{L}$-rough sets in an easier, unified way.

Let $\langle U, \underline{E}, \overline{E}\rangle$ be an $\mathbf{L}$-*approximation space*; i.e. U is a universe, $\underline{E}$ and $\overline{E}$ are $\mathbf{L}$-equivalences on U.[2] An $\mathbf{L}$-*rough set* in an $\mathbf{L}$-approximation space $\langle U, \underline{E}, \overline{E}\rangle$ is a pair of $\mathbf{L}$-sets $\langle \underline{A}, \overline{A}\rangle \in (\mathbf{L}\times\mathbf{L}^{-1})^U$ such that $\underline{A}$ is compatible with $\underline{E}$ and $\overline{A}$ is compatible with $\overline{E}$. The set $\underline{A}$ is called a *lower approximation* and the set $\overline{A}$ is called an *upper approximation*. Analogously, $\mathbf{L}$-rough relations are $(\mathbf{L}\times\mathbf{L}^{-1})$-sets in $X \times Y$ compatible with $\underline{E}$ and $\overline{E}$ (in the same way as above).

The set operations are defined componentwise, i.e.

$$\begin{aligned} \bigcap_{i\in I}\langle \underline{A}, \overline{A}\rangle &= \langle \bigcap_{i\in I}\underline{A}, {\bigcap_{i\in I}}^{-1}\overline{A}\rangle = \langle \bigcap_{i\in I}\underline{A}, \bigcup_{i\in I}\overline{A}\rangle,\\ \bigcup_{i\in I}\langle \underline{A}, \overline{A}\rangle &= \langle \bigcup_{i\in I}\underline{A}, {\bigcup_{i\in I}}^{-1}\overline{A}\rangle = \langle \bigcup_{i\in I}\underline{A}, \bigcap_{i\in I}\overline{A}\rangle. \end{aligned}$$

Similarly, the graded subsethood is then applied componentwise

$$S(\langle \underline{A}, \overline{A}\rangle, \langle \underline{B}, \overline{B}\rangle) = S(\underline{A}, \underline{B}) \wedge S^{-1}(\overline{A}, \overline{B}) = S(\underline{A}, \underline{B}) \wedge S(\overline{B}, \overline{A})$$

and the crisp subsethood is then defined using the graded subsethood:

$$\langle \underline{A}, \overline{A}\rangle \subseteq \langle \underline{B}, \overline{B}\rangle \text{ iff } S(\langle \underline{A}, \overline{A}\rangle, \langle \underline{B}, \overline{B}\rangle) = 1, \text{ iff } \underline{A} \subseteq \underline{B} \text{ and } \overline{B} \subseteq \overline{A}.$$

An $\mathbf{L}$-rough set $\langle \underline{A}, \overline{A}\rangle$ is called *natural* if $\underline{A} \subseteq \overline{A}$.

[1]In terms of Rough Set Theory, compatible sets are called definable sets.

[2]We assume setting with different equivalence for lower and upper approximation similarly as in [29].

Remark 2.3.

(a) Naturally, we want the lower approximation to be a subset of the upper approximation. Nonetheless, the results covered in the present paper work well even for non-natural **L**-rough sets and **L**-rough relations.

(b) Concept lattices we study in this paper contain concepts with both natural and non-natural intents. The presence of non-natural intents can be seen as inconvenient for practical applications. Fortunately, the concept lattice can be cut to two iceberg lattices[3]—one containing only concepts with natural intents, one containing only concepts with non-natural intents; we show this in Section 4. Figures 5 and 8 depict this cut.

3. L-rough contexts and L-rough concept lattices

In our approach we identify the generalized **L**-approximation space $\langle X, Y, \underline{I}, \overline{I}\rangle$ with a tabular data. That is, X and Y are sets of objects and attributes, respectively, and $\underline{I}$, $\overline{I}$ represent lower and upper approximations of an incidence relation, respectively. The meaning of $\underline{I}$ and $\overline{I}$ is as follows: $\underline{I}(x, y)$ (resp. $\overline{I}(x, y)$) is the truth degree to which the object x surely (resp. possibly) has the attribute y. The quadruple $\langle X, Y, \underline{I}, \overline{I}\rangle$ is called an **L**-*rough context*.

The **L**-rough context induces two operators defined as follows.

Definition 3.1. Let $\langle X, Y, \underline{I}, \overline{I}\rangle$ be an **L**-rough context. Define **L**-*rough concept-forming operators* as

$$\begin{aligned} A^{\triangle} &= \langle A^{\uparrow_{\underline{I}}}, A^{\cap_{\overline{I}}}\rangle, \\ \langle \underline{B}, \overline{B}\rangle^{\triangledown} &= \underline{B}^{\downarrow_{\underline{I}}} \cap \overline{B}^{\cup_{\overline{I}}} \end{aligned} \tag{12}$$

for $A \in \mathbf{L}^X, \underline{B}, \overline{B} \in \mathbf{L}^Y$. Fixed points of $\langle \triangle, \triangledown\rangle$, i.e. tuples $\langle A, \langle \underline{B}, \overline{B}\rangle\rangle \in \mathbf{L}^X \times (\mathbf{L} \times \mathbf{L}^{-1})^Y$ such that $A^{\triangle} = \langle \underline{B}, \overline{B}\rangle$ and $\langle \underline{B}, \overline{B}\rangle^{\triangledown} = A$, are called **L**-*rough concepts*. The $\underline{B}$ and $\overline{B}$ are called *lower intent approximation* and *upper intent approximation*, respectively.

The interpretation of $A^{\triangle}$ is "the set of all attributes surely shared by all objects in A and the set all attributes which at least one object possibly has"; while the interpretation of $\langle \underline{B}, \overline{B}\rangle^{\triangledown}$ is "the set of all objects, such that they surely share all attributes in $\underline{B}$ and have no other attributes than those in $\overline{B}$."

To simplify notation in this paper, we declare that in what follows, the operators $\uparrow, \downarrow$ are always induced by $\underline{I}$ and $\cap, \cup$ are always induced by $\overline{I}$; unless stated otherwise. Also, we identify $\langle A, \langle \underline{B}, \overline{B}\rangle\rangle$ with $\langle A, \underline{B}, \overline{B}\rangle$. Fig. 4 depicts schema of an **L**-rough concept for $L = \{0, 1\}$.

Theorem 3.2. The pair $\langle \triangle, \triangledown\rangle$ of **L**-rough concept-forming operators is a $(\mathbf{L}, \mathbf{L} \times \mathbf{L}^{-1})$-Galois connection between X and Y.

[3]By *iceberg lattice* we mean a meet- or a join-semilattice accompanied by the greatest or the lowest element, respectively.

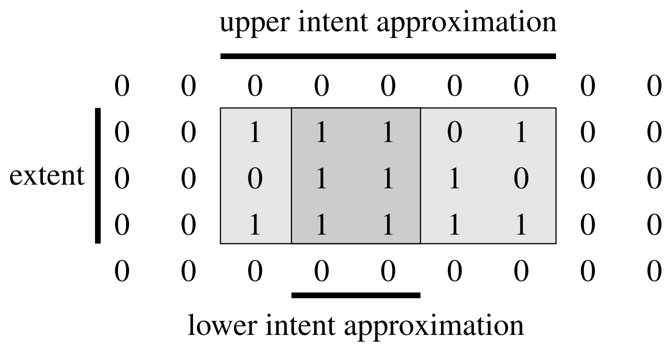

Figure 4. Schema of an **L**-rough concept (in the case $L = \{0, 1\}$).

Proof:
We use the fact, that $\langle\uparrow, \downarrow\rangle$ is an $(\mathbf{L}, \mathbf{L})$-Galois connection between X and Y, and $\langle\cap, \cup\rangle$ is an $(\mathbf{L}, \mathbf{L}^{-1})$-Galois connection between X and Y: For each $A \in \mathbf{L}^X, \langle \underline{B}, \overline{B}\rangle \in (\mathbf{L} \times \mathbf{L}^{-1})^Y$ we have

$$\begin{aligned} S(A, \langle \underline{B}, \overline{B}\rangle^{\triangledown}) &= \bigwedge_{x\in X} \left(A(x) \rightarrow \left(\underline{B}^{\downarrow}(x) \wedge \overline{B}^{\cup}(x)\right)\right) \\ &= \bigwedge_{x\in X} \left(A(x) \rightarrow \underline{B}^{\downarrow}(x)\right) \wedge \bigwedge_{x\in X} \left(A(x) \rightarrow \overline{B}^{\cup}(x)\right) \\ &= S(A, \underline{B}^{\downarrow}) \wedge S(A, \overline{B}^{\cup}) \\ &= S(\underline{B}, A^{\uparrow}) \wedge S(A^{\cap}, \overline{B}) \\ &= S(\langle \underline{B}, \overline{B}\rangle, A^{\vartriangle}). \end{aligned}$$

□

From Theorem 3.2 we can obtain following properties [6].

Corollary 3.3. *The* **L**-*rough concept-forming operators satisfy the following properties for each* $A, A_1, A_2 \in \mathbf{L}^X, \langle \underline{B}, \overline{B}\rangle, \langle \underline{B_1}, \overline{B_1}\rangle, \langle \underline{B_2}, \overline{B_2}\rangle \in (\mathbf{L} \times \mathbf{L}^{-1})^Y$:

(a) $A \subseteq A^{\vartriangle\triangledown}$ and $\langle \underline{B}, \overline{B}\rangle \subseteq \langle \underline{B}, \overline{B}\rangle^{\triangledown\vartriangle}$,

(b) $S(A_1, A_2) \leqslant S(A_2^{\vartriangle}, A_1^{\vartriangle})$ and $S(\langle \underline{B_1}, \overline{B_1}\rangle, \langle \underline{B_2}, \overline{B_2}\rangle) \leqslant S(\langle \underline{B_2}, \overline{B_2}\rangle^{\triangledown}, \langle \underline{B_1}, \overline{B_1}\rangle^{\triangledown})$,

(c) $A_1 \subseteq A_2$ implies $A_2^{\vartriangle} \subseteq A_1^{\vartriangle}$,
and $\langle \underline{B_1}, \overline{B_1}\rangle \subseteq \langle \underline{B_2}, \overline{B_2}\rangle$ implies $\langle \underline{B_2}, \overline{B_2}\rangle^{\triangledown} \subseteq \langle \underline{B_1}, \overline{B_1}\rangle^{\triangledown}$,

(d) $A^{\vartriangle} = A^{\vartriangle\triangledown\vartriangle}$ and $\langle \underline{B}, \overline{B}\rangle^{\triangledown} = \langle \underline{B}, \overline{B}\rangle^{\triangledown\vartriangle\triangledown}$.

We denote the set of all fixed-points of $\langle\vartriangle, \triangledown\rangle$, analogously to (7), as $\mathcal{B}^{\vartriangle\triangledown}(X, Y, \underline{I}, \overline{I})$ and we call it **L**-*rough concept lattice*. We define a partial order $\leqslant$ on $\mathcal{B}^{\vartriangle\triangledown}(X, Y, \underline{I}, \overline{I})$ by means of subsethood of extents; i.e.

$$\langle A_1, \underline{B_1}, \overline{B_1}\rangle \leqslant \langle A_2, \underline{B_2}, \overline{B_2}\rangle \quad \text{iff} \quad A_1 \subseteq A_2 \tag{13}$$

for each $\langle A_1, \underline{B_1}, \overline{B_1}\rangle, \langle A_2, \underline{B_2}, \overline{B_2}\rangle \in \mathcal{B}^{\vartriangle\triangledown}(X, Y, \underline{I}, \overline{I})$. By Corollary 3.3 (c,d), we can equivalently define $\leqslant$ by means of subsethood of intents

$$\langle A_1, \underline{B_1}, \overline{B_1}\rangle \leqslant \langle A_2, \underline{B_2}, \overline{B_2}\rangle \quad \text{iff} \quad \langle \underline{B_2}, \overline{B_2}\rangle \subseteq \langle \underline{B_1}, \overline{B_1}\rangle$$
$$\text{(i.e. iff } \underline{B_2} \subseteq \underline{B_1} \text{ and } \overline{B_1} \subseteq \overline{B_2}\text{).}$$

Later in this section, we state an analogy of so-called Main theorem on concept lattices [27] saying, among others, that $\mathcal{B}^{\vartriangle\triangledown}(X, Y, \underline{I}, \overline{I})$ is a complete lattice. That justifies the word 'lattice' in the name '**L**-rough concept lattice.' We prove the Main theorem by showing that the present setting is covered by the general framework introduced in [8] (the framework is summarized in Appendix A). The theorem then follows from results on Cartesian representation for general concept lattices published in [15].

Theorem 3.4. Consider the aggregation structure $\langle \mathbf{L}, \mathbf{L}\times\mathbf{L}^{-1}, \mathbf{L}\times\mathbf{L}^{-1}, \square\rangle$ with $\square$ defined by

$$a \,\square\, \langle \underline{b}, \overline{b}\rangle = \langle a \otimes \underline{b}, a \rightarrow \overline{b}\rangle$$

for each $a, \underline{b}, \overline{b} \in L$, and let $\langle X, Y, \underline{I}, \overline{I}\rangle$ be a $(\mathbf{L}\times\mathbf{L}^{-1})$-context. Then the pair of derived operators $\langle ⓐ, ⓥ\rangle$ is equal to **L**-rough concept-forming operators $\langle \vartriangle, \triangledown\rangle$.

Proof:
From the related definitions; compare Definition 3.1 and Definition 6.2 in Appendix A. □

Theorem 3.5. (Main theorem on L-rough concept lattices)

(a) **L**-rough concept lattice $\mathcal{B}^{\vartriangle\triangledown}(X, Y, \underline{I}, \overline{I})$ is a complete lattice with suprema and infima defined as follows

$$\bigwedge_i \langle A_i, \underline{B}_i, \overline{B}_i\rangle = \langle \bigcap_i A_i, \langle \bigcup_i \underline{B}_i, \bigcap_i \overline{B}_i\rangle^{\triangledown\vartriangle}\rangle,$$
$$\bigvee_i \langle A_i, \underline{B}_i, \overline{B}_i\rangle = \langle (\bigcup_i A_i)^{\vartriangle\triangledown}, \bigcap_i \underline{B}_i, \bigcup_i \overline{B}_i\rangle.$$

(b) Moreover, a complete lattice $\mathbf{V} = \langle V, \leqslant\rangle$ is isomorphic to $\mathcal{B}^{\vartriangle\triangledown}(X, Y, \underline{I}, \overline{I})$ iff there are mappings

$$\gamma : X \times L \rightarrow V \quad \text{and} \quad \mu : Y \times L \times L \rightarrow V$$

such that $\gamma(X \times L)$ is supremally dense in $\mathbf{V}$, $\mu(Y \times L \times L)$ is infimally dense in $\mathbf{V}$, and

$$a \otimes \underline{b} \leqslant \underline{I}(x, y) \text{ and } \overline{I}(x, y) \leqslant a \rightarrow \overline{b} \quad \text{is equivalent to} \quad \gamma(x, a) \leqslant \mu(y, \underline{b}, \overline{b})$$

for all $x \in X, y \in Y, a, \underline{b}, \overline{b} \in L$.

Proof:
It is a specification of Theorem from Appendix A. □

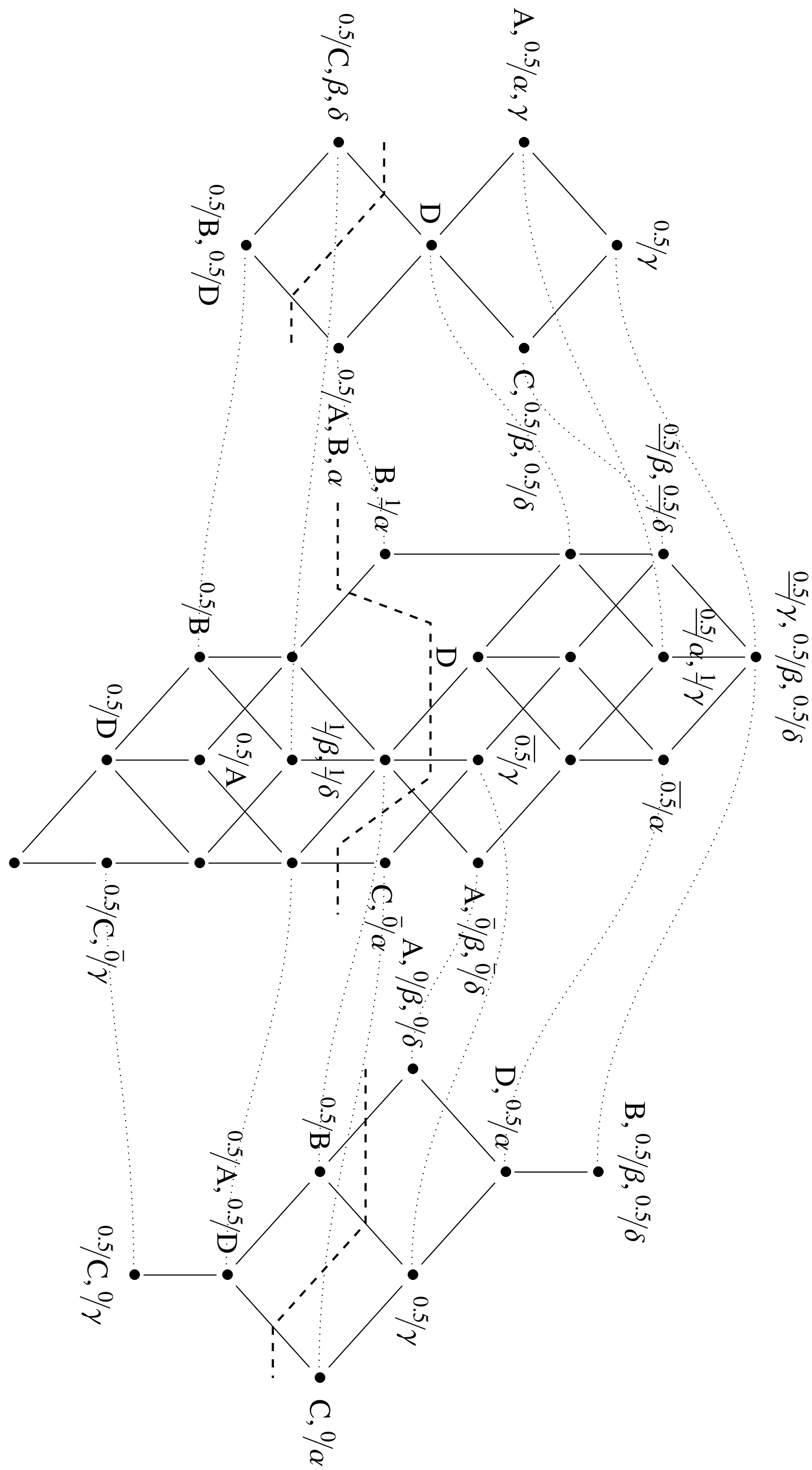

Figure 5. $\mathcal{B}^{\Delta\nabla}(X,Y,I,I)$ (middle) and positions of original concepts in $\mathcal{B}^{\uparrow\downarrow}(X,Y,I)$ (left) and $\mathcal{B}^{\cap\cup}(X,Y,I)$ (right) with **L** being a three-element Łukasiewicz chain. Dashed lines separate natural concept (above the lines) and non-natural concepts (below the lines).

When drawing an $\mathbf{L}$-rough concept lattice we label nodes as in $\mathcal{B}^{\uparrow\downarrow}$ for lower intent approximations and $\mathcal{B}^{\cap\cup}$ for upper intent approximations. We write ${}^{\underline{a}}/y$ or ${}^{\overline{a}}/y$ instead of just ${}^{a}/y$ to distinguish them. Fig. 5 (middle) shows an $\mathbf{L}$-rough concept lattice for the $\mathbf{L}$-context from Fig. 1.

The next theorem says, that if $\langle \underline{I}, \overline{I} \rangle$ is a natural $\mathbf{L}$-rough relation, then the concept-forming operator ${}^{\triangle}$ maps normal $\mathbf{L}$-sets to natural $\mathbf{L}$-rough sets.

Theorem 3.6. For natural $\mathbf{L}$-rough relation $\langle \underline{I}, \overline{I} \rangle$ we have if $A \in \mathbf{L}^X$ is normal $\mathbf{L}$-set then $A^{\triangle}$ is natural $\mathbf{L}$-rough set.

Proof:
Since A is normal, there is $x \in X$ such that $A(x) = 1$. Then we have $A^{\uparrow}(y) = \bigwedge_{x \in X} A(x) \to \underline{I}(x,y) \leqslant A(x) \to \underline{I}(x,y) = \underline{I}(x,y) \leqslant \overline{I}(x,y) = A(x) \otimes \overline{I}(x,y) \leqslant \bigvee_{x \in X} A(x) \otimes \overline{I}(x,y) = A^{\cap}(y)$ for each $y \in Y$. □

Directly from Theorem 3.6 we have the following corollary

Corollary 3.7. For an $\mathbf{L}$-rough context with natural $\langle \underline{I}, \overline{I} \rangle$ we have that normal extents have natural intents.

It is worth nothing, that a converse of Theorem 3.6 does not hold true generally as the following example shows.

Example 3.8. Assume $\mathbf{L}$-rough context with one object x and one attribute y, $\mathbf{L}$being three-element Łukasiewicz chain, and $\underline{I}(x,y) = 0, \overline{I}(x,y) = 1$. Now consider

$$\begin{aligned} \langle \{{}^{0.5}/y\}, \{{}^{0.5}/y\} \rangle^{\triangledown}(x) &= (\{{}^{0.5}/y\}^{\downarrow} \cap \{{}^{0.5}/y\}^{\cup})(x) \\ &= (\{{}^{0.5}/y\}(y) \to \underline{I}(x,y)) \wedge (\overline{I}(x,y) \to \{{}^{0.5}/y\}(y)) \\ &= (0.5 \to 0) \wedge (1 \to 0.5) = 0.5. \end{aligned}$$

Therefore, the natural $\mathbf{L}$-rough set is mapped to the non-normal $\mathbf{L}$-set.

Since $\langle \triangle, \triangledown \rangle$ is defined via $\langle \uparrow, \downarrow \rangle$ and $\langle \cap, \cup \rangle$, one can expect that there is a strong relationship between the associated concept lattices. In the rest of this section, we examine such relationship.

Theorem 3.9. For $\mathcal{S} \subseteq \mathbf{L}^X$, let $[\mathcal{S}]$ denote an $\mathbf{L}$-closure span of $\mathcal{S}$, i.e. the smallest $\mathbf{L}$-closure system containing $\mathcal{S}$. We have

$$[\mathrm{Ext}^{\uparrow\downarrow}(X, Y, \underline{I}) \cup \mathrm{Ext}^{\cap\cup}(X, Y, \overline{I})] = \mathrm{Ext}^{\triangle\triangledown}(X, Y, \underline{I}, \overline{I}).$$

Proof:
"$\supseteq$": Let $A \in \mathrm{Ext}^{\triangle\triangledown}(X, Y, \underline{I}, \overline{I})$ and let $\langle \underline{B}, \overline{B} \rangle = A^{\triangle}$. Then we have $A = \underline{B}^{\downarrow} \cap \overline{B}^{\cup} \in [\mathrm{Ext}^{\uparrow\downarrow}(X, Y, \underline{I}) \cup \mathrm{Ext}^{\cap\cup}(X, Y, \overline{I})]$ since $\underline{B}^{\downarrow} \in \mathrm{Ext}^{\uparrow\downarrow}(X, Y, \underline{I})$ and $\overline{B}^{\cup} \in \mathrm{Ext}^{\cap\cup}(X, Y, \overline{I})$.

"$\subseteq$": Let $A \in \mathrm{Ext}^{\uparrow\downarrow}(X, Y, \underline{I})$. Then $A = A \cap X = \langle A^{\uparrow}, Y \rangle^{\triangledown} \in \mathrm{Ext}^{\triangle\triangledown}(X, Y, \underline{I}, \overline{I})$. Similarly for $A \in \mathrm{Ext}^{\cap\cup}(X, Y, \overline{I})$. □

From Theorem 3.9 one can also observe that no extent from $\mathrm{Ext}^{\uparrow\downarrow}(X,Y,\underline{I})$ and $\mathrm{Ext}^{\cap\cup}(X,Y,\overline{I})$ is lost.

Corollary 3.10. $\mathrm{Ext}^{\uparrow\downarrow}(X,Y,\underline{I}) \subseteq \mathrm{Ext}^{\Delta\nabla}(X,Y,\underline{I},\overline{I})$ and $\mathrm{Ext}^{\cap\cup}(X,Y,\overline{I}) \subseteq \mathrm{Ext}^{\Delta\nabla}(X,Y,\underline{I},\overline{I})$.

In addition, no concept from $\mathcal{B}^{\uparrow\downarrow}(X,Y,\underline{I})$ and $\mathcal{B}^{\cap\cup}(X,Y,\overline{I})$ is lost.

Corollary 3.11. For each $\langle A,\underline{B}\rangle \in \mathcal{B}^{\uparrow\downarrow}(X,Y,\underline{I})$ there is $\langle A,\underline{B},A^{\cap}\rangle \in \mathcal{B}^{\Delta\nabla}(X,Y,\underline{I},\overline{I})$. For each $\langle A,\overline{B}\rangle \in \mathcal{B}^{\cap\cup}(X,Y,\overline{I})$ there is $\langle A,A^{\uparrow},\overline{B}\rangle \in \mathcal{B}^{\Delta\nabla}(X,Y,\underline{I},\overline{I})$.

Moreover, $\mathcal{B}^{\Delta\nabla}(X,Y,\underline{I},\emptyset)$ is isomorphic to $\mathcal{B}^{\uparrow\downarrow}(X,Y,\underline{I})$ and $\langle A,\underline{B},\emptyset\rangle \mapsto \langle A,\underline{B}\rangle$ is the isomorphism.

$\mathcal{B}^{\Delta\nabla}(X,Y,X\times Y,\overline{I})$ is isomorphic to $\mathcal{B}^{\cap\cup}(X,Y,\overline{I})$ and $\langle A,Y,\overline{B}\rangle \mapsto \langle A,\overline{B}\rangle$ is the isomorphism.

Remark 3.12. One can observe from Fig. 5 that in $\mathrm{Ext}^{\Delta\nabla}(X,Y,\underline{I},\overline{I})$ there exist extents which are present neither in $\mathrm{Ext}^{\uparrow\downarrow}(X,Y,\underline{I})$ nor in $\mathrm{Ext}^{\cap\cup}(X,Y,\overline{I})$. On the other hand, upper intent approximations are exactly those from $\mathrm{Int}^{\cap\cup}(X,Y,\overline{I})$ and lower intent approximations are exactly those from $\mathrm{Int}^{\uparrow\downarrow}(X,Y,\underline{I})$.

With results on mutual reducibility from [14] we can state the following theorem on representation of $\mathcal{B}^{\Delta\nabla}$ by $\mathcal{B}^{\uparrow\downarrow}$.

Theorem 3.13. For $\langle X,Y,\underline{I},\overline{I}\rangle$, consider $\mathbf{L}$-context $\langle X,Y\times L,J\rangle$ where J is defined by

$$J(x,\langle y,a\rangle) = \begin{cases} \underline{I}(x,y) & \text{if } a=1,\\ \overline{I}(x,y)\rightarrow a & \text{otherwise.}\end{cases}$$

Then we have that $\mathcal{B}^{\uparrow\downarrow}(X,Y\times L,J)$ is isomorphic to $\mathcal{B}^{\Delta\nabla}(X,Y,\underline{I},\overline{I})$ as a lattice.
In addition,

$$\mathrm{Ext}^{\uparrow\downarrow}(X,Y\times L,J) = \mathrm{Ext}^{\Delta\nabla}(X,Y,\underline{I},\overline{I}).$$

Theorem 3.13 shows how we can obtain an $\mathbf{L}$-concept lattice formed by $\langle\uparrow,\downarrow\rangle$ which is isomorphic to $\mathbf{L}$-rough concept lattice of given $\mathbf{L}$-rough context. On the other hand, from results in [13] we can state that there exist such $\mathbf{L}$-rough contexts that are not isomorphic to any $\mathbf{L}$-concept lattice formed by $\langle\cap,\cup\rangle$.

	$\langle\alpha,1\rangle$	$\langle\beta,1\rangle$	$\langle\gamma,1\rangle$	$\langle\delta,1\rangle$	$\langle\alpha,0.5\rangle$	$\langle\beta,0.5\rangle$	$\langle\gamma,0.5\rangle$	$\langle\delta,0.5\rangle$	$\langle\alpha,0\rangle$	$\langle\beta,0\rangle$	$\langle\gamma,0\rangle$	$\langle\delta,0\rangle$
A	0.5	0	1	0	1	1	0.5	1	0.5	1	0	1
B	1	0.5	1	0.5	0.5	1	0.5	1	0	0.5	0	0.5
C	0	0.5	0.5	0.5	1	1	1	1	1	0.5	0.5	0.5
D	0.5	0.5	1	0.5	1	1	0.5	1	0.5	0.5	1	0.5

Figure 6. $\mathbf{L}$-context J from Theorem 3.13 of the $\mathbf{L}$-rough context $\langle X,Y,\underline{I},\overline{I}\rangle$ from Fig. 1.

Example 3.14. For illustration, in Fig. 6 we show a representation described in Theorem 3.13 of **L**-rough context $\langle X, Y, I, I\rangle$ from Fig. 1.

The last result in this section is representation theorem on rough **L**-concept lattices.

Theorem 3.15. The **L**-rough concept lattice $\mathcal{B}^{\triangle\triangledown}(X, Y, \underline{I}, \overline{I})$ is isomorphic to the ordinary concept lattice $\mathcal{B}^{\uparrow\downarrow}(X \times L, Y \times L \times L, I^{\times})$ where

$$\langle\langle x, a\rangle, \langle y, \underline{b}, \overline{b}\rangle\rangle \in I^{\times} \text{ iff } a \otimes \underline{b} \leqslant \underline{I}(x, y) \text{ and } a \to \overline{b} \geqslant \overline{I}(x, y).$$

Proof:
It is a specification of Theorem from Appendix A. □

4. Rough approximation of L-rough concept lattices

In this section we study an application of **L**-equivalences to influence both granularity and size of **L**-rough concept lattices.

First we show that for any $A \in L^X$, the $A^{\triangle}$ is an **L**-rough set in **L**-approximation space $\langle Y, E_{\underline{I}}, E_{\overline{I}}\rangle$, where $E_{\underline{I}}$ and $E_{\overline{I}}$ are Leibniz **L**-equivalences induced by $\underline{I}$ and $\overline{I}$, respectively.

Theorem 4.1. Let $\langle X, Y, \underline{I}, \overline{I}\rangle$ be an **L**-rough context, $\langle \underline{B}, \overline{B}\rangle$ be its arbitrary intent, and $E_{\underline{I}}, E_{\overline{I}}$ be the equivalences induced by $\underline{I}$ and $\overline{I}$, respectively. Then $\underline{B}$ is compatible with $E_{\underline{I}}$ and $\overline{B}$ is compatible with $E_{\overline{I}}$.

Proof:
Let A be the extent corresponding to $\langle \underline{B}, \overline{B}\rangle$, i.e. $A = \langle \underline{B}, \overline{B}\rangle^{\triangledown}$. We have

$$\begin{aligned}
\underline{B}(y_1) \otimes E_{\underline{I}}(y_1, y_2) &= A^{\uparrow}(y_1) \otimes E_{\underline{I}}(y_1, y_2) \\
&= \Big(\bigwedge_{x\in X} A(x) \to \underline{I}(x, y_1)\Big) \otimes \Big(\bigwedge_{x\in X} \underline{I}(x, y_1) \leftrightarrow \underline{I}(x, y_2)\Big) \\
&\leqslant \bigwedge_{x\in X} (A(x) \to \underline{I}(x, y_1)) \otimes (\underline{I}(x, y_1) \leftrightarrow \underline{I}(x, y_2)) \\
&\leqslant \bigwedge_{x\in X} (A(x) \to \underline{I}(x, y_1)) \otimes (\underline{I}(x, y_1) \to \underline{I}(x, y_2)) \\
&\leqslant \bigwedge_{x\in X} A(x) \to \underline{I}(x, y_2) = \underline{B}(y_2).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\overline{B}(y_1) \otimes E_{\overline{I}}(y_1, y_2) &= A^{\cap}(y_1) \otimes E_{\overline{I}}(y_1, y_2) \\
&= \Big(\bigvee_{x \in X} A(x) \otimes \overline{I}(x, y_1)\Big) \otimes \Big(\bigwedge_{x \in X} \overline{I}(x, y_1) \leftrightarrow \overline{I}(x, y_2)\Big) \\
&\leqslant \bigvee_{x \in X} A(x) \otimes \overline{I}(x, y_1) \otimes (\overline{I}(x, y_1) \leftrightarrow \overline{I}(x, y_2)) \\
&\leqslant \bigvee_{x \in X} A(x) \otimes \overline{I}(x, y_1) \otimes (\overline{I}(x, y_1) \leftrightarrow \overline{I}(x, y_2)) \\
&\leqslant \bigvee_{x \in X} A(x) \otimes \overline{I}(x, y_1) \otimes (\overline{I}(x, y_1) \rightarrow \overline{I}(x, y_2)) \\
&\leqslant \bigvee_{x \in X} A(x) \otimes \overline{I}(x, y_2) = \overline{B}(y_2).
\end{aligned}$$

□

Remark 4.2. [6] considers **L**-equivalences induced by formal **L**-contexts and standard **L**-concept lattices, and showed that they are equal. Analogous equality can be proved for object-oriented and attribute-oriented concept lattices [22, 35].

As **L**-rough concept lattices combine standard and attribute-oriented concept lattices, this property is naturally inherited. We demonstrate that in this section.

L-rough concept lattice $\mathcal{B}^{\Delta\nabla}(X, Y, \underline{I}, \overline{I})$ induces **L**-equivalences $\underline{E}_{\mathrm{Int}^{\Delta\nabla}}, \overline{E}_{\mathrm{Int}^{\Delta\nabla}}$ on Y as follows:

$$\begin{aligned}
\underline{E}_{\mathrm{Int}^{\Delta\nabla}}(y_1, y_2) &= \bigwedge_{\langle \underline{B}, \overline{B}\rangle \in \mathrm{Int}^{\Delta\nabla}(X,Y,\underline{I},\overline{I})} \underline{B}(y_1) \leftrightarrow \underline{B}(y_2), \\
\overline{E}_{\mathrm{Int}^{\Delta\nabla}}(y_1, y_2) &= \bigwedge_{\langle \underline{B}, \overline{B}\rangle \in \mathrm{Int}^{\Delta\nabla}(X,Y,\underline{I},\overline{I})} \overline{B}(y_1) \leftrightarrow \overline{B}(y_2).
\end{aligned}$$

In words, two attributes are similar w.r.t. $\underline{E}_{\mathrm{Int}^{\Delta\nabla}}$ (resp. $\overline{E}_{\mathrm{Int}^{\Delta\nabla}}$) if they are not separated by any lower (resp. upper) intent approximation in $\mathcal{B}^{\Delta\nabla}(X, Y, \underline{I}, \overline{I})$.

For **L**-rough context $\langle X, Y, \underline{I}, \overline{I}\rangle$ we have

$$\underline{E}_{\mathrm{Int}^{\Delta\nabla}} = E_{\underline{I}} \quad \text{and} \quad \overline{E}_{\mathrm{Int}^{\Delta\nabla}} = E_{\overline{I}}.$$

Remark 4.3. Clearly, in the case $\underline{I} = \overline{I}$ we also have $E_{\underline{I}} = E_{\overline{I}}$; whence we obtain the setting with a single **L**-relation for both lower and upper approximation.

As we have $\underline{I} = \underline{I} \triangleright E_{\underline{I}}$ and $\overline{I} = \overline{I} \circ E_{\overline{I}}$, we can assume the **L**-rough context to be $\langle X, Y, \underline{I} \triangleright E_{\underline{I}}, \overline{I} \circ E_{\overline{I}}\rangle$ and consider replacing the two induced **L**-equivalences $E_{\underline{I}}, E_{\overline{I}}$ with rougher ones to influence granularity of intents.

In this section we consider **L**-rough contexts $\langle X, Y, \underline{I} \triangleright \underline{E}, \overline{I} \circ \overline{E}\rangle$ where $\underline{E} \supseteq E_{\underline{I}}$, $\overline{E} \supseteq E_{\overline{I}}$. We will denote $E = \langle \underline{E}, \overline{E}\rangle$.

Obviously, the **L**-rough concept-forming operators (denoted here as $\langle \triangle_E, \triangledown_E \rangle$) then become

$$\begin{aligned} A^{\triangle_E} &= \langle A^{\uparrow_{\underline{I} \triangleright \underline{E}}}, A^{\cap_{\overline{I} \circ \overline{E}}} \rangle, \\ \langle \underline{B}, \overline{B} \rangle^{\triangledown_E} &= \underline{B}^{\downarrow_{\underline{I} \triangleright \underline{E}}} \cap \overline{B}^{\cup_{\overline{I} \circ \overline{E}}}. \end{aligned} \tag{14}$$

For this special case, we denote **L**-rough concept lattice and associated set of its extents and intents by $\mathcal{B}^{\triangle_E \triangledown_E}(X, Y, \underline{I}, \overline{I})$, $\mathrm{Ext}^{\triangle_E \triangledown_E}(X, Y, \underline{I}, \overline{I})$ and $\mathrm{Int}^{\triangle_E \triangledown_E}(X, Y, \underline{I}, \overline{I})$, respectively.

As before, we want to know how the roughness of E influences the corresponding **L**-rough concepts. In other words, we are interested in description of **L**-rough concept-forming operators $\langle \triangle_E, \triangledown_E \rangle$ using operators (10).

Theorem 4.4. Let $\langle X, Y, \underline{I}, \overline{I} \rangle$ be an **L**-context, $E = \langle \underline{E}, \overline{E} \rangle$ be a pair of **L**-equivalences on Y. Then

$$\begin{aligned} A^{\triangle_E} &= \langle A^{\uparrow_{\underline{I}} \Downarrow_E}, A^{\cap_{\overline{I}} \Uparrow_E} \rangle, \\ \langle \underline{B}, \overline{B} \rangle^{\triangledown_E} &= \underline{B}^{\Uparrow_E \downarrow_{\underline{I}}} \cap \overline{B}^{\Downarrow_E \cup_{\overline{I}}}. \end{aligned}$$

In addition, if $\langle \underline{B}, \overline{B} \rangle \in \mathrm{Int}^{\triangle \triangledown}(X, Y, \underline{I}, \overline{I})$, then $\langle \underline{B} \triangleright \underline{E}, \overline{B} \circ \overline{E} \rangle \in \mathrm{Int}^{\triangle_E \triangledown_E}(X, Y, \underline{I}, \overline{I})$.

Proof:
Directly from Lemma 2.2 and (11). □

Remark 4.5. Note that by Theorem 8, we can interpret $\mathcal{B}^{\triangle_E \triangledown_E}(X, Y, \underline{I}, \overline{I})$ as an **L**-rough concept lattice of $\langle X, Y, \underline{I}, \overline{I} \rangle$ with the requirement that both lower and upper approximations of every intent must be compatible with E.

The following theorem shows that a rougher **L**-equivalence relation leads to a reduction of size of the **L**-rough concept lattices. Furthermore, this reduction preserves extents.

Theorem 4.6. Let $\langle X, Y, \underline{I}, \overline{I} \rangle$ be an **L**-rough context, and $E_1 = \langle \underline{E}_1, \overline{E}_1 \rangle, E_2 = \langle \underline{E}_2, \overline{E}_2 \rangle$ be **L**-equivalences on Y, such that $\underline{E}_1 \subseteq \underline{E}_2, \overline{E}_1 \subseteq \overline{E}_2$. Then

$$|\mathcal{B}^{\triangle_{E_2} \triangledown_{E_2}}(X, Y, I)| \leqslant |\mathcal{B}^{\triangle_{E_1} \triangledown_{E_1}}(X, Y, \underline{I}, \overline{I})|. \tag{15}$$

In addition, we have

$$\mathrm{Ext}^{\triangle_{E_2} \triangledown_{E_2}}(X, Y, \underline{I}, \overline{I}) \subseteq \mathrm{Ext}^{\triangle_{E_1} \triangledown_{E_1}}(X, Y, \underline{I}, \overline{I}). \tag{16}$$

Proof:
First, note that for the pairs of **L**-equivalences $E_1, E_2 \in \mathbf{L}^{Y \times Y}$ such that $\underline{E}_1 \subseteq \underline{E}_2, \overline{E}_1 \subseteq \overline{E}_2$, we have

$$\underline{E}_2 = \underline{E}_1 \circ \underline{E}_2 = \underline{E}_2 \circ \underline{E}_1 \text{ and } \overline{E}_2 = \overline{E}_1 \circ \overline{E}_2 = \overline{E}_2 \circ \overline{E}_1. \tag{17}$$

By Corollary 3.3(d), for arbitrary extent $A \in \mathrm{Ext}^{\vartriangle_{E_2}\triangledown_{E_2}}(X, Y, \underline{I}, \overline{I})$ there is an **L**-rough set $\langle \underline{B}, \overline{B} \rangle$ such that $A = \langle \underline{B}, \overline{B} \rangle^{\triangledown_{E_2}}$. Now, we have

$$
\begin{aligned}
\langle \underline{B}, \overline{B} \rangle^{\triangledown_{E_2}} &= \underline{B}^{\downarrow_{\underline{I} \triangleright \underline{E}_2}} \cap \overline{B}^{\cup_{\overline{I} \circ \overline{E}_2}} && \text{by (14)} \\
&= \underline{B}^{\downarrow_{\underline{I} \triangleright (\underline{E}_2 \circ \underline{E}_1)}} \cap \overline{B}^{\cup_{\overline{I} \circ (\overline{E}_2 \circ \overline{E}_1)}} && \text{due (17)} \\
&= \underline{B}^{\downarrow_{(\underline{I} \triangleright \underline{E}_2) \triangleright \underline{E}_1}} \cap \overline{B}^{\cup_{(\overline{I} \circ \overline{E}_2) \circ \overline{E}_1}} && \text{by Lemma 2.1} \\
&= (\underline{B} \triangleright \underline{E}_2)^{\downarrow_{\underline{I} \triangleright \underline{E}_1}} \cap (\overline{B} \circ \overline{E}_2)^{\cup_{\overline{I} \circ \overline{E}_1}} && \text{by Theorem 4.4} \\
&= \langle \underline{B} \triangleright \underline{E}_2, \overline{B} \circ \overline{E}_2 \rangle^{\triangledown_{E_1}} && \text{by (14).}
\end{aligned}
$$

Thus, A is also an extent in $\mathrm{Ext}^{\vartriangle_{E_1}\triangledown_{E_1}}(X, Y, \underline{I}, \overline{I})$. That proves the inclusion (16). The inequality (15) is then a direct consequence of (16). □

Example 4.7. Fig. 8 shows the rough **L**-concept lattice of the **L**-context in Fig. 1 and rough **L**-concept lattice approximated using the **L**-equivalence relation on Y in Fig. 7. To demonstrate the second part of Theorem 4.6, the concepts with the same extents in the two lattices are connected in Fig. 8.

	α	β	γ	δ
α	1	0.5	0	0
β	0.5	1	0	0
γ	0	0	1	0.5
δ	0	0	0.5	1

Figure 7. **L**-equivalence used in Example 4.7.

Remark 4.8. For approximation by general **L**-relations we can prove a weaker version of Theorem 4.6: let $\langle X, Y, \underline{I}, \overline{I} \rangle$ be an **L**-rough context and $\underline{R}, \overline{R} \in L^{Y \times Y}$, then we have

$$
\mathrm{Ext}^{\vartriangle\triangledown}(X, Y, \underline{I}, \overline{I}) \subseteq \mathrm{Ext}^{\vartriangle\triangledown}(X, Y, \underline{I} \triangleright \underline{R}, \overline{I} \circ \overline{R}). \qquad (18)
$$

5. Reduction of size

The number of the **L**-rough concepts obtained from a given data is usually too large. When used as a tool of exploratory data analysis, the resulting concept lattice may be incomprehensible and unreadable for a user due its size. When used as a method of preprocessing data, the resulting number of **L**-rough concepts may be too large to be efficiently processed by other algorithms.

For this reason, reducing the size of a concept lattice has become one of the most recognized problems in FCA. In Section 4, we showed application of **L**-equivalences to obtain rougher intents, and thus smaller concept lattice. Here, we describe two methods of reduction of size which were previously studied for both standard and attribute-oriented **L**-concept lattices and can be easily adjusted for the present setting.

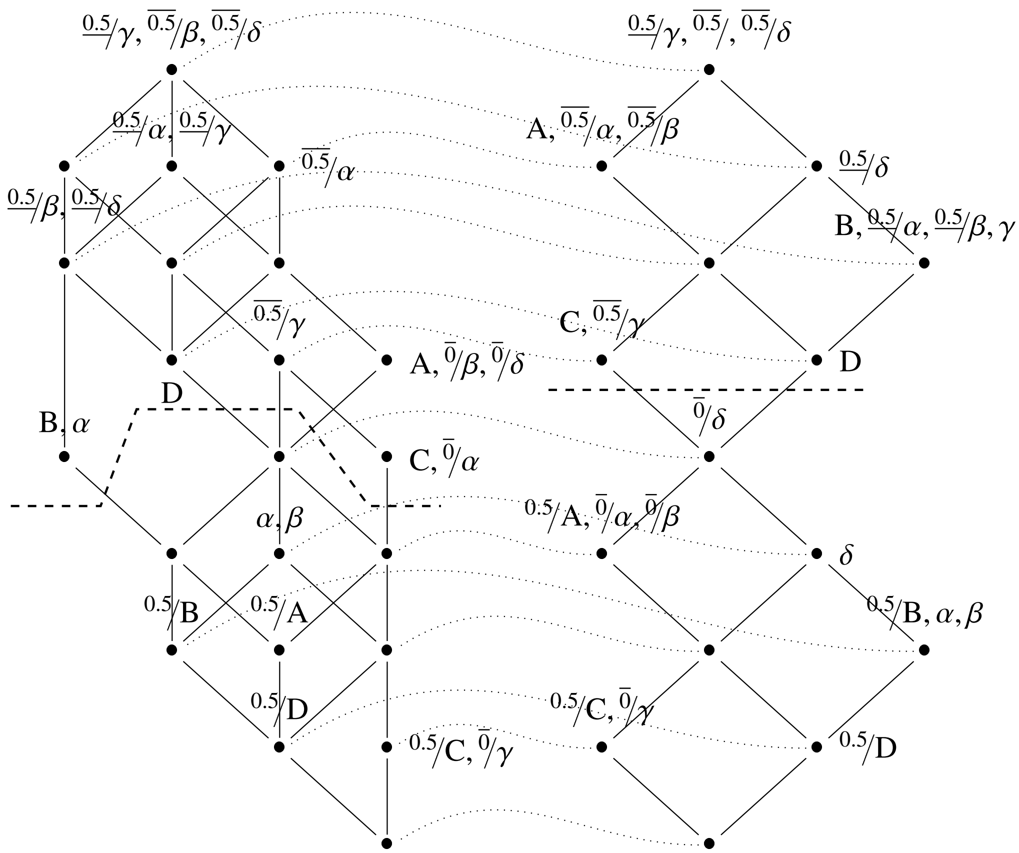

Figure 8. Rough **L**-concept lattices $\mathcal{B}^{\Delta\nabla}(X, Y, I, I)$ (left) and $\mathcal{B}^{\Delta_E \nabla_E}(X, Y, I)$ (right) with **L** being three-element Łukasiewicz chain. The corresponding extents are connected. Dashed lines separate natural concepts (above the lines) and non-natural concepts (below the lines).

Truth-Stressing and Truth-Depressing Hedges

In [16, 34] the authors addressed the problem of size of **L**-concept lattices $\mathcal{B}^{\uparrow\downarrow}$ and $\mathcal{B}^{\cap\cup}$ by parametrization of the concept-forming operators with particular unary operation—a truth-stressing or truth-depressing hedge. Selection of the hedge influences the size of concept lattice. In this part, we show how the same idea can be applied in the present setting.

The truth-stressing hedges were studied from the point of fuzzy logic as logical connectives "very true", see [31]. Our approach is close to that in [31]. A *truth-stressing hedge* is a mapping $^* : L \to L$ satisfying the following conditions

$$1^* = 1, \quad a^* \leqslant a, \quad a \leqslant b \text{ implies } a^* \leqslant b^*, \quad a^{**} = a^* \tag{19}$$

for each $a, b \in L$; i.e. it is an interior operator on **L** which preserves one.

A *truth-depressing hedge* is a mapping ${}^{\square} : L \to L$ such that following conditions are satisfied

$$0^{\square} = 0, \quad a \leqslant a^{\square}, \quad a \leqslant b \text{ implies } a^{\square} \leqslant b^{\square}, \quad a^{\square\square} = a^{\square} \tag{20}$$

for each $a, b \in L$; i.e. it is a closure operator on $\mathbf{L}$ which preserves zero. The truth-depressing hedge is a (truth function of) logical connective "slightly true" [48].

The truth-stressing and truth-depressing hedges were used to parametrize antitone as well as isotone concept-forming operators:

$$A^{\uparrow_*}(y) = \bigwedge_{x \in X} A(x)^* \to I(x,y), \quad B^{\downarrow_*}(x) = \bigwedge_{y \in Y} B(y)^* \to I(x,y), \tag{21}$$

$$A^{\cap_*}(y) = \bigvee_{x \in X} A(x)^* \otimes I(x,y), \quad B^{\cup_{\square}}(x) = \bigwedge_{y \in Y} I(x,y) \to B(y)^{\square}; \tag{22}$$

for more details see e.g. [3, 5, 11, 18, 34]

We naturally extend application of truth-stressing/truth-depressing hedges to $\mathbf{L}$-sets:

$$A^*(x) = A(x)^*$$

for all $x \in U$.

Let $\heartsuit, \diamondsuit$ be truth-stressing hedges on $\mathbf{L}$ and let $\spadesuit$ be a truth-depressing hedge on $\mathbf{L}$. We parametrize the $\mathbf{L}$-rough concept-forming operators as follows

$$A^{\blacktriangle} = \langle A^{\uparrow_\heartsuit}, A^{\cap_\heartsuit} \rangle, \qquad \langle \underline{B}, \overline{B} \rangle^{\blacktriangle\blacktriangledown} = \underline{B}^{\downarrow_\diamondsuit} \cap \overline{B}^{\cup_\spadesuit}. \tag{23}$$

for $A \in \mathbf{L}^X, \underline{B} \subseteq \overline{B} \in \mathbf{L}^Y$.

With the concept-forming operators (23) one obtains $\mathbf{L}$-rough concept lattice with smaller or equal number of $\mathbf{L}$-rough concepts than with (23). Using various combinations of truth-stressing and truth-depressing hedges, one can influence its size. Reader can find detailed results on use of hedges in this setting in [4].

Factorization by Tolerances

Complete tolerance on lattice $\mathbf{K} = \langle K, \leqslant \rangle$ is a reflexive, symmetric relation $\phi \subseteq K \times K$ such that $a_i \phi b_i$ for each $i \in In$ (In being an index set) implies $(\bigwedge_{i \in In} a_i)\, \phi\, (\bigwedge_{i \in In} b_i)$, and $(\bigvee_{i \in In} a_i)\, \phi\, (\bigvee_{i \in In} b_i)$. A maximal set B with $a \phi b$ for each $a, b \in K$ is called a block of tolerance ϕ.

Czédli [23] showed that blocks of tolerance are intervals in $\mathbf{K}$ and that they form a complete lattice. We call this complete lattice a factorization of $\mathbf{K}$ by ϕ. Wille [51] proved that in the complete tolerances on a concept lattice $\mathcal{B}(X, Y, I)$ (in crisp setting) are in one-to-one correspondence with block relations of formal context $\langle X, Y, I \rangle$, i.e. relations $J \subseteq X \times Y$, s.t. $\mathrm{Ext}(X, Y, J) \subseteq \mathrm{Ext}(X, Y, I)$ and $\mathrm{Int}(X, Y, J) \subseteq \mathrm{Int}(X, Y, I)$. Belohlavek [6] used a particular group of tolerances for factorization of standard concept lattice $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$, namely, a-cuts of a similarity of formal $\mathbf{L}$-concepts. The similarity $\approx$ of formal $\mathbf{L}$-concepts is defined by means of equivalence of extents (or equivalently, intents):

$$\begin{aligned} \langle A_1, B_1 \rangle \approx \langle A_2, B_2 \rangle &= S(A_1, A_2) \wedge S(A_2, A_1) \\ &= S(B_1, B_2) \wedge S(B_2, B_1) \end{aligned} \tag{24}$$

for each $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}^{\uparrow\downarrow}(X, Y, I)$. For $a \in L$, an a-cut of $\approx$ is a (crisp) relation $\approx^a$ given by

$$\langle A_1, B_1\rangle \approx^a \langle A_2, B_2\rangle \quad \text{iff} \quad (\langle A_1, B_1\rangle \approx \langle A_2, B_2\rangle) \geqslant a \tag{25}$$

for each $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}^{\uparrow\downarrow}(X, Y, I)$.

For factorization of $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$ by a-cut of similarity one does not need to compute the blocks of tolerance, since it is isomorphic to $\mathcal{B}^{\uparrow\downarrow}(X, Y, a \to I)$ [10]. In [22] one can find analogous results for object-oriented and attribute-oriented concept lattices.

In [36] we generalize the results of [23, 51]. While a study of block relations for present setting is yet to be developed we can easily apply approach of [6, 10].

Define similarity of **L**-rough concepts analogously as in (24):

$$\begin{aligned}\langle A_1, B_1\rangle \approx \langle A_2, B_2\rangle &= S(A_1, A_2) \wedge S(A_2, A_1)\\ &= S(\underline{B}_1, \underline{B}_2) \wedge S(\underline{B}_2, \underline{B}_1) \wedge S(\overline{B}_1, \overline{B}_2) \wedge S(\overline{B}_2, \overline{B}_1)\end{aligned} \tag{26}$$

It can be showed that its a-cuts are complete tolerances on $\mathcal{B}^{\Delta\nabla}(X, Y, \underline{I}, \overline{I})$. The factorization of the **L**-rough concept lattice $\mathcal{B}^{\Delta\nabla}(X, Y, \underline{I}, \overline{I})$ is then isomorphic to $\mathcal{B}^{\Delta\nabla}(X, Y, a \to \underline{I}, a \otimes \overline{I})$.

6. Future research & conclusions

The present results on **L**-rough concept analysis open several ways to continue the research. Here we summarize three of them.

Optimal Decompositions of L-rough Relations. Belohlavek studied in [8] decompositions of matrices with entries from residuated lattices in a general framework of aggregation structures. Main result there are that formal concepts are optimal and universal factors for the decomposition. Since by Theorem 3.4 the general framework covers **L**-rough contexts, we can apply these results here.

We concern with a decomposition of an **L**-rough relation $\langle \underline{I}, \overline{I}\rangle$ between X and Y to product

$$\langle \underline{I}, \overline{I}\rangle = R * \langle \underline{S}, \overline{S}\rangle \tag{27}$$

of an **L**-relation R between X and F and an **L**-rough relation $\langle \underline{S}, \overline{S}\rangle$ between F and Y with $|F|$ being as small as possible. The operation $*$ in (27) is defined as

$$R * \langle \underline{S}, \overline{S}\rangle = \langle R \circ \underline{S}, R \triangleleft \overline{S}\rangle. \tag{28}$$

By [8] the **L**-rough concepts of $\langle X, Y, \underline{I}, \overline{I}\rangle$ can be used to find the optimal decomposition.

In addition, [2] considers decomposition of **L**-relation to product of an ordinary relation and **L**-relation. This approach can be adapted to the present setting to concern with decompositions of the form (27) where R is and ordinary relation between F and Y.

[2] also provides a greedy algorithm to find an approximation of decomposition. The fact that we work with the lower and upper approximation instead of a single **L**-relation gives us more freedom for the optimization condition in the algorithm, for example we can use weights to say that we prefer lower or upper approximation more. That deserves more study and experimentation.

L-rough Attribute Implications. Besides concept lattices, second main output of FCA are attribute implications—if-then rules describing the formal context. In the setting of **L**-rough contexts, the attribute implications have the form

$$\langle \underline{A}, \overline{A} \rangle \Rightarrow \langle \underline{B}, \overline{B} \rangle.$$

The intended meaning is: "*if an object has all attributes in* **L***-rough set* $\langle \underline{A}, \overline{A} \rangle$ *(i.e. it has all attributes from* $\underline{A}$ *and no other attributes than those in* $\overline{A}$*) then it has all attributes in* **L***-rough set* $\langle \underline{B}, \overline{B} \rangle$*.*"

The general framework of aggregation structures does not handle attribute implications. Proper study of the **L**-rough attribute implications could provide some clues for further development of the framework.

More Connections of RST with FCA. We intend to study problems of Rough Set Theory in terms of the Formal Concept Analysis. We believe that the present contribution is a great start of such work.

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Appendix A – General Framework

The notion of a sup-preserving aggregation structure has been introduced in [8] and studied further in [9], see also [1, 17, 37, 39, 40] for related works, to which we refer for more details.

Supremum-preserving aggregation structure (or just aggregation structure) is a quadruple $\mathcal{L} = \langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \square \rangle$ where $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3$ are complete lattices; $\square$ is a mapping $\square : L_1 \times L_2 \to L_3$ which commutes with suprema of $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3$, i.e.

$$\left(\bigvee\nolimits_{1\, j\in J} a_j\right) \square\, b = \bigvee\nolimits_{3\, j\in J} \left(a_j \,\square\, b\right),$$
$$a \,\square \left(\bigvee\nolimits_{2\, j\in J} b_j\right) = \bigvee\nolimits_{3\, j\in J} \left(a \,\square\, b_j\right),$$

for all $a, a_j \in L_1, b, b_j \in L_2$.

We denote the operations on $\mathbf{L}_i$ by adding the subscript i. For example, the infima, suprema, the least, and the greatest element in $\mathbf{L}_2$ are denoted by $\bigwedge_2, \bigvee_2, 0_2$, and 1_2. For mnemonic reasons we use analogical convention to denote the elements of L_i, that is, the elements of L_i are denoted as a_i, b_i, etc. Unless stated otherwise $\mathcal{L} = \langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \square \rangle$.

Define $\underset{\rightarrow}{\square} : L_1 \times L_3 \to L_2$ and $\underset{\leftarrow}{\square} : L_3 \times L_2 \to L_1$ as

$$a_1 \underset{\rightarrow}{\square} a_3 = \bigvee\nolimits_2 \{a_2 \mid a_1 \square a_2 \leqslant_3 a_3\},$$
$$a_3 \underset{\leftarrow}{\square} a_2 = \bigvee\nolimits_1 \{a_1 \mid a_1 \square a_2 \leqslant_3 a_3\}.$$

Example 6.1. (a) $\mathbf{L_1} = \mathbf{L_2} = \mathbf{L_3} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ is complete residuated lattice; $\square$ is $\otimes$:

$$a_1 \underset{\rightarrow}{\square} a_3 = a_1 \to a_3,$$
$$a_3 \underset{\leftarrow}{\square} a_2 = a_2 \to a_3.$$

(b) $\mathbf{L_1} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$, $\mathbf{L_2} = \mathbf{L_3} = \langle L, \leqslant^{-1}\rangle$; $\square$ is $\to$:

$$a_1 \underset{\rightarrow}{\square} a_3 = a_1 \otimes a_3,$$
$$a_3 \underset{\leftarrow}{\square} a_2 = a_3 \to a_2.$$

(c) $\mathbf{L_2} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$, $\mathbf{L_1} = \mathbf{L_3} = \langle L, \leqslant^{-1}\rangle$; $\square$ is $\leftarrow$:

$$a_1 \underset{\rightarrow}{\square} a_3 = a_3 \to a_1,$$
$$a_3 \underset{\leftarrow}{\square} a_2 = a_2 \otimes a_3.$$

Definition 6.2. Let $J \in L_1^{X\times F}, K \in L_2^{F\times Y}, I \in L_3^{X\times Y}$. Define relational compositions as follows

$$(J \boxdot K)(x,y) = \bigvee\nolimits_{3}{}_{f\in F} J(x,f) \,\square\, K(f,y),$$
$$(J^{-1} \underset{\rightarrow}{\boxdot} I)(f,y) = \bigwedge\nolimits_{2}{}_{x\in X} J(x,f) \underset{\rightarrow}{\square} I(x,y),$$
$$(I \underset{\leftarrow}{\boxdot} K^{-1})(x,f) = \bigwedge\nolimits_{1}{}_{y\in Y} I(x,y) \underset{\leftarrow}{\square} K(f,y).$$

To enable us to apply the relational compositions to $\mathbf{L}$-sets, we identify $\mathbf{L}^X$ with $\mathbf{L}^{\{1\}\times X}$ and with $\mathbf{L}^{X\times\{1\}}$.

Concept-forming operators

Let $\langle X, Y, I \rangle$ be a $\mathbf{L}_3$-context. Define concept-forming operators as follows. Let $A \in L_1^{\{1\}\times X}$, $B \in L_2^{Y\times\{1\}}$. Define

$$A^{\owedge} = A \underset{\longrightarrow}{\boxdot} I \quad \text{and} \quad B^{\ovee} = I \underset{\longleftarrow}{\boxdot} B \tag{29}$$

In what follows, we write just $A(x)$ instead of $A(1, x)$, and $B(y)$ instead of $B(y, 1)$.

Note that for $A \in L_1^X$ we have

$$A^{\owedge}(y) = (A \underset{\longrightarrow}{\boxdot} I)(y) = \bigwedge_{x\in X}{}_2\, A(x) \underset{\longrightarrow}{\square} I(x, y) = \bigwedge_{x\in X}{}_2\, A(x) \underset{\longrightarrow}{\square} I_y(x) = A \underset{\longrightarrow}{\boxdot} I_y,$$

where $I_y \in \mathbf{L}_3^X$ is defined by $I_y(x) = I(x, y)$. Similarly, for $B \in L_2^Y$ we have $B^{\ovee}(x) = I_x \underset{\longleftarrow}{\boxdot} B$ where $I_x \in \mathbf{L}_3^Y$ is defined by $I_x(y) = I(x, y)$.

Example 6.3. With settings as in Example 6.1, one obtains following concept-forming operators:

(a)

$$A^{\owedge} = A^{\uparrow} \quad \text{and} \quad B^{\ovee} = B^{\downarrow},$$

(b)

$$A^{\owedge} = A^{\cap} \quad \text{and} \quad B^{\ovee} = B^{\cup},$$

(c)

$$A^{\owedge} = A^{\wedge} \quad \text{and} \quad B^{\ovee} = B^{\vee}.$$

Furthermore, $\mathcal{B}(X, Y, I)$ denotes the set of all formal concepts of I formed by $\langle \owedge, \ovee \rangle$, i.e.

$$\mathcal{B}(X, Y, I) = \{\langle A, B \rangle \in {L_1}^X \times {L_2}^Y \mid A^{\owedge} = B,\ B^{\ovee} = A\}.$$

Theorem 6.4. ([15])
Let $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \square \rangle$ be a supremum-preserving aggregation structure. Let $\langle X, Y, I \rangle$ be an $\mathbf{L}_3$-context $\langle X, Y, I \rangle$.

(1) $\mathcal{B}(X, Y, I)$ equipped with $\leqslant$ is a complete lattice with infima and suprema described as:

$$\bigwedge_{j\in J} \langle A_j, B_j \rangle = \left\langle \bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{\ovee\owedge} \right\rangle,$$

$$\bigvee_{j\in J} \langle A_j, B_j \rangle = \left\langle (\bigcup_{j\in J} A_j)^{\owedge\ovee}, \bigcap_{j\in J} B_j \right\rangle.$$

(2) Moreover, a complete lattice $\mathbf{V} = \langle V, \leqslant \rangle$ is isomorphic to $\mathcal{B}(X, Y, I)$ iff there are mappings $\gamma : X \times L_1 \to V$ and $\mu : Y \times L_2 \to V$ such that $\gamma(X \times L_1)$ is supremally dense in $\mathbf{V}$, $\mu(Y \times L_2)$ is infimally dense in $\mathbf{V}$, and $a \,\square\, b \leqslant_3 I(x, y)$ is equivalent to $\gamma(x, a) \leqslant \mu(y, b)$ for all $x \in X, y \in Y, a \in L_1, b \in L_2$.

Theorem 6.5. ([15])
The concept lattice $\mathcal{B}(X, Y, I)$ over $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \Box \rangle$ is isomorphic to the ordinary concept lattice $\mathcal{B}(X \times L_1, Y \times L_2, I^{\times})$ where $I^{\times} \subseteq (X \times L_1) \times (Y \times L_2)$ is defined by

$$\langle\langle x, a\rangle, \langle y, b\rangle\rangle \in I^{\times} \text{ iff } a \Box b \leqslant_3 I(x, y).$$

The papers F-H represent an elevation of some results known in the ordinary setting of FCA into the graded setting. We consider all three kinds of concept-forming operators which are of interest in the graded setting (standard, attribute-oriented, and object-oriented).

F Complete Relations on Fuzzy Complete Lattices

[45] Jan Konecny and Michal Krupka. Complete relations on fuzzy complete lattices. *Fuzzy Sets and Systems*, 320:64–80, 2017.

We focus on complete fuzzy tolerances. A (crisp) tolerance on a set is a reflexive and symmetric binary relation. A block of a tolerance is a set whose elements are pairwise related. A maximal block is a block which is maximal w.r.t. set inclusion. The set of all maximal blocks of a tolerance is called the factor set. One of basic results on tolerances on complete lattices in the basic setting is that complete lattices can be factorized by complete tolerances [28, 65]. That is, an ordering on the set of all maximal blocks of a complete tolerance can be introduced in a natural way, such that the factor set, together with this ordering, is again a complete lattice.

We show that this result hold true for complete **L**-tolerances on completely lattice **L**-ordered sets. More precisely, we use the usual definition of fuzzy tolerance and corresponding factor set and introduce an **L**-order on the factor set of a completely lattice **L**-ordered set by a complete **L**-tolerance, such that the new **L**-order is again a complete lattice **L**-order. To prove this main result, we more deeply investigate properties of complete **L**-tolerances. We use similar techniques to those used in classical ordered sets. However, we also introduce a result that is new even in the classical case: we show that complete fuzzy tolerances are in one-to-one correspondence with so-called extensive isotone fuzzy Galois connections.

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Complete relations on fuzzy complete lattices

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Abstract

We generalize the notion of complete binary relation on complete lattice to residuated lattice valued ordered sets and show its properties. Then we focus on complete fuzzy tolerances on fuzzy complete lattices and prove they are in one-to-one correspondence with extensive isotone Galois connections. Finally, we prove that any fuzzy complete lattice factorized by a complete fuzzy tolerance is again a fuzzy complete lattice.

Keywords: Fuzzy tolerance; Fuzzy order; Fuzzy Galois connection; Factorization

1. Introduction

In classical algebra, a complete relation on a complete lattice is a relation which preserves arbitrary infima and suprema. For instance, a binary relation $\sim$ on a complete lattice $\mathbf{U}$ is complete if for each system $\{\langle u_i, v_i\rangle\}_{i\in I}$ of pairs of elements from $\mathbf{U}$, $u_i \sim v_i$ for each $i \in I$ implies $\bigwedge_{i\in I} u_i \sim \bigwedge_{i\in I} v_i$ and $\bigvee_{i\in I} u_i \sim \bigvee_{i\in I} v_i$.

One of the goals of this paper is to define a notion of complete relation for fuzzy sets. That is, we need to state an appropriate condition for completeness of a fuzzy relation on a set possessing an appropriate structure of a complete lattice in fuzzy sense. However, the above definition cannot be used as is.

As it turns out, there is an equivalent condition to that of completeness of a relation on a complete lattice, that involves extending relations between sets to relations between power sets (i.e. sets of all subsets). This situation is known from theory of *power algebras* [1] which offers a natural way to extend a binary relation R on a set X to a binary relation R^+ on the power set 2^X.

This extension allows us formulate the following equivalent condition for completeness of binary relations: a binary relation $\sim$ on a complete lattice $\mathbf{U}$ is complete if and only if for any two subsets V_1, V_2 in $\mathbf{U}$, $V_1 \sim^+ V_2$ implies $\bigwedge V_1 \sim \bigwedge V_2$ and $\bigvee V_1 \sim \bigvee V_2$.

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In [2], Georgescu extended the theory of power algebras to a fuzzy setting. He shows a way of extending any fuzzy n-ary relation R on a set X to a fuzzy n-ary relation on the set of all fuzzy sets in X. In this paper, we use these results to define a notion of a complete binary fuzzy relation on a complete fuzzy lattice.

As a general framework, we use **L**-valued fuzzy sets where **L** is a complete residuated lattice, thus covering [0, 1]-valued fuzzy sets with arbitrary left-continuous t-norm on [0, 1] as a special case. Under this framework, we use a notion of **L**-ordered set which is, basically, a set with an **L**-relation satisfying requirements of reflexivity, antisymmetry and transitivity. A complete fuzzy lattice, or, more precisely, a completely lattice **L**-ordered set, is then an **L**-ordered set whose each **L**-subset has a (properly defined) infimum and supremum.

L-valued fuzzy sets, completely lattice **L**-ordered sets and other standard basic notions of fuzzy set theory (e.g. isotone **L**-Galois connections and **L**-closure and **L**-interior operators) are introduced in Sec. 2.

Section 3 is devoted to some basic parts of the Georgescu's theory of fuzzy power structures and its applications to **L**-ordered sets. We start with recalling the notion of power binary **L**-relations and their basic properties and then we prove some results on power relations of **L**-orders. Sec. 4 contains our definition of complete binary **L**-relation on completely lattice **L**-ordered set. We also prove some basic properties of complete **L**-relations.

In the main part of the paper, Sec. 5, we focus on complete fuzzy tolerances. A (crisp) tolerance on a set is a reflexive and symmetric binary relation. A block of a tolerance is a set whose elements are pairwise related. A maximal block is a block which is maximal w.r.t. set inclusion. The set of all maximal blocks of a tolerance is called the factor set. One of basic results on tolerances on complete lattices is that complete lattices can be factorized by complete tolerances [3,4]. That is, there can be introduced in a natural way an ordering on the set of all maximal blocks of a complete tolerance, such that the factor set, together with this ordering, is again a complete lattice.

We show that the same holds for complete **L**-tolerances on completely lattice **L**-ordered sets. More precisely, we use the usual definition of fuzzy tolerance and corresponding factor set and introduce an **L**-order on the factor set of completely lattice **L**-ordered set by a complete **L**-tolerance such that the new **L**-order is again a complete lattice **L**-order.

To prove this main result, we investigate properties of complete **L**-tolerances. We use similar techniques to those used in classical ordered sets. We also introduce a result that is new even in the classical case: we show that complete fuzzy tolerances are in one-to-one correspondence with so-called extensive isotone fuzzy Galois connections.

Note that factorization of complete lattices, either in ordinary or fuzzy setting, has been studied in the past [4–8] as it is useful for reducing dimensionality of concept lattices. The present paper can be viewed as a contribution to this area.

The paper is an extended and thoroughly rewritten version of a part of [9]. We fix some inaccuracies that appeared in the original paper.

2. Preliminaries

2.1. Residuated lattices and fuzzy sets

A *complete residuated lattice* [7,10,11] is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that

(i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, i.e. a partially ordered set in which arbitrary infima and suprema exist, $0 = \bigwedge L$, $1 = \bigvee L$;
(ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation which is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$;
(iii) $\otimes$ and $\rightarrow$ satisfy adjointness, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$.

The partial order of **L** is denoted by $\leq$. Throughout the paper, **L** denotes an arbitrary complete residuated lattice.

Elements of L are called truth degrees. $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication".

Common examples of complete residuated lattices include those defined on [0, 1], (i.e. $L = [0, 1]$), $\wedge$ being minimum, $\vee$ maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\rightarrow$.

The three most important pairs of adjoint operations on the unit interval are

Łukasiewicz:
$$a \otimes b = \max(a+b-1, 0)$$
$$a \to b = \min(1-a+b, 1)$$

Gödel:
$$a \otimes b = \min(a, b)$$
$$a \to b = \begin{cases} 1 & a \le b \\ b & \text{otherwise} \end{cases}$$

Goguen (product):
$$a \otimes b = a \cdot b$$
$$a \to b = \begin{cases} 1 & a \le b \\ \frac{b}{a} & \text{otherwise} \end{cases}$$

An **L**-*set* (or *fuzzy set*) A in a universe set X is a mapping assigning to each $x \in X$ a truth degree $A(x) \in L$. The set of all **L**-sets in a universe X is denoted L^X.

Operations with **L**-sets are defined element-wise. For instance, *the union of* **L**-*sets* $A, B \in L^X$ is the **L**-set $A \cup B$ in X satisfying $(A \cup B)(x) = A(x) \vee B(x)$ for each $x \in X$. An **L**-set $A \in L^X$ is also denoted $\{^{A(x)}/x \mid x \in X\}$. If for all $y \in X$ distinct from $x_1, x_2, \dots, x_n$ we have $A(y) = 0$, we also write $\{^{A(x_1)}/x_1, {}^{A(x_2)}/x_1, \dots, {}^{A(x_n)}/x_n\}$.

Binary **L**-relations (binary fuzzy relations) between X and Y can be thought of as **L**-sets in the universe $X \times Y$. That is, a *binary* **L**-*relation* $I \in L^{X\times Y}$ *between a set* X *and a set* Y is a mapping assigning to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$ (a degree to which x and y are related by I). The *inverse relation* I^{-1} *to the* **L**-*relation* I is an **L**-set in $Y \times X$ and is defined by $I^{-1}(y, x) = I(x, y)$. Binary **L**-relations between X and X are called simply binary **L**-relations *on* X.

The *composition* $R \circ T$ *of binary* **L**-*relations* $R \in L^{X\times Y}$ *and* $T \in L^{Y\times Z}$ [12] is a binary **L**-relation between X and Z defined by

$$(R \circ T)(x, z) = \bigvee_{y\in Y} R(x, y) \otimes T(y, z). \tag{1}$$

L-sets in a set X can be naturally identified with binary **L**-relations between $\{1\}$ and X, resp. X and $\{1\}$. Thus, we can also consider composition of an **L**-set and a binary **L**-relation and even composition of two **L**-sets: for $A, A_1, A_2 \in L^X$, $B \in L^Y$ and $R \in L^{X\times Y}$ we have

$$(A \circ R)(y) = \bigvee_{x\in X} A(x) \otimes R(x, y), \qquad (R \circ B)(x) = \bigvee_{y\in Y} R(x, y) \otimes B(y) \tag{2}$$

and

$$A_1 \circ A_2 = \bigvee_{x\in X} A_1(x) \otimes A_2(x). \tag{3}$$

An **L**-set $A \in L^X$ is called *crisp* if $A(x) \in \{0, 1\}$ for each $x \in X$. Crisp **L**-sets can be identified with ordinary sets. For a crisp **L**-set A we also write $x \in A$ for $A(x) = 1$ and $x \notin A$ for $A(x) = 0$. An **L**-set $A \in L^X$ is called *empty* (denoted by $\emptyset$) if $A(x) = 0$ for each $x \in X$. For $a \in L$ and $A \in L^X$, $a \otimes A \in L^X$ and $a \to A \in L^X$ are defined by

$$(a \otimes A)(x) = a \otimes A(x) \text{ and } (a \to A)(x) = a \to A(x).$$

For an **L**-set $A \in L^X$ and $a \in L$, the *a-cut of* A is a crisp subset ${}^aA \subseteq X$ such that $x \in {}^aA$ iff $a \le A(x)$. This definition applies also to binary **L**-relations, whose a-cuts are classical (crisp) binary relations.

For a universe X we define an **L**-relation of *graded subsethood* $L^X \times L^X \to L$ by:

$$\mathrm{S}(A, B) = \bigwedge_{x\in X} A(x) \to B(x). \tag{4}$$

Graded subsethood generalizes the classical subsethood relation $\subseteq$. Indeed, in the crisp case (i.e. $L = \{0, 1\}$) $\mathrm{S}(A, B) = 1$ iff $x \in A$ implies $x \in B$ for each $x \in X$. Described verbally, $\mathrm{S}(A, B)$ represents the degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $\mathrm{S}(A, B) = 1$. As a consequence, we have $A \subseteq B$ iff $A(x) \le B(x)$ for each $x \in X$.

We set

$$A \approx^X B = \mathrm{S}(A, B) \wedge \mathrm{S}(B, A). \tag{5}$$

The value $A \approx^X B$ is interpreted as the degree to which the $\mathbf{L}$-sets A and B are similar.

A binary $\mathbf{L}$-relation R on a set X is called *reflexive* if $R(x,x) = 1$ for any $x \in X$, *symmetric* if $R(x, y) = R(y, x)$ for any $x, y \in X$, and *transitive* if $R(x, y) \otimes R(y, z) \leq R(x, z)$ for any $x, y, z \in X$. R is called an $\mathbf{L}$-*tolerance* if it is reflexive and symmetric, $\mathbf{L}$-*equivalence* if it is reflexive, symmetric and transitive. If R is an $\mathbf{L}$-equivalence such that for any $x, y \in X$ from $R(x, y) = 1$ it follows $x = y$ then R is called an $\mathbf{L}$-*equality* on X. $\mathbf{L}$-equalities are often denoted by $\approx$. The similarity $\approx^X$ of $\mathbf{L}$-sets (5) is an $\mathbf{L}$-equality on L^X.

Let $\sim$ be an $\mathbf{L}$-equivalence on X. We say that an $\mathbf{L}$-set A in X is *compatible with* $\sim$ (or *extensional w.r.t.* $\sim$) if for any $x, x' \in X$ it holds

$$A(x) \otimes (x \sim x') \leq A(x'). \tag{6}$$

A binary $\mathbf{L}$-relation R on X is *compatible with* $\sim$ if for each $x, x', y, y' \in X$,

$$R(x, y) \otimes (x \sim x') \otimes (y \sim y') \leq R(x', y'). \tag{7}$$

Zadeh's extension principle [13] allows extending any mapping $f : X \to Y$ to $\mathbf{L}$-sets in X by setting for each $A \in L^X$

$$f(A)(y) = \bigvee_{x \in X, f(x)=y} A(x). \tag{8}$$

Note that, historically, the symbol for the mapping f is used in the notation $f(A)$. This cannot cause any confusion and is in accordance with the classical notation for the image of a set w.r.t. a mapping.

In this paper, we use well-known properties of residuated lattices and fuzzy structures which can be found e.g. in [7,10].

2.2. $\mathbf{L}$-ordered sets

In this and two subsequent sections, we recall basic definitions and results of the theory of $\mathbf{L}$-ordered sets. All results are either trivial or standard. Basic references are [14,7] and the references therein. Some results appear also in other sources (e.g. [15–17]).

An $\mathbf{L}$-*order* on a set U with an $\mathbf{L}$-equality $\approx$ is a binary $\mathbf{L}$-relation $\preceq$ on U which is compatible with $\approx$, reflexive, transitive and satisfies $(u \preceq v) \wedge (v \preceq u) \leq u \approx v$ for any $u, v \in U$ (*antisymmetry*). The tuple $\mathbf{U} = \langle\langle U, \approx\rangle, \preceq\rangle$ is called an $\mathbf{L}$-*ordered set*. An immediate consequence of the definition is that for any $u, v \in U$ it holds

$$u \approx v = (u \preceq v) \wedge (v \preceq u). \tag{9}$$

Note that as discovered by Yao [18], for any binary $\mathbf{L}$-relation $\preceq$ which is reflexive, transitive and satisfies

$$(u \preceq v) \wedge (v \preceq u) \quad \text{implies} \quad u = v, \tag{10}$$

$\langle\langle U, \approx\rangle, \preceq\rangle$ with $\approx$ given by (9) is an $\mathbf{L}$-ordered set. This alternative possibility of defining fuzzy ordered sets can be used if no $\mathbf{L}$-equality is given on U in advance. Fan [19] was first who defined fuzzy ordered sets this way.

If $\mathbf{U} = \langle\langle U, \approx\rangle, \preceq\rangle$ is an $\mathbf{L}$-ordered set then the tuple $\langle U, {}^1\!\preceq\rangle$, where ${}^1\!\preceq$ is the 1-cut of $\preceq$, is a (partially) ordered set. We sometimes write $\leq$ instead of ${}^1\!\preceq$ and use the symbols $\wedge$, $\bigwedge$ resp. $\vee$, $\bigvee$ for denoting infima resp. suprema in $\langle U, {}^1\!\preceq\rangle$.

For two $\mathbf{L}$-ordered sets $\mathbf{U} = \langle\langle U, \approx_U\rangle, \preceq_U\rangle$ and $\mathbf{V} = \langle\langle V, \approx_V\rangle, \preceq_V\rangle$, a mapping $f : U \to V$ is *isotone* (or *order-preserving*) if $(u_1 \preceq_U u_2) \leq (f(u_1) \preceq_V f(u_2))$ for any $u_1, u_2 \in V$. The mapping f is called an *isomorphism of* $\mathbf{U}$ *and* $\mathbf{V}$ if it is a bijection and $(u_1 \preceq_U u_2) = (f(u_1) \preceq_V f(u_2))$ for any $u_1, u_2 \in V$. $\mathbf{U}$ and $\mathbf{V}$ are then called *isomorphic*.

In classical theory of ordered sets, a subset V of an ordered set is called a lower set if for each element u such that there is $v \in V$ satisfying $u \leq v$ it holds $u \in V$. Equivalently, for a lower set V it holds: if $u \leq v$ then $v \in V$ implies $u \in V$.

Analogously, for an **L**-ordered set **U**, an **L**-set $V \in L^U$ is called a *lower set* (resp. an *upper set*) if for each $u, v \in U$ it holds

$$u \preceq v \leq V(v) \to V(u) \qquad \text{(resp. } u \preceq v \leq V(u) \to V(v)\text{)}. \tag{11}$$

The lower (resp. *upper*) *set of an* **L**-*set* $V \in L^U$ is the **L**-set $\downarrow V$ (resp. $\uparrow V$), defined by

$$\downarrow V(u) = (\preceq \circ V)(u) = \bigvee_{v \in U} (u \preceq v) \otimes V(v), \tag{12}$$

$$\uparrow V(u) = (V \circ \preceq)(u) = \bigvee_{v \in U} V(v) \otimes (v \preceq u). \tag{13}$$

In a similar manner we define the lower and upper cone of $V \in L^U$. For any $v \in U$ we set

$$\mathscr{L}V(v) = \bigwedge_{u \in U} V(u) \to (v \preceq u), \qquad \mathscr{U}V(v) = \bigwedge_{u \in U} V(u) \to (u \preceq v). \tag{14}$$

The right-hand side of the first equation is the degree of "For each $u \in U$, if u is in V then v is less than or equal to u", and similarly for the second equation. Thus, $\mathscr{L}V(v)$ ($\mathscr{U}V(v)$) can be seen as the degree to which v is less (greater) than or equal to each element of V, that is *the degree to which v is a lower* (*upper*) *bound of V*.

In the case $\mathscr{L}V(v) = 1$ (resp. $\mathscr{U}V(v) = 1$) we say simply v is a *lower* (*upper*) *bound of V*. $\mathscr{L}V$ (resp. $\mathscr{U}V$) is called the **L**-*set of lower bounds* (resp. *upper bounds*) *of V*, or *the lower cone* (resp. *the upper cone*) of V.

Directly from definition we obtain the following property of cones of singleton **L**-sets:

$$\mathscr{L}\{v\}(u) = u \preceq v, \qquad \mathscr{U}\{v\}(u) = v \preceq u. \tag{15}$$

If $u, v \in U$, $v \leq u$, then the **L**-set $[\![v, u]\!] = \mathscr{U}\{v\} \cap \mathscr{L}\{u\}$ is called an **L**-*interval* (or simply an *interval*) in **U**.

We set $[v, u] = {}^1[\![v, u]\!]$. Thus, $[v, u]$ denotes the classical interval with respect to the 1-cut of $\preceq$: $[v, u] = \{u' \mid v \leq u' \leq u\}$.

An **L**-set $V \in L^U$ is *convex* if $V = \downarrow V \cap \uparrow V$. The "$\subseteq$" inclusion always holds as the lower set as well as upper set of V always contains V as a subset. For any $V \in L^U$, each of the following **L**-sets is convex: $\downarrow V$, $\uparrow V$, $\mathscr{L}V$, $\mathscr{U}V$. Every **L**-interval $[\![v, u]\!]$ in **U** is convex as well. Every convex **L**-set in **U** is compatible with $\approx$.

2.3. *Completely lattice* **L**-*ordered sets*

For any **L**-set $V \in L^U$ there exists at most one element $u \in U$ such that $\mathscr{L}V(u) \wedge \mathscr{U}(\mathscr{L}V)(u) = 1$ (resp. $\mathscr{U}V(u) \wedge \mathscr{L}(\mathscr{U}V)(u) = 1$) [14,7]. If there is such an element, we call it *the infimum of V* (resp. *the supremum of V*) and denote $\inf V$ (resp. $\sup V$); otherwise we say that the infimum (resp. supremum) does not exist.

If $\inf V$ exists and $V(\inf V) = 1$ then it is called *the minimum of V* and denoted $\min V$. Similarly, if $\sup V$ exists and $V(\sup V) = 1$ then we call it *the maximum of V* and denote $\max V$.

An **L**-ordered set **U** is called *completely lattice* **L**-*ordered* if for each $V \in L^U$ both $\inf V$ and $\sup V$ exist.

An important example of a completely lattice **L**-ordered set is the following. For a set X, the tuple $\langle\langle L^X, \approx^X\rangle, \mathrm{S}\rangle$ is a completely lattice **L**-ordered set with infima and suprema given by

$$(\inf V)(u) = \bigwedge_{W \in L^X} V(W) \to W(u), \qquad (\sup V)(u) = \bigvee_{W \in L^X} V(W) \otimes W(u). \tag{16}$$

This fact follows easily e.g. from the main theorem of fuzzy concept lattices (fuzzy order version) [7,14].

The following holds for infima and suprema of **L**-intervals:

$$v = \min[\![v, u]\!], \qquad u = \max[\![v, u]\!]. \tag{17}$$

Evidently, $u = \inf V$ iff for each $v \in U$

$$V(v) \leq u \preceq v, \qquad \mathscr{L}V(v) \leq v \preceq u. \tag{18}$$

Similarly, $u = \sup V$ iff for each $v \in U$

$$V(v) \leq v \preceq u, \qquad \mathscr{L}V(v) \leq u \preceq v. \tag{19}$$

Thus, infimum (supremum) of V is a lower (upper) bound of V and, in the same time, an upper bound of $\mathscr{L}V$ (a lower bound of $\mathscr{U}V$).

First inequalities in (18) and (19) together with (15) imply

Lemma 1. *If* $\inf V$ *exists then* $V \subseteq \mathscr{U}\{\inf V\}$. *If* $\sup V$ *exists then* $V \subseteq \mathscr{L}\{\sup V\}$.

In some papers (e.g. [20]), (18) (resp. (19)) is used as definition of infimum (resp. supremum). The following result [20] is a simple consequence of definition.

Lemma 2. *For* $u \in U$ *it holds* $u = \inf V$ *iff for each* $v \in U$

$$\mathscr{L}V(v) = v \preceq u \tag{20}$$

and $u = \sup V$ *iff for each* $v \in U$

$$\mathscr{U}V(v) = u \preceq v. \tag{21}$$

The following is a trivial consequence of [7, Lemma 4.54].

Lemma 3. *For any* $\mathbf{L}$*-sets* V_1, V_2 *in a completely lattice* $\mathbf{L}$*-ordered set it holds*

$$\mathrm{S}(V_1, V_2) \leq \inf V_2 \preceq \inf V_1, \qquad \mathrm{S}(V_1, V_2) \leq \sup V_1 \preceq \sup V_2 \tag{22}$$

2.4. *Isotone* $\mathbf{L}$*-Galois connections*

Basics of the theory of isotone Galois connections come from [21], other references are [22,16,15].

An *isotone* $\mathbf{L}$*-Galois connection* between $\mathbf{L}$-ordered sets $\mathbf{U}$ and $\mathbf{V}$ is a pair $\langle f, g\rangle$ where $f : U \to V$, $g : V \to U$ are mappings such that for each $u \in U$, $v \in V$ it holds

$$f(u) \preceq v = u \preceq g(v). \tag{23}$$

An isotone Galois connection between $\mathbf{U}$ and $\mathbf{U}$ is called simply an isotone Galois connection *on* $\mathbf{U}$.

By isotone $\mathbf{L}$-Galois connection *between sets* X *and* Y we understand an isotone $\mathbf{L}$-Galois connection between completely lattice $\mathbf{L}$-ordered sets $\mathbf{L}^X$ and $\mathbf{L}^Y$ (16).

The following theorem summarizes well-known basic properties of isotone $\mathbf{L}$-Galois connections. Proofs can be found e.g. in [22,16,15].

Theorem 1 *(Basic properties of isotone* $\mathbf{L}$*-Galois connections). Let* $\langle f, g\rangle$ *be an isotone* $\mathbf{L}$*-Galois connection between* $\mathbf{L}$*-ordered sets* $\mathbf{U}$ *and* $\mathbf{V}$*. Then*

(a) $u \leq g(f(u))$ *for each* $u \in U$, $f(g(v)) \leq v$ *for each* $v \in V$.

(b) f *and* g *are isotone.*

(c) $f(g(f(u))) = f(u)$, $g(f(g(v))) = g(v)$.

(d) Let $\mathbf{U}$ *and* $\mathbf{V}$ *be completely lattice* $\mathbf{L}$*-ordered sets. For* $U' \in L^U$ *and* $V' \in L^V$ *we have*

$$f(\inf U') \leq \inf f(U'), \qquad g(\sup V') \geq \sup g(V').$$

Let $\langle f, g\rangle$ be an isotone $\mathbf{L}$-Galois connection between $\mathbf{U}$ and $\mathbf{V}$. A pair $\langle u, v\rangle$ where $u \in U$ and $v \in V$ is called a *fixpoint of* $\langle f, g\rangle$ if $f(u) = v$ and $g(v) = u$.

Suppose $\langle u_1, v_1\rangle$, $\langle u_2, v_2\rangle$ are two fixpoints of $\langle f, g\rangle$. We have by (23),

$$u_1 \preceq u_2 = u_1 \preceq g(v_2) = f(u_1) \preceq v_2 = v_1 \preceq v_2$$

and by (9),

$$u_1 \approx u_2 = v_1 \approx v_2.$$

We denote the set of all fixpoints of $\langle f, g\rangle$ by $\mathrm{Fix}_{\langle f,g\rangle}$. For **L**-relations $\approx_{\mathrm{Fix}_{\langle f,g\rangle}}$ and $\preceq_{\mathrm{Fix}_{\langle f,g\rangle}}$ defined on $\mathrm{Fix}_{\langle f,g\rangle}$ by

$$\langle u_1, v_1\rangle \approx_{\mathrm{Fix}_{\langle f,g\rangle}} \langle u_2, v_2\rangle = u_1 \approx u_2 \quad (= v_1 \approx v_2), \tag{24}$$

$$\langle u_1, v_1\rangle \preceq_{\mathrm{Fix}_{\langle f,g\rangle}} \langle u_2, v_2\rangle = u_1 \preceq u_2 \quad (= v_1 \preceq v_2), \tag{25}$$

we obtain an **L**-ordered set $\mathbf{Fix}_{\langle f,g\rangle} = \langle\langle \mathrm{Fix}_{\langle f,g\rangle}, \approx_{\mathrm{Fix}_{\langle f,g\rangle}}\rangle, \preceq_{\mathrm{Fix}_{\langle f,g\rangle}}\rangle$. In the rest of the paper, we will usually write $\approx$ instead of $\approx_{\mathrm{Fix}_{\langle f,g\rangle}}$ and $\preceq$ instead of $\preceq_{\mathrm{Fix}_{\langle f,g\rangle}}$.

We denote the set of all isotone Galois connections between **L**-ordered sets **U** and **V** by $\mathrm{IGal}(\mathbf{U}, \mathbf{V})$ and consider the following binary **L**-relations $\approx_{\mathrm{IGal}(\mathbf{U},\mathbf{V})}$, $\preceq_{\mathrm{IGal}(\mathbf{U},\mathbf{V})}$ on $\mathrm{IGal}(\mathbf{U}, \mathbf{V})$:

$$\langle f_1, g_1\rangle \approx_{\mathrm{IGal}(\mathbf{U},\mathbf{V})} \langle f_2, g_2\rangle = \bigwedge_{u\in U} (f_2(u) \approx f_1(u)) \wedge \bigwedge_{v\in V} (g_1(v) \approx g_2(v)), \tag{26}$$

$$\langle f_1, g_1\rangle \preceq_{\mathrm{IGal}(\mathbf{U},\mathbf{V})} \langle f_2, g_2\rangle = \bigwedge_{u\in U} (f_2(u) \preceq f_1(u)) \wedge \bigwedge_{v\in V} (g_1(v) \preceq g_2(v)). \tag{27}$$

Lemma 4. *$\langle\langle \mathrm{IGal}(\mathbf{U}, \mathbf{V}), \approx_{\mathrm{IGal}(\mathbf{U},\mathbf{V})}\rangle, \preceq_{\mathrm{IGal}(\mathbf{U},\mathbf{V})}\rangle$ is an **L**-ordered set.*

Proof. Straightforward. □

For an **L**-ordered set **U**, an isotone **L**-Galois connection $\langle f, g\rangle$ on **U** is called *extensive* if

$$f(u) \leq u \quad \text{or equivalently} \quad g(u) \geq u \tag{28}$$

for each $u \in U$. The set of all extensive isotone **L**-Galois connections on **U** is denoted $\mathrm{EIGal}(\mathbf{U})$.

2.5. **L**-closure and **L**-interior operators

We recall briefly basic definitions and results on **L**-closure and **L**-interior operators we will need in what follows. More details can be found in [23,14,21,15].

For an **L**-ordered set **U**, a mapping $C : U \to U$ is called an **L**-*closure operator* if the following holds for each $u, u_1, u_2 \in U$:

$$C(u) \geq u, \tag{29}$$

$$C(C(u)) = C(u), \tag{30}$$

$$u_1 \preceq u_2 \leq C(u_1) \preceq C(u_2). \tag{31}$$

A mapping $I : U \to U$ is called an **L**-*interior operator* if for each $u, u_1, u_2 \in U$,

$$I(u) \leq u, \tag{32}$$

$$I(I(u)) = I(u), \tag{33}$$

$$u_1 \preceq u_2 \leq I(u_1) \preceq I(u_2). \tag{34}$$

By **L**-closure (resp. **L**-interior) operator *on a set X* we mean an **L**-closure (resp. **L**-interior) operator on the completely lattice **L**-ordered set $\mathbf{L}^X$ (16).

An element $u \in U$ is a *fixpoint of C* (resp. *fixpoint of I*) if $C(u) = u$ (resp. $I(u) = u$). The set of all fixpoints of C (resp. I) will be denoted Fix_C (resp. Fix_I). The sets Fix_C and Fix_I inherit a structure of an **L**-ordered set from **U**.

The following result has been proved in [23, Theorem 17] for **L**-closure operators; the version for **L**-interior operators is dual.

Theorem 2. *Let **U** be a completely lattice **L**-ordered set. Then Fix_C is closed w.r.t. arbitrary infima (i.e. for any **L**-set $V \in L^U$, $V \subseteq \mathrm{Fix}_C$, we have $\inf V \in \mathrm{Fix}_C$) and Fix_I is closed w.r.t. arbitrary suprema (i.e. for any **L**-set $V \in L^U$, $V \subseteq \mathrm{Fix}_I$, we have $\sup V \in \mathrm{Fix}_I$). Consequently, Fix_C and Fix_I are completely lattice **L**-ordered sets.*

A subset $V \subseteq U$ which is closed w.r.t. arbitrary infima (resp. suprema) is called an **L**-*closure* (resp. **L**-*interior*) *system in* **U**. The above theorem says that Fix_C (resp. Fix_I) is an **L**-closure (resp. **L**-interior) system in **U**. In the case $\mathbf{U} = \mathbf{L}^X$ for some set X we also talk about **L**-closure (resp. **L**-interior) system *in* X.

Let $\langle f, g\rangle$ be an isotone **L**-Galois connection on **U**. From Theorem 1 it easily follows that the composition C given by $C(u) = g(f(u))$ is an **L**-closure operator on U and the composition I, $I(v) = f(g(v))$ is an **L**-interior operator on V.

We have the following simple result for the **L**-ordered sets of fixpoints of these operators and of the **L**-Galois connection $\langle f, g\rangle$ itself. It has been proved in [21] for isotone **L**-Galois connections between completely lattice **L**-ordered sets $\mathbf{L}^X$ and $\mathbf{L}^Y$. As we shall see in the proof, it follows easily from Theorem 2.

Theorem 3. *Let* **U** *be a completely lattice* **L***-ordered set. Then the* **L***-ordered sets* $\text{Fix}_{\langle f,g\rangle}$, Fix_C, Fix_I *are isomorphic. Consequently,* $\text{Fix}_{\langle f,g\rangle}$ *is a completely lattice* **L***-ordered set. The isomorphism* $\text{Fix}_{\langle f,g\rangle} \to \text{Fix}_C$ *is given by* $\langle u, v\rangle \to u$ *and the isomorphism* $\text{Fix}_{\langle f,g\rangle} \to \text{Fix}_I$ *is given by* $\langle u, v\rangle \to v$.

Proof. Evidently, $\langle u, f(u)\rangle$ is a fixpoint of $\langle f, g\rangle$ iff u is a fixpoint of C. Thus, the assignment $\langle u, v\rangle \to u$ is a bijection between $\text{Fix}_{\langle f,g\rangle}$ and Fix_C. (25) implies it is an isomorphism. Similarly for I.

The fact that $\text{Fix}_{\langle f,g\rangle}$ is a completely lattice **L**-ordered set now follows from Theorem 2. □

3. Power structures of L-ordered sets

A power structure [1] is an algebraic structure constructed by "lifting" operations and relations on a (ordinary) set to its power set, i.e. the set of all its (ordinary) subsets. The theory goes back to Frobenius and recently [2] has been generalized to a fuzzy setting.

In this section, we recall basic definitions and results from [2] to the extent we need in this paper. We also show some results from [7,24]. Then we prove some properties of power structures of fuzzy ordered sets we will need later.

Note that in [2], fuzzy power structures are studied under the framework of continuous t-norms. However, it is straightforward to generalize the results we use in this paper to complete residuated lattices.

Let R be a binary **L**-relation on a set X. We set for any **L**-sets $A, B \in L^X$

$$R^{\to}(A, B) = \mathrm{S}(A, R \circ B) = \bigwedge_{x\in X}\left(A(x) \to \bigvee_{y\in X} R(x, y) \otimes B(y)\right), \tag{35}$$

$$\begin{aligned} R^{\leftarrow}(A, B) &= (R^{-1})^{\to}(B, A) = \mathrm{S}(B, R^{-1} \circ A) = \mathrm{S}(B, A \circ R) \\ &= \bigwedge_{y\in X}\left(B(y) \to \bigvee_{x\in X} R(x, y) \otimes A(x)\right). \end{aligned} \tag{36}$$

Since $\mathrm{S}(A, R \circ B)$ is the degree to which A is a subset of $R \circ B$, $R^{\to}(A, B)$ can be viewed as the degree to which each element of A is related to an element of B. We set

$$R^{+}(A, B) = R^{\to}(A, B) \wedge R^{\leftarrow}(A, B), \tag{37}$$

obtaining a binary **L**-relation, called *power* **L**-*relation*, R^+ on the set L^X. In the following, we prove some basic properties of the power **L**-relation R^+ for R being a binary **L**-relation on a set X and later on an **L**-ordered set $\langle\langle U, \approx\rangle, \preceq\rangle$.

The following result is straightforward and has been proved in [7, Theorem 4.41].

Lemma 5. *For any binary* **L***-relation* $R \in L^X$ *it holds*

1. *if* R *is reflexive then so is* R^+,
2. *if* R *is symmetric then so is* R^+,
3. *if* R *is transitive then so is* R^+.

The following has been proved in [24, Theorem 2].

Theorem 4. *For any two* $\mathbf{L}$*-relations* $R, Q \in L^X$ *it holds*

$$R^+ \circ Q^+ \subseteq (R \circ Q)^+. \tag{38}$$

In the next two theorems we show some basic properties of power relations of $\mathbf{L}$-equivalences. In the first of them we use the $\mathbf{L}$-relation $\approx^X$ on $\mathbf{L}$-sets in X defined by (5). Notice that this $\mathbf{L}$-relation does not depend on $\sim$.

Theorem 5. *Let* $\sim$ *be an* $\mathbf{L}$*-equivalence on a set* X*,* $M \subseteq L^X$ *a subset containing only* $\mathbf{L}$*-sets compatible with* $\sim$*. Then* $\sim^+$ *is equal to* $\approx^X$ *on* M *and is therefore an* $\mathbf{L}$*-equality on* M*.*

Proof. Let $A, B \in M$. By compatibility, $\bigvee_{x' \in X}(x \sim x') \otimes A(x') \leq A(x)$. As $(x \sim x) \otimes A(x) = A(x)$, the opposite inequality also holds true and we have $\sim \circ A = A$. Similarly, $\sim \circ B = B$. Thus,

$$\begin{aligned} A \sim^+ B &= (A \sim^{\rightarrow} B) \wedge (A \sim^{\leftarrow} B) = \mathrm{S}(A, \sim \circ B) \wedge \mathrm{S}(B, \sim \circ A) = \mathrm{S}(A, B) \wedge \mathrm{S}(B, A) \\ &= A \approx^X B. \qquad \square \end{aligned}$$

Theorem 6. *Let* R *be compatible with an* $\mathbf{L}$*-equivalence* $\sim$ *on* X*. Then* R^+ *is compatible with the power* $\mathbf{L}$*-equivalence* $\sim^+$*.*

Proof. By Lemma 5, $\sim^+$ is indeed an $\mathbf{L}$-equivalence. Compatibility of R with $\sim$ means $\sim \circ R \circ \sim \subseteq R$. By Theorem 4, $\sim^+ \circ R^+ \circ \sim^+ \subseteq (\sim \circ R \circ \sim)^+ \subseteq R^+$. $\square$

The following is our main result on power relations of $\mathbf{L}$-orders.

Theorem 7. *Let* $\mathbf{U} = \langle\langle U, \approx\rangle, \preceq\rangle$ *be an* $\mathbf{L}$*-ordered set,* $M \subseteq L^U$ *a subset containing only convex* $\mathbf{L}$*-sets in* $\mathbf{U}$*. Then* $\langle\langle M, \approx^+\rangle, \preceq^+\rangle$ *is an* $\mathbf{L}$*-ordered set.*

Proof. Since convex $\mathbf{L}$-sets are compatible with $\approx$, $\approx^+$ is an $\mathbf{L}$-equality by Theorem 5. By Theorem 6, $\preceq^+$ is compatible with $\approx^+$ and by Lemma 5, $\preceq^+$ is reflexive and transitive.

Let $V_1, V_2 \in L^U$ be convex. We have

$$\begin{aligned} (V_1 \preceq^+ V_2) \wedge (V_2 \preceq^+ V_1) &= (V_1 \preceq^{\rightarrow} V_2) \wedge (V_1 \preceq^{\leftarrow} V_2) \wedge (V_2 \preceq^{\rightarrow} V_1) \wedge (V_2 \preceq^{\leftarrow} V_1) \\ &= (V_1 \preceq^{\rightarrow} V_2) \wedge (V_2 \preceq^{\rightarrow} V_1) \wedge (V_2 \preceq^{\leftarrow} V_1) \wedge (V_1 \preceq^{\leftarrow} V_2) \\ &= \mathrm{S}(V_1, \downarrow V_2) \wedge \mathrm{S}(V_1, \uparrow V_2) \wedge \mathrm{S}(V_2, \downarrow V_1) \wedge \mathrm{S}(V_2, \uparrow V_1) \\ &= \mathrm{S}(V_1, \downarrow V_2 \cap \uparrow V_2) \wedge \mathrm{S}(V_2, \downarrow V_1 \cap \uparrow V_1) = \mathrm{S}(V_1, V_2) \wedge \mathrm{S}(V_2, V_1) = V_1 \approx^X V_2 \\ &= V_1 \approx^+ V_2 \end{aligned}$$

(the last equality following by Theorem 5), proving antisymmetry. $\square$

The following two lemmas show values of power relations $\approx^+$ and $\preceq^+$ can be computed efficiently on intervals.

Lemma 6. *Let* $V_1, V_2 \in L^U$ *be two* $\mathbf{L}$*-sets in an* $\mathbf{L}$*-ordered set having minimum and maximum,* $\min V_1 = u_1$*,* $\max V_1 = v_1$*,* $\min V_2 = u_2$*,* $\max V_2 = v_2$*. Then* $V_1 \preceq^+ V_2 = (u_1 \preceq u_2) \wedge (v_1 \preceq v_2)$*.*

Proof. We have $\{v_2\} \subseteq V_2$, which implies $\downarrow\{v_2\} \subseteq \downarrow V_2$. On the other hand, since $w \preceq v_2 = \downarrow\{v_2\}(w)$, the first inequality in (19) gives $V_2 \subseteq \downarrow\{v_2\}$, which yields $\downarrow V_2 \subseteq \downarrow\downarrow\{v_2\} = \downarrow\{v_2\}$. Thus, $\downarrow V_2(w) = w \preceq v_2$ for each w.

Now,

$$\begin{aligned} (V_1 \preceq^{\rightarrow} V_2) &= \bigwedge_{w \in U} V_1(w) \rightarrow \downarrow V_2(w) = \bigwedge_{w \in U} V_1(w) \rightarrow (w \preceq v_2) = \mathscr{U} V_1(v_2) \\ &= v_1 \preceq v_2 \end{aligned}$$

by (21).

One can prove similarly that $(V_1 \preceq^{\leftarrow} V_2) = u_1 \preceq u_2$ and obtain the desired equality. $\square$

Lemma 7. *Let $V_1 = [\![u_1, v_1]\!]$ and $V_2 = [\![u_2, v_2]\!]$ be intervals in an* **L**-*ordered set. Then $V_1 \approx^+ V_2 = (u_1 \approx u_2) \wedge (v_1 \approx v_2)$.*

Proof. According to Theorem 7, $\langle\langle M, \approx^+\rangle, \preceq^+\rangle$ with $M = \{V_1, V_2\}$ is an **L**-ordered set. Therefore, by (9) and Lemma 6,

$$V_1 \approx^+ V_2 = (V_1 \preceq^+ V_2) \wedge (V_2 \preceq^+ V_1) = (u_1 \preceq u_2) \wedge (v_1 \preceq v_2) \wedge (u_2 \preceq u_1) \wedge (v_2 \preceq v_1)$$
$$= (u_1 \approx u_2) \wedge (v_1 \approx v_2). \qquad \square$$

4. Complete L-relations

In classical theory of complete lattices (see e.g. [5]), a binary relation R on a complete lattice **U** is called complete if for each system $\{\langle u_j, v_j\rangle\}_{j\in J}$ of pairs of elements of U from $u_j \, R \, v_j$ for each $j \in J$ it follows $\left(\bigwedge_{j\in J} u_j\right) R \left(\bigwedge_{j\in J} v_j\right)$ and $\left(\bigvee_{j\in J} u_j\right) R \left(\bigvee_{j\in J} v_j\right)$.

It can be easily checked that the following condition is equivalent to the above condition of completeness of R: if $V_1, V_2 \subseteq U$ are such that for each $v_1 \in V_1$ there is $v_2 \in V_2$ satisfying $v_1 \, R \, v_2$ and for each $v_2 \in V_2$ there is $v_1 \in V_1$ satisfying $v_1 \, R \, v_2$ then $\left(\bigwedge V_1\right) R \left(\bigwedge V_2\right)$ and $\left(\bigvee V_1\right) R \left(\bigvee V_2\right)$.

This leads us to the following definition. A binary **L**-relation R on a completely lattice **L**-ordered set $\mathbf{U} = \langle\langle U, \approx\rangle, \preceq\rangle$ is called *complete* if it is compatible with $\approx$ and for any two **L**-sets $V_1, V_2 \in L^U$ it holds

$$R^+(V_1, V_2) \leq R(\inf V_1, \inf V_2), \tag{39}$$

$$R^+(V_1, V_2) \leq R(\sup V_1, \sup V_2). \tag{40}$$

The following are basic properties of complete relations on a completely lattice **L**-ordered set $\mathbf{U} = \langle\langle U, \approx\rangle, \preceq\rangle$.

Lemma 8. *If R is complete then so is R^{-1}.*

Proof. By definition,

$$(R^{-1})^+(V_1, V_2) = (R^{-1})^\rightarrow(V_1, V_2) \wedge (R^{-1})^\leftarrow(V_1, V_2) = R^\leftarrow(V_2, V_1) \wedge R^\rightarrow(V_2, V_1)$$
$$= R^+(V_2, V_1) \leq R(\inf V_2, \inf V_1) = R^{-1}(\inf V_1, \inf V_2),$$

and similarly for suprema. $\square$

Theorem 8. *The system of all complete binary* **L**-*relations on* **U** *is an* **L**-*closure system in the set $U \times U$, hence a completely lattice* **L**-*ordered set.*

Proof. We will show that 1. if R_j, $j \in J$, are complete then so is $\bigcap_{j\in J} R_j$ and 2. for each $a \in L$ and R complete the shift $a \rightarrow R$ is complete as well. Since the system of all binary **L**-relations that are compatible with $\approx$ is an **L**-closure system, there is no need to prove compatibility of the relations.

1. We have

$$\left(\bigcap_j R_j\right) \circ V(v) = \bigvee_{w\in U} \left(\bigwedge_j R_j(v, w)\right) \otimes V(w) \leq \bigwedge_j \bigvee_{w\in U} R_j(v, w) \otimes V(w)$$
$$= \bigwedge_j (R_j \circ V)(v).$$

Thus, $(\bigcap_j R_j) \circ V \subseteq \bigcap_j (R_j \circ V)$. Now,

$$\left(\bigcap_j R_j\right)^\rightarrow (V_1, V_2) = \mathrm{S}\left(V_1, \left(\bigcap_j R_j\right) \circ V_2\right) \leq \mathrm{S}\left(V_1, \bigcap_j \left(R_j \circ V_2\right)\right) = \bigwedge_j S(V_1, R_j \circ V_2)$$
$$= \bigwedge_j (R_j)^\rightarrow(V_1, V_2)$$

and, finally,

$$\left(\bigcap_j R_j\right)^{+}(V_1, V_2) = \left(\bigcap_j R_j\right)^{\rightarrow}(V_1, V_2) \wedge \left(\bigcap_j R_j\right)^{\leftarrow}(V_1, V_2)$$
$$\leq \bigwedge_j (R_j)^{\rightarrow}(V_1, V_2) \wedge (R_j)^{\leftarrow}(V_1, V_2) = \bigwedge_j (R_j)^{+}(V_1, V_2)$$
$$\leq \bigwedge_j R_j(\inf V_1, \inf V_2) = \left(\bigcap_j R_j\right)(\inf V_1, \inf V_2).$$

Similarly for suprema.

2. We have

$$((a \to R) \circ V)(v) = \bigvee_{w \in U} (a \to R(v, w)) \otimes V(w) \leq a \to \bigvee_{w \in U} R(v, w) \otimes V(w)$$
$$= a \to (R \circ V)(v).$$

Thus, $(a \to R) \circ V \subseteq a \to (R \circ V)$. Now,

$$(a \to R)^{\rightarrow}(V_1, V_2) = \mathrm{S}(V_1, (a \to R) \circ V_2) \leq \mathrm{S}(V_1, a \to (R \circ V_2)) = a \to \mathrm{S}(V_1, R \circ V_2)$$

and, finally,

$$(a \to R)^{+}(V_1, V_2) = (a \to R)^{\rightarrow}(V_1, V_2) \wedge (a \to R)^{\leftarrow}(V_1, V_2)$$
$$\leq (a \to R^{\rightarrow}(V_1, V_2)) \wedge (a \to (R^{\leftarrow}(V_1, V_2)) = a \to (R^{\rightarrow}(V_1, V_2) \wedge R^{\leftarrow}(V_1, V_2))$$
$$\leq a \to R(\inf V_1, \inf V_2) = (a \to R)(\inf V_1, \inf V_2).$$

Similarly for suprema. □

Lemma 9. *The following holds for each $V_1, V_2 \in L^U$:*

$$V_1 \preceq^{\rightarrow} V_2 \leq \sup V_1 \preceq \sup V_2, \qquad V_1 \preceq^{\leftarrow} V_2 \leq \inf V_1 \preceq \inf V_2.$$

Proof. We have by (22) and Lemma 3,

$$V_1 \preceq^{\rightarrow} V_2 = \mathrm{S}(V_1, \preceq \circ V_2) = \mathrm{S}(V_1, \downarrow V_2) \leq \sup V_1 \preceq \sup \downarrow V_2.$$

Now, by direct computation one obtains $\mathscr{U} \downarrow V_2 = \mathscr{U} V_2$ which gives $\sup V_1 \preceq \sup \downarrow V_2 = \sup V_1 \preceq \sup V_2$ and proves the first part. The second part is obtained similarly. □

Theorem 9. *The* **L***-relations $\preceq$ and $\approx$ on* **U** *are complete.*

Proof. By Lemma 9, for each $V_1, V_2 \in L^U$, $V_1 \preceq^{+} V_2 \leq (V_1 \preceq^{\rightarrow} V_2) \leq \sup V_1 \preceq \sup V_2$ and $V_1 \preceq^{+} V_2 \leq V_1 \preceq^{\leftarrow} V_1 \leq \inf V_1 \preceq \inf V_2$, proving completeness of $\preceq$.

Since $\approx = \preceq \cap \succeq$, completeness of $\approx$ follows from Lemma 8 and Theorem 8. □

5. Complete tolerances

5.1. Basic properties

Recall that an **L**-tolerance on a set X is a reflexive and symmetric binary **L**-relation on X. For an **L**-tolerance $\sim$ on a set X, an **L**-set $B \in L^X$ is called a *block of* $\sim$ if for each $x_1, x_2 \in X$ it holds $B(x_1) \otimes B(x_2) \leq (x_1 \sim x_2)$. A block B is called *maximal* if for each block B' from $B \subseteq B'$ it follows $B = B'$. The set of all maximal blocks of $\sim$ always exists by Zorn's lemma, is called *the factor set of X by* $\sim$ and denoted by $X/\sim$.

Further we set for each $x \in X$, $[\![x]\!]_\sim(y) = x \sim y$, obtaining an **L**-set $[\![x]\!]_\sim$ called *the class of $\sim$ determined by x*.

Let $\sim$ be a complete **L**-tolerance on a completely lattice **L**-ordered set $\mathbf{U} = \langle\langle U, \approx\rangle, \preceq\rangle$. From reflexivity of $\sim$ we have $V \subseteq \sim \circ V$ for each $V \in L^U$ and from symmetry $\sim^{-1} = \sim$.

For each $u \in U$ we set

$$u_\sim = \inf[\![u]\!]_\sim, \qquad u^\sim = \sup[\![u]\!]_\sim. \tag{41}$$

We denote the system of all complete **L**-tolerances on a completely lattice **L**-ordered set **U** by CTol **U** and consider it together with the **L**-equality $\approx^{U\times U}$ and **L**-order S.

Theorem 10. CTol **U** *is an* **L***-closure system in the set* $U \times U$*, hence a completely lattice* **L***-ordered set.*

Proof. Evidently, if $\sim$ is an **L**-tolerance then so is $a \to \sim$ for each $a \in L$ and if $\sim_j$, $j \in J$, are **L**-tolerances then $\bigcap_{j\in J} \sim_j$ is also an **L**-tolerance. Thus, the theorem follows from Theorem 8. □

5.2. From complete tolerances to isotone Galois connections

Lemma 10. *For each* $u \in U$, $u \sim u_\sim = u \sim u^\sim = 1$.

Proof. Let $V_1 = \{u\}$, $V_2 = [\![u]\!]_\sim$. Since $V_1 \subseteq V_2$, we have $V_1 \sim^\to V_2 = 1$. Further, $(\sim \circ V_1)(v) = v \sim u = [\![u]\!]_\sim(v)$. Thus, $V_1 \sim^\leftarrow V_2 = \mathrm{S}([\![u]\!]_\sim, [\![u]\!]_\sim) = 1$. Now,

$$V_1 \sim^+ V_2 = (V_1 \sim^\to V_2) \wedge (V_1 \sim^\leftarrow V_2) = 1$$

and by completeness of $\sim$, $1 = \inf V_1 \sim \inf V_2 = u \sim u_\sim$ and $1 = \sup V_1 \sim \sup V_2 = u \sim u^\sim$. □

Lemma 11. *For each* $u \in U$ *it holds*

$$u_\sim{}^\sim \geq u, \qquad u^\sim{}_\sim \leq u. \tag{42}$$

Proof. By Lemma 10, $[\![u]\!]_\sim(u_\sim) = 1$. This means that also $[\![u_\sim]\!]_\sim(u) = 1$. Since $u_\sim{}^\sim = \sup[\![u_\sim]\!]_\sim$, we have the first inequality.

The second inequality is analogous. □

Lemma 12. *For each* $u, v \in U$ *it holds*

$$(u \preceq v) \leq (u_\sim \preceq v_\sim), \qquad (u \preceq v) \leq (u^\sim \preceq v^\sim). \tag{43}$$

Proof. Let $a = u \preceq v$, $V_1 = \{{}^a/u, v\}$, $V_2 = \{{}^a/u^\sim, v^\sim\}$ (recall that V_1 is the **L**-set satisfying $V_1(u) = a$, $V_1(v) = 1$ and $V_1(w) = 0$ for every $w \notin \{u, v\}$; similarly for V_2). By Lemma 10, $u \sim u^\sim = v \sim v^\sim = 1$. Thus, $V_1 \subseteq \sim \circ V_2$, $V_2 \subseteq \sim \circ V_1$ and we have $V_1 \sim^+ V_2 = 1$. By completeness of $\sim$, $\sup V_1 \sim \sup V_2 = 1$.

By direct computation (e.g. using (21)), $\sup V_1 = v$. Thus, $v \sim \sup V_2 = 1$, which means $\sup V_2 \leq v^\sim$. On the other hand, since $V_2(v^\sim) = 1$, we have $\sup V_2 \geq v^\sim$, whence $\sup V_2 = v^\sim$. Using the first inequality in (19), $a = V_2(u^\sim) \leq u^\sim \preceq v^\sim$ and the second inequality in (43) is proved.

The first inequality is proved similarly. □

Theorem 11. *The pair* $\langle {}_\sim, {}^\sim \rangle$ *is an extensive isotone Galois connection on* **U**.

Proof. Let $u, v \in U$. We have by Lemma 12, Lemma 11, and transitivity of $\preceq$,

$$(u_\sim \preceq v) \leq (u_\sim{}^\sim \preceq v^\sim) \leq (u \preceq v^\sim).$$

The converse inequality $(u \preceq v^\sim) \leq (u_\sim \preceq v)$ is proved analogously. Together, $\langle {}_\sim, {}^\sim \rangle$ is an isotone Galois connection.

Extensivity of $\langle {}_\sim, {}^\sim \rangle$ follows trivially from reflexivity of $\preceq$. □

5.3. Structure of maximal blocks

Lemma 13. *If $\langle u, v\rangle$ is a fixpoint of $\langle {}_\sim, {}^\sim\rangle$ then $[\![v, u]\!]$ is a block of $\sim$.*

Proof. 1. We will prove that for each w,

$$[\![v, u]\!](w) \le u \sim w \tag{44}$$

(i.e. "if w belongs to $[\![v, u]\!]$ then it is similar to u").

Let $a = w \preceq u$, $b = v \preceq w$, $V_1 = \{u, {}^a/w\}$, $V_2 = \{{}^b/v, w\}$. By direct computation (e.g. using (21)), $\sup V_1 = u$ and $\sup V_2 = w$.

Now using Lemma 10,

$$(\sim \circ V_1)(v) = ((v \sim u) \otimes V_1(u)) \vee ((v \sim w) \otimes V_1(w)) = (1 \otimes 1) \vee ((v \sim w) \otimes a) = 1,$$
$$(\sim \circ V_1)(w) = ((w \sim u) \otimes V_1(u)) \vee ((w \sim w) \otimes V_1(w)) = ((w \sim u) \otimes 1) \vee (1 \otimes a) \ge a,$$
$$(\sim \circ V_2)(u) = ((u \sim v) \otimes V_2(v)) \vee ((u \sim w) \otimes V_2(w)) = (1 \otimes b) \vee ((u \sim w) \otimes 1) \ge b,$$
$$(\sim \circ V_2)(w) = ((w \sim v) \otimes V_2(v)) \vee ((w \sim w) \otimes V_2(w)) = ((w \sim v) \otimes b) \vee (1 \otimes 1) = 1.$$

Thus,

$$\mathrm{S}(V_1, \sim \circ V_2) = (V_1(u) \to (\sim \circ V_2)(u)) \wedge (V_1(w) \to (\sim \circ V_2)(w)) \ge b,$$
$$\mathrm{S}(V_2, \sim \circ V_1) = (V_2(v) \to (\sim \circ V_1)(v)) \wedge (V_2(w) \to (\sim \circ V_1)(w)) \ge a$$

and by completeness of $\sim$,

$$[\![v, u]\!](w) = a \wedge b \le \mathrm{S}(V_1, \sim \circ V_2) \wedge \mathrm{S}(V_2, \sim \circ V_1) = V_1 \sim^{+} V_2 \le \sup V_1 \sim \sup V_2 = u \sim w,$$

proving (44).

2. Let $w_1, w_2 \in U$, $a_1 = [\![v, u]\!](w_1)$, $a_2 = [\![v, u]\!](w_2)$, $b_1 = w_1 \preceq u$, $b_2 = w_2 \preceq u$. By (44), $a_1 \le b_1$, $a_2 \le b_2$. Let $V_1 = \{{}^{b_1}/u, w_1\}$, $V_2 = \{{}^{b_2}/u, w_2\}$. By similar direct calculations as above we obtain

$$a_1 \otimes a_2 \le a_1 \otimes b_2 = (b_1 \to b_2) \wedge (1 \to b_2 \otimes a_1) \le V_1 \sim^{\to} V_2.$$

Similarly, $a_1 \otimes a_2 \le V_1 \sim^{\leftarrow} V_2$ and

$$a_1 \otimes a_2 \le V_1 \sim^{+} V_2 \le \inf V_1 \sim \inf V_2 = w_1 \sim w_2,$$

proving $[\![v, u]\!]$ is a block. □

To understand the following lemma, recall that $B \cup \{\inf B\}$ denotes the union of the **L**-set B and the singleton **L**-set $\{\inf B\}$.

Lemma 14. *If B is a block of $\sim$ then so is $B \cup \{\inf B\}$.*

Proof. Let $u = \inf B$. It suffices to prove $B(v) \le u \sim v$ for each $v \in U$.

Let $V = \{v\}$. By adjointness and definition of block, $B(w) \to (v \sim w) \ge B(v)$ for each $w \in U$. Therefore, $B \sim^{\to} V = \bigwedge_{w \in U} B(w) \to (v \sim w) \ge B(v)$, and by direct calculation, $B \sim^{\leftarrow} V = B(v)$. Thus,

$$B(v) \le B \sim^{+} V \le \inf B \sim \inf V = u \sim v$$

and the lemma is proved. □

Lemma 15. *For each block B of $\sim$ there is a fixpoint $\langle u, v\rangle$ of $\langle {}_\sim, {}^\sim\rangle$ such that $B \subseteq [\![v, u]\!]$.*

Proof. Let $w = \inf B$, $u = w^{\sim}$, $v = u_{\sim}$. Since $\langle {}_\sim, {}^\sim\rangle$ is an isotone Galois connection (Theorem 11), $\langle u, v\rangle$ is a fixpoint. By Lemma 14, the **L**-set $B' = B \cup \{w\}$ is again a block. By the first inequality in (18), $B' \subseteq \mathscr{U}\{w\} \subseteq \mathscr{U}\{v\}$. Since B' is a block, for each w' it holds $B'(w') \le w' \sim w$. Thus, by definition of class, $B' \subseteq [\![w]\!]_\sim$, whence $\sup B' \le \sup[\![w]\!]_\sim = u$ (22). This yields $B' \subseteq \mathscr{L}\{u\}$ and we can conclude $B \subseteq B' \subseteq \mathscr{U}\{v\} \cap \mathscr{L}\{u\} = [\![v, u]\!]$. □

Theorem 12. *Maximal blocks of* $\sim$ *are exactly intervals* $[\![v, u]\!]$*, where* $\langle u, v\rangle$ *are fixpoints of* $\langle {}_\sim, {}^\sim\rangle$.

Proof. Follows from the above lemmas. □

5.4. Structure of classes

Theorem 13. *For each* $u \in U$*, the class* $[\![u]\!]_\sim$ *is equal to the interval* $[\![u_\sim, u^\sim]\!]$.

Proof. By Lemma 1, $[\![u]\!]_\sim \subseteq \mathscr{U}\{\inf[\![u]\!]_\sim\} = \mathscr{U}\{u_\sim\}$ and similarly $[\![u]\!]_\sim \subseteq \mathscr{L}\{u^\sim\}$. Thus, $[\![u]\!]_\sim \subseteq [\![u_\sim, u^\sim]\!]$.

Let $u' \in U$, $a = [\![u_\sim, u^\sim]\!](u') = (u_\sim \preceq u') \wedge (u' \preceq u^\sim)$. We will show that the **L**-set $V = \{{}^a/u', u\}$ is a block. For the lower cone of V we have

$$\mathscr{L}V(w) = (w \preceq u) \wedge (a \to (w \preceq u')). \tag{45}$$

Let $v = \inf V$. Lemma 1 gives $V \subseteq \mathscr{U}\{v\}$. By Lemma 11, $v^\sim{}_\sim \leq v$, whence $V \subseteq \mathscr{U}\{v^\sim{}_\sim\}$.

Now consider membership degrees of u and u' in the lower cone $\mathscr{L}\{v^\sim\}$. Since $a \leq u_\sim \preceq u'$ then (45) $\mathscr{L}V(u_\sim) = 1 \wedge (a \to (u_\sim \preceq u')) = 1$. Thus, $1 = u_\sim \preceq v = u \preceq v^\sim = \mathscr{L}\{v^\sim\}(u)$, obtaining $\mathscr{L}\{v^\sim\}(u) = 1$.

For $\mathscr{L}\{v^\sim\}(u')$ we first notice by (45), $\mathscr{L}V(u'_\sim) = u'_\sim \preceq u = u' \preceq u^\sim \geq a$. By Lemma 1 and (15), $\mathscr{L}V(u'_\sim) = \mathscr{L}\{v\}(u'_\sim) = u'_\sim \preceq v$ and $\mathscr{L}\{v^\sim\}(u') = u' \preceq v^\sim = u'_\sim \preceq v \geq a$. Thus, $V \subseteq \mathscr{L}\{v^\sim\}$.

Together, $V \subseteq \mathscr{U}\{v^\sim{}_\sim\} \cap \mathscr{L}\{v^\sim\} = [\![v^\sim{}_\sim, v^\sim]\!]$. By Theorem 12, $[\![v^\sim{}_\sim, v^\sim]\!]$ is a block. Thus, V is also a block and by definition of block we obtain $[\![u_\sim, u^\sim]\!](u') = V(u') = V(u') \otimes V(u) \leq u' \sim u = [\![u]\!]_\sim(u')$. Thus, $[\![u_\sim, u^\sim]\!] \subseteq [\![u]\!]_\sim$ and the theorem is proved. □

The following is an important consequence of Theorem 13 which we will use later to prove our main result.

Lemma 16. *For each* $u, v \in U$ *we have*

$$u \sim v = (u_\sim \preceq v) \wedge (v \preceq u^\sim). \tag{46}$$

Proof. The right-hand side is equal to $[\![u_\sim, u^\sim]\!](v)$, which is by Theorem 13 equal to $[\![u]\!]_\sim(v) = u \sim v$. □

We use the above results in the proof of the following lemma. By Theorem 11, for each complete **L**-tolerance $\sim$ on **U** the pair $\langle {}_\sim, {}^\sim\rangle$ is an isotone **L**-Galois connection. Thus, we can **L**-order such **L**-Galois connections by the **L**-relation $\preceq_{\mathrm{IGal}(\mathbf{U},\mathbf{U})}$ (27).

Lemma 17. *For any two complete* **L***-tolerances* $\sim_1$, $\sim_2$ *on* **U** *we have*

$$\mathrm{S}(\sim_1, \sim_2) = \langle {}_{\sim_1}, {}^{\sim_1}\rangle \preceq_{\mathrm{IGal}(\mathbf{U},\mathbf{U})} \langle {}_{\sim_2}, {}^{\sim_2}\rangle.$$

Proof. By definitions of S and $\preceq_{\mathrm{IGal}(\mathbf{U},\mathbf{U})}$ we have to prove the following equality:

$$\bigwedge_{u,v\in U} (u \sim_1 v) \to (u \sim_2 v) = \bigwedge_{u\in U} (u_{\sim_2} \preceq u_{\sim_1}) \wedge \bigwedge_{u\in U} (u^{\sim_1} \preceq u^{\sim_2}). \tag{47}$$

We will proceed by proving both inequalities "$\leq$" and "$\geq$".

"$\leq$": Since $u \sim_1 u^{\sim_1} = 1$ (Lemma 10), the left-hand side of (47) is $\leq \bigwedge_{u\in U}(u \sim_1 u^{\sim_1}) \to (u \sim_2 u^{\sim_1}) = \bigwedge_{u\in U} u \sim_2 u^{\sim_1}$. Now by Theorem 13 and (15) we have

$$\begin{aligned} u \sim_2 u^{\sim_1} &= [\![u^{\sim_1}]\!]_{\sim_2}(u) = \mathscr{L}\{(u^{\sim_1})^{\sim_2}\}(u) \wedge \mathscr{U}\{(u^{\sim_1})_{\sim_2}\}(u) \\ &\leq \mathscr{U}\{(u^{\sim_1})_{\sim_2}\}(u) = (u^{\sim_1})_{\sim_2} \preceq u = u^{\sim_1} \preceq u^{\sim_2}. \end{aligned}$$

Thus, $\bigwedge_{u,v\in U}(u \sim_1 v) \to (u \sim_2 v) \leq \bigwedge_{u\in U} u^{\sim_1} \preceq u^{\sim_2}$. The inequality $\bigwedge_{u,v\in U}(u \sim_1 v) \to (u \sim_2 v) \leq \bigwedge_{u\in U} u_{\sim_2} \preceq u_{\sim_1}$ is proved similarly.

"$\geq$": by Theorem 13 and (15) again and by transitivity of $\preceq$ we have

$$(u^{\sim_1} \preceq u^{\sim_2}) \otimes (u \sim_1 v) = (u^{\sim_1} \preceq u^{\sim_2}) \otimes [\![u]\!]_{\sim_1}(v) \quad = (u^{\sim_1} \preceq u^{\sim_2}) \otimes \big((v \preceq u^{\sim_1}) \wedge (u_{\sim_1} \preceq v)\big)$$
$$\leq (u^{\sim_1} \preceq u^{\sim_2}) \otimes (v \preceq u^{\sim_1}) \leq v \preceq u^{\sim_2}.$$

Similarly $(u_{\sim_2} \preceq u_{\sim_1}) \otimes (u \sim_1 v) \leq (u_{\sim_2} \preceq v)$, thereby (Theorem 13 and (15))

$$u \sim_2 v = [\![u]\!]_{\sim_2}(v) = (u_{\sim_2} \preceq v) \wedge (v \preceq u^{\sim_2}) \geq \big((u^{\sim_1} \preceq u^{\sim_2}) \otimes (u \sim_1 v)\big) \wedge \big((u_{\sim_2} \preceq u_{\sim_1}) \otimes (u \sim_1 v)\big)$$
$$\geq \big((u^{\sim_1} \preceq u^{\sim_2}) \wedge (u_{\sim_2} \preceq u_{\sim_1})\big) \otimes (u \sim_1 v).$$

By adjointness,

$$(u \sim_1 v) \to (u \sim_2 v) \geq (u^{\sim_1} \preceq u^{\sim_2}) \wedge (u_{\sim_2} \preceq u_{\sim_1}),$$

yielding the "$\geq$" part of (47). □

5.5. *From extensive isotone Galois connections to complete tolerances*

Let $\langle f, g\rangle$ be an extensive isotone **L**-Galois connection on a completely lattice **L**-ordered set $\mathbf{U} = \langle\langle U, \approx\rangle, \preceq\rangle$. We set for each $u, v \in U$,

$$u \sim_{\langle f,g\rangle} v = (f(u) \preceq v) \wedge (v \preceq g(u)). \tag{48}$$

The following theorem summarizes main properties of the **L**-relation $\sim_{\langle f,g\rangle}$.

Theorem 14. *$\sim_{\langle f,g\rangle}$ is a complete tolerance such that for each $u \in U$,*

$$u_{\sim_{\langle f,g\rangle}} = f(u), \qquad u^{\sim_{\langle f,g\rangle}} = g(u). \tag{49}$$

Proof. The **L**-relation $\sim_{\langle f,g\rangle}$ is evidently reflexive and symmetric, hence an **L**-tolerance.

Set $R(u, v) = u \preceq g(v)$. We have $u \sim_{\langle f,g\rangle} v = R(u, v) \wedge R^{-1}(u, v)$. Thus, by Lemma 8 and Theorem 8, in order to prove completeness of $\sim_{\langle f,g\rangle}$ it is sufficient to prove that R is complete.

Let $V \in L^U$. Using the obvious inequality $V(w) \leq g(V)(g(w))$ we have

$$\begin{aligned}(R \circ V)(v) &= \bigvee_{w\in U} R(v, w) \otimes V(w) = \bigvee_{w\in U} (v \preceq g(w)) \otimes V(w)\\ &\leq \bigvee_{w\in U} (v \preceq g(w)) \otimes g(V)(g(w)) \leq \bigvee_{w'\in U} (v \preceq w') \otimes g(V)(w')\\ &= (\preceq \circ\, g(V))(v)\end{aligned}$$

and

$$\begin{aligned}(R^{-1} \circ V)(v) &= \bigvee_{w\in U} R(w, v) \otimes V(w) = \bigvee_{w\in U} (v \succeq f(w)) \otimes V(w)\\ &\leq \bigvee_{w\in U} (v \succeq f(w)) \otimes f(V)(f(w)) \leq \bigvee_{w'\in U} (v \succeq f(w)) \otimes f(V)(w')\\ &= (\succeq \circ\, f(V))(v),\end{aligned}$$

whence $R^{\to}(V_1, V_2) = \mathrm{S}(V_1, R \circ V_2) \leq \mathrm{S}(V_1, \preceq \circ\, g(V_2)) = V_1 \preceq^{\to} g(V_2)$ and $R^{\leftarrow}(V_1, V_2) = \mathrm{S}(V_2, R^{-1} \circ V_1) \leq \mathrm{S}(V_2, \succeq \circ\, f(V_1)) = f(V_1) \preceq^{\leftarrow} V_2$.

Now by Lemma 9 and Theorem 1 (d),

$$\begin{aligned}R^{+}(V_1, V_2) &\leq R^{\to}(V_1, V_2) \leq V_1 \preceq^{\to} g(V_2) \leq \sup V_1 \preceq \sup g(V_2)\\ &\leq \sup V_1 \preceq g(\sup V_2) = R(\sup V_1, \sup V_2),\\ R^{+}(V_1, V_2) &\leq R^{\leftarrow}(V_1, V_2) \leq f(V_1) \preceq^{\leftarrow} V_2 \leq \inf f(V_1) \preceq \inf V_2\\ &\leq f(\inf V_1) \preceq \inf V_2 = R(\inf V_1, \inf V_2),\end{aligned}$$

proving completeness of R.

To prove (49), we notice that for each $u \in U$ the class $[\![u]\!]_{\sim_{\langle f,g\rangle}}$ is equal to the interval $[\![f(u), g(u)]\!]$:

$$\begin{aligned}[\![u]\!]_{\sim_{\langle f,g\rangle}}(v) &= u \sim_{\langle f,g\rangle} v = (f(u) \preceq v) \wedge (v \preceq g(u)) \\ &= \mathscr{U}\{f(u)\}(v) \wedge \mathscr{L}\{g(u)\}(v) = [\![f(u), g(u)]\!](v).\end{aligned}$$

Now, $u_{\sim_{\langle f,g\rangle}} = \inf[\![f(u), g(u)]\!] = f(u)$ and $u^{\sim_{\langle f,g\rangle}} = \sup[\![f(u), g(u)]\!] = g(u)$. □

5.6. Factorization theorem, representation theorem

By Theorem 12, the factor set $U/\sim$ consists of intervals. Therefore, by Theorem 7, the tuple $\mathbf{U}/\sim = \langle\langle U/\sim, \approx^+\rangle, \preceq^+\rangle$ is an $\mathbf{L}$-ordered set. By Theorem 11, the pair $\langle {}_\sim, {}^\sim\rangle$ is an extensive isotone Galois connection. The following theorem connects $\mathbf{U}/\sim$ with the completely lattice $\mathbf{L}$-ordered set $\mathrm{Fix}_{\langle \sim, \sim\rangle}$.

Theorem 15 *(Factorization theorem). The* $\mathbf{L}$*-ordered set* $\mathbf{U}/\sim$ *is isomorphic to the completely lattice* $\mathbf{L}$*-ordered set* $\mathrm{Fix}_{\langle \sim, \sim\rangle}$ *and is therefore a completely lattice* $\mathbf{L}$*-ordered set as well. The isomorphism is given by* $[\![v, u]\!] \to \langle u, v\rangle$.

Proof. Follows directly from Lemma 7, 6 and definition of $\mathbf{L}$-order on $\mathrm{Fix}_{\langle \sim, \sim\rangle}$. □

The second main result is that complete tolerances on completely lattice $\mathbf{L}$-ordered sets can be represented by extensive isotone Galois connections.

Theorem 16 *(Representation theorem). The mapping*

$$\sim \mapsto \langle {}_\sim, {}^\sim\rangle$$

is an isomorphism between $\mathrm{CTol}\,\mathbf{U}$ *and* $\mathrm{EIGal}(\mathbf{U})$*. Its inverse is*

$$\langle f, g\rangle \mapsto \sim_{\langle f,g\rangle}.$$

$\mathrm{CTol}\,\mathbf{U}$ *and* $\mathrm{EIGal}(\mathbf{U})$ *are both completely lattice* $\mathbf{L}$*-ordered sets.*

Proof. Follows from Theorem 11, Lemma 16, Theorem 14, Lemma 17, and Theorem 10. □

6. Conclusion

We introduced a notion of complete binary fuzzy relation on complete fuzzy lattice (completely lattice fuzzy ordered set). The notion leads in ordinary (crisp) case to the classical notion of complete relation on complete lattice, but re-formulated in terms of the theory of power structures. We proved some basic properties of power structures of fuzzy ordered sets.

In the main part of the paper, we defined complete fuzzy binary relations and complete fuzzy tolerances and investigated their properties. Our main results are covered by Theorem 15 and 16. We show that a fuzzy complete lattice can be factorized by means of a complete fuzzy tolerance and that there is a naturally-defined structure of fuzzy complete lattice on the factor set. This result corresponds to the known result from the ordinary case [3,4].

In addition, we found an isomorphism between the fuzzy ordered sets of all complete fuzzy tolerances and extensive isotone fuzzy Galois connections on a fuzzy complete lattice. This result is useful for testing fuzzy tolerances for completeness.

Our future research will focus on applying results from this paper to formal concept analysis of data with fuzzy attributes [7]. In ordinary setting, there is a correspondence between complete tolerances on a concept lattice and so called block relations of the associated formal context [4,5]. Theorem 15 and 16 will help establish a link between complete fuzzy tolerances on a fuzzy concept lattice and (properly defined) block relations on the formal context. This will allow generalize results from [4,5] to fuzzy concept lattices.[2]

One consequence of our results is that the condition of compatibility from the definition of complete relation on a completely lattice $\mathbf{L}$-ordered set (Sec. 4) is redundant for $\mathbf{L}$-tolerances. This leads to an open problem, namely, whether the condition of compatibility follows from the other conditions of the definition.

[2] Meanwhile, these results have been published in [25].

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G Block Relations in Formal Fuzzy Concept Analysis

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One of the main problems in FCA, especially in the graded setting, is to reduce a concept lattice of a formal context to an appropriate size to make it graspable and understandable by a human user. A natural way to do it is to substitute the formal context by its block relation which is equivalent to factorization of the concept lattice by a complete tolerance. We generalize the known results on the correspondence of block relations of formal contexts and complete tolerances on concept lattices to the graded setting.

We provide a definition of block **L**-relation—a convenient generalization of the notion of block-relation from [66]. We show, that the block **L**-relations are in one-to-one correspondence to particular automorphisms on concept lattices. We describe the structure of systems of all block **L**-relations. All the results are considered for all three kinds of concept-forming operators.

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Block relations in formal fuzzy concept analysis

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ABSTRACT

One of the main problems in formal concept analysis (especially in fuzzy setting) is to reduce a concept lattice of a formal context to appropriate size to make it graspable and understandable. A natural way to do it is to substitute the formal context by its block relation which is equivalent to factorization of the concept lattice by a complete tolerance. We generalize known results on the correspondence of block relations of formal contexts and complete tolerances on concept lattices to fuzzy setting and we provide an illustrative example of using block relations to reduce the size of a concept lattice.

1. Introduction

The present paper studies block relations and related structures in formal fuzzy concept analysis.

Formal concept analysis (FCA) [16,14] is a method of exploratory data analysis based on a formalization of a philosophical view of conceptual knowledge. The basic notion in FCA is that of a formal concept which consists of two sets: extent – a set of all objects sharing the same attributes, and intent – a set of all the shared attributes. This definition of formal concept comes from traditional (Port-Royal) logic [1,22]. The input data for FCA (in basic setting), called a formal context, is a flat table in which rows represent objects and columns represent attributes. Entries of the table contain 1 (or $\times$), which means that the corresponding object has the corresponding attribute, or 0 (blank) which means the opposite. The main output is a hierarchy of formal concepts in the table.

In everyday life we use concepts which are not sharply bounded (e.g. 'great dancer' or 'middle aged man'). In terms of FCA, the formal concepts do not divide objects and attributes sharply into those which are covered and which are not; it is rather a matter of degree. There are several approaches to generalize formal concept analysis to work with graded data [8,5,29,27,21,13]. Many of them are based on the Zadeh's theory of fuzzy sets [33]. Our work follows approach of [8].

One of the main problems in FCA (especially in fuzzy setting) is to reduce the size of a concept lattice to make it graspable and understandable. One method to achieve it is to use a block relation and obtain rougher data which contain a smaller number of formal concepts. That (in the crisp case) corresponds to particular factorization of the associated concept lattice [32,16], or to particular automorphism of the concept lattice. We generalize these known results to fuzzy setting.

In [26] we have studied a generalization of bonds—intercontextual structures binding two fuzzy contexts—and related morphisms of associated concept lattices. A few specific cases of these structures (in the crisp case) deserve a special attention at their generalization to fuzzy setting. For instance, infomorphisms, scale measures, and presently studied block relations are such cases. We study block relations in fuzzy setting for two main reasons. First, in the crisp setting they

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correspond to another interesting notion—complete tolerances on associated concept lattices [32,16]. Second, they provide a natural way to reduce a concept lattice [28].

The paper is structured as follows. Section 2 recalls notions used in the paper. In Sections 3.1–3.3 we separately study three instances of block **L**-relations. In addition, Sections 2.4, 3.1, 3.3, and 3.5 contain a central running example of this paper.

2. Preliminaries

2.1. Residuated lattices, fuzzy sets, and fuzzy relations

We use complete residuated lattices as basic structures of truth degrees. A complete residuated lattice is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that

(i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, i.e. a partially ordered set in which arbitrary infima and suprema exist;
(ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation which is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$;
(iii) $\otimes$ and $\rightarrow$ satisfy adjointness, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$.

0 and 1 denote the least and greatest element, respectively. The partial order of **L** is denoted by $\leq$. Throughout this work, **L** denotes an arbitrary complete residuated lattice.

Elements a of L are called truth degrees. Operations $\otimes$ (multiplication) and $\rightarrow$ (residuum) play the role of (truth functions of) “fuzzy conjunction” and “fuzzy implication”. Furthermore, we define the complement of $a \in L$ as

$$\neg a = a \rightarrow 0 \tag{1}$$

An **L**-set A in a universe set X is a mapping assigning to each $x \in X$ some truth degree $A(x) \in L$ [20,19]. The set of all **L**-sets in a universe X is denoted L^X.

Operations with **L**-sets are defined componentwise. For instance, the intersection of **L**-sets $A, B \in L^X$ is an **L**-set $A \cap B$ in X such that $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in X$, etc.

Intersection and union of two **L**-sets can be generalized to any number of **L**-sets and even to **L**-sets of **L**-sets. For an **L**-set $U : L^X \rightarrow L$, the intersection $\bigcap U$ and union $\bigcup U$ of U are **L**-sets in X, defined by

$$\bigcap U(x) = \bigwedge_{A \in L^X} U(A) \rightarrow A(x), \qquad \bigcup U(x) = \bigvee_{A \in L^X} U(A) \otimes A(x), \tag{2}$$

for any $x \in X$.

We often use the following notation to specify fuzzy sets. If $x_1, x_2, \ldots, x_n \in X$ are pairwise distinct and $a_1, a_2, \ldots, a_n \in L$ then $\{{}^{a_1}/x_1, {}^{a_2}/x_2, \ldots, {}^{a_n}/x_n\}$ denotes the **L**-set A given by $A(x) = a_k$ if $x = x_k$ for some $k \in \{1, 2, \ldots, n\}$ and $A(x) = 0$ otherwise. More generally, for an index set K, let for each $k \in K$, $a_k \in L$ and $x_k \in X$ be pairwise distinct. We denote by $\{{}^{a_k}/x_k \mid k \in K\}$ the **L**-set A satisfying $A(x) = a_k$ if $x = x_k$ for some $k \in K$ and $A(x) = 0$ otherwise.

Sometimes, it is useful to allow repeated occurrences of elements of X in this notation. In this case the membership degree of each element is obtained as supremum of all its listed degrees: if $A = \{{}^{a_1}/x_1, {}^{a_2}/x_2, \ldots, {}^{a_n}/x_n\}$ then

$$A(x) = \bigvee \{a_k \mid k \in \{1, \ldots, n\} \text{ and } x_k = x\}$$

and if $A = \{{}^{a_k}/x_k \mid k \in K\}$ then

$$A(x) = \bigvee \{a_k \mid k \in K \text{ and } x_k = x\}.$$

An **L**-set $A \in L^X$ is called crisp if $A(x) \in \{0, 1\}$ for each $x \in X$. Crisp **L**-sets can be identified with ordinary sets. For a crisp A, we also write $x \in A$ for $A(x) = 1$ and $x \notin A$ for $A(x) = 0$. An **L**-set $A \in L^X$ is called empty (denoted by $\emptyset$) if $A(x) = 0$ for each $x \in X$. For $a \in L$ and $A \in L^X$, the **L**-sets $a \otimes A \in L^X$, $a \rightarrow A$, $A \rightarrow a$, and $\neg A$ in X are defined by

$$(a \otimes A)(x) = a \otimes A(x), \tag{3}$$

$$(a \rightarrow A)(x) = a \rightarrow A(x), \tag{4}$$

$$(A \rightarrow a)(x) = A(x) \rightarrow a, \tag{5}$$

$$\neg A(x) = A(x) \rightarrow 0. \tag{6}$$

For $A \in L^X$ the **L**-sets $a \otimes A, a \rightarrow A, A \rightarrow a$ are called a-multiplication, a-shift, and a-complement, respectively. By (2) we have $a \otimes A = \bigcup \{{}^{a}/A\}$ and $a \rightarrow A = \bigcap \{{}^{a}/A\}$.

Given $A, B \in L^X$, we define the subsethood degree

$$S(A, B) = \bigwedge_{x \in X} \big(A(x) \to B(x)\big), \tag{7}$$

which generalizes the ordinary subsethood relation $\subseteq$. $S(A, B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A, B) = 1$. As a consequence, $A \subseteq B$ iff $A(x) \le B(x)$ for each $x \in X$.

Further, we set

$$A \approx^X B = S(A, B) \wedge S(B, A). \tag{8}$$

The value $A \approx^X B$ is interpreted as the degree to which the sets A and B are equal.

2.2. Binary L-relations

Binary **L**-relations between X and Y can be thought of as **L**-sets in the universe $X \times Y$. That is, a binary **L**-relation $I \in L^{X \times Y}$ between a set X and a set Y is a mapping assigning to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$ (the degree to which x and y are related by I).

A binary **L**-relation R on a set X (i.e. $R \in L^{X \times X}$) is called reflexive if $R(x, x) = 1$ for any $x \in X$, symmetric if $R(x, y) = R(y, x)$ for any $x, y \in X$, and transitive if $R(x, y) \otimes R(y, z) \le R(x, z)$ for any $x, y, z \in X$. R is called an **L**-tolerance if it is reflexive and symmetric, and an **L**-equivalence if it is reflexive, symmetric and transitive. If R is an **L**-equivalence such that for any $x, y \in X$, $R(x, y) = 1$ implies $x = y$, then R is called an **L**-equality on X.

Let $\sim$ be an **L**-equivalence on X. We say that a binary **L**-relation R on X is compatible with $\sim$ if for each $x, x', y, y' \in X$,

$$R(x, y) \otimes (x \sim x') \otimes (y \sim y') \le R(x', y').$$

Various composition operators for binary **L**-relations were extensively studied by Bandler and Kohout, see e.g. [23]; we will use the following three composition operators, defined for relations $A \in L^{X \times F}$ and $B \in L^{F \times Y}$:

$$(A \circ B)(x, y) = \bigvee_{f \in F} A(x, f) \otimes B(f, y), \tag{9}$$

$$(A \triangleleft B)(x, y) = \bigwedge_{f \in F} A(x, f) \to B(f, y), \tag{10}$$

$$(A \triangleright B)(x, y) = \bigwedge_{f \in F} B(f, y) \to A(x, f). \tag{11}$$

All of them have natural verbal descriptions. For instance, $(A \circ B)(x, y)$ is the truth degree of the proposition "*there is a factor f such that f applies to object x and attribute y is a manifestation of f*"; $(A \triangleleft B)(x, y)$ is the truth degree of "*for every factor f, if f applies to object x then attribute y is a manifestation of f*". Note also that for $L = \{0, 1\}$, $A \circ B$ coincides with the well-known composition of binary relations.

We will occasionally use some of the following properties concerning associativity of several composition operators.

Theorem 1. *(See [8].)* Associativity of composition operators. *We have*

$$R \circ (S \circ T) = (R \circ S) \circ T, \tag{12}$$

$$R \triangleleft (S \triangleright T) = (R \triangleleft S) \triangleright T, \tag{13}$$

$$R \triangleleft (S \triangleleft T) = (R \circ S) \triangleleft T, \tag{14}$$

$$R \triangleright (S \circ T) = (R \triangleright S) \triangleright T. \tag{15}$$

Distributivity of composition operators. *We have*

$$(\bigcup_i R_i) \circ S = \bigcup_i (R_i \circ S), \quad \text{and} \quad R \circ (\bigcup_i S_i) = \bigcup_i (R \circ S_i), \tag{16}$$

$$(\bigcap_i R_i) \triangleright S = \bigcap_i (R_i \triangleright S), \quad \text{and} \quad R \triangleright (\bigcup_i S_i) = \bigcap_i (R \triangleright S_i), \tag{17}$$

$$(\bigcup_i R_i) \triangleleft S = \bigcap_i (R_i \triangleleft S), \quad \text{and} \quad R \triangleleft (\bigcap_i S_i) = \bigcap_i (R \triangleleft S_i). \tag{18}$$

2.3. **L**-ordered sets

An **L**-order on a set U with an **L**-equality $\approx$ is a binary **L**-relation $\preceq$ on U which is compatible with $\approx$, reflexive, transitive and satisfies $(u \preceq v) \wedge (v \preceq u) \leq u \approx v$ for any $u, v \in U$ (antisymmetry). The tuple $\mathbf{U} = \langle\langle U, \approx\rangle, \preceq\rangle$ is called an **L**-ordered set. An immediate consequence of the definition is that for any $u, v \in U$ it holds

$$u \approx v = (u \preceq v) \wedge (v \preceq u). \tag{19}$$

If $\mathbf{U} = \langle\langle U, \approx\rangle, \preceq\rangle$ is an **L**-ordered set then the tuple $\langle U, {}^1\!\preceq\rangle$, where ${}^1\!\preceq$ is the 1-cut of $\preceq$, is a (partially) ordered set. We sometimes write $\leq$ instead of ${}^1\!\preceq$ and use the symbols $\wedge$, $\bigwedge$ resp. $\vee$, $\bigvee$ for denoting infima resp. suprema in $\langle U, {}^1\!\preceq\rangle$.

For two **L**-ordered sets $\mathbf{U} = \langle\langle U, \approx_U\rangle, \preceq_U\rangle$ and $\mathbf{V} = \langle\langle V, \approx_V\rangle, \preceq_V\rangle$, a mapping $f : U \to V$ is isotone, if $(u_1 \preceq_U u_2) \leq (f(u_1) \preceq_V f(u_2))$, and an embedding, if $(u_1 \preceq_U u_2) = (f(u_1) \preceq_V f(u_2))$, for any $u_1, u_2 \in V$.

A mapping $f : U \to V$ is called an isomorphism of **U** and **V**, if it is both, a bijection and an embedding. **U** and **V** are then called isomorphic.

An antitone mapping and dual embedding are defined by $(u_1 \preceq_U u_2) \leq (f(u_2) \preceq_V f(u_1))$ and $(u_1 \preceq_U u_2) = (f(u_2) \preceq_V f(u_1))$, respectively. A dual isomorphism is a bijection which is a dual embedding.

Let **U** be an **L**-ordered set. For any $W \in L^U$ and $w \in U$ we set

$$\mathcal{L}W(w) = \bigwedge_{u \in U} W(u) \to (w \preceq u), \qquad \mathcal{U}W(w) = \bigwedge_{u \in U} W(u) \to (u \preceq w). \tag{20}$$

The right-hand side of the first equation is the degree of "For each $u \in U$, if u is in W, then w is less than or equal to u", and similarly for the second equation. Thus, $\mathcal{L}W(w)$ ($\mathcal{U}W(w)$) can be seen as the degree to which w is less (greater) than or equal to each element of W. The **L**-set $\mathcal{L}W$ (resp. $\mathcal{U}W$) is called the lower cone (resp. the upper cone) of W.

For $u, v \in U$, $u \leq v$, the **L**-set $[\![u, v]\!] = \mathcal{U}\{u\} \cap \mathcal{L}\{v\}$ is called the **L**-interval with bounds u and v. We have

$$[\![u, v]\!](w) = (u \preceq w) \wedge (w \preceq v). \tag{21}$$

Let **U** be an **L**-ordered set. For any **L**-set $W \in L^U$ there exists at most one element $u \in U$ such that $\mathcal{L}W(u) \wedge \mathcal{U}(\mathcal{L}W)(u) = 1$ (resp. $\mathcal{U}W(u) \wedge \mathcal{L}(\mathcal{U}W)(u) = 1$) [5,8]. If there is such an element, we call it the infimum of W (resp. the supremum of W) and denote $\inf W$ (resp. $\sup W$); otherwise we say that the infimum (resp. supremum) does not exist.

U is called completely lattice **L**-ordered, if for each $W \in L^U$, both $\inf W$ and $\sup W$ exist.

An important example of a completely lattice **L**-ordered set is the tuple $\mathbf{L}^X = \langle\langle L^X, \approx^X\rangle, S\rangle$, where X is an arbitrary set and $\approx^X$ and S are given by (8) and (7), respectively. Infima and suprema in $\mathbf{L}^X$ are intersections and unions: for any $M \in L^{L^X}$ we have

$$\inf M = \bigcap M, \qquad \sup M = \bigcup M. \tag{22}$$

2.4. Formal fuzzy concept analysis

An **L**-context is a triplet $\langle X, Y, I\rangle$ where X and Y are (ordinary) sets and $I \in L^{X \times Y}$ is an **L**-relation between X and Y. Elements of X are called objects, elements of Y are called attributes, I is called an incidence relation. $I(x, y) = a$ is read: "The object x has the attribute y to the degree a."

Consider the following pairs of operators, called concept-forming operators, induced by an **L**-context $\langle X, Y, I\rangle$. First, the pair $\langle{}^{\uparrow}, {}^{\downarrow}\rangle$ of operators ${}^{\uparrow} : L^X \to L^Y$ and ${}^{\downarrow} : L^Y \to L^X$ is defined, for all $A \in L^X$ and $B \in L^Y$, by

$$A^{\uparrow}(y) = \bigwedge_{x \in X} A(x) \to I(x, y), \quad B^{\downarrow}(x) = \bigwedge_{y \in Y} B(y) \to I(x, y). \tag{23}$$

Second, the pair $\langle{}^{\cap}, {}^{\cup}\rangle$ of operators ${}^{\cap} : L^X \to L^Y$ and ${}^{\cup} : L^Y \to L^X$ is defined by

$$A^{\cap}(y) = \bigvee_{x \in X} A(x) \otimes I(x, y), \quad B^{\cup}(x) = \bigwedge_{y \in Y} I(x, y) \to B(y). \tag{24}$$

Third, the pair $\langle{}^{\wedge}, {}^{\vee}\rangle$ of operators ${}^{\wedge} : L^X \to L^Y$ and ${}^{\vee} : L^Y \to L^X$ is defined by

$$A^{\wedge}(y) = \bigwedge_{x \in X} I(x, y) \to A(x), \quad B^{\vee}(x) = \bigvee_{y \in Y} B(y) \otimes I(x, y). \tag{25}$$

Throughout this paper we often need to emphasize which **L**-relation induces particular pair of operators. In such cases we write the associated **L**-relation into the subscript; for instance we write $\uparrow_I$ instead of just $\uparrow$ to emphasize that it is a concept-forming operator induced by the **L**-relation I.

	c1	c2	c3	c4	c5
BV	0.4	0.4	0.2	0.4	0.6
LH	0.8	0.8	0.8	1	1
MD	0.8	0.4	1	0.8	1
TSS	0.4	0.4	0.8	0.6	0.4

Fig. 1. Formal context of movies ("Blue Velvet" (BV), "Lost Highway" (LH), "Mulholland Drive" (MD), and "The Straight Story" (TSS)), reviewers (David Sterritt (c1), Desson Thomson (c2), Jonathan Rosenbaum (c3), Owen Gleiberman (c4) and Roger Ebert (c5)) and their ratings on the 6-element Łukasiewicz chain $L = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$. Data taken from www.metacritic.com on March 20, 2011.

Remark 1. Notice that the three pairs of concept-forming operators can be interpreted as compositions relations. Applying the isomorphisms $L^{1\times X} \cong L^X$ and $L^{Y\times 1} \cong L^Y$ whenever necessary, one could write them, alternatively, as follows:

$$\begin{array}{lll} A^{\uparrow} = A \triangleleft I & A^{\cap} = A \circ I & A^{\wedge} = A \triangleright I \\ B^{\downarrow} = I \triangleright B & B^{\cup} = I \triangleleft B & B^{\vee} = I \circ B \end{array}$$

Furthermore, denote the corresponding sets of fixpoints by $\mathcal{B}^{\Delta\nabla}(X, Y, I)$, where $\mathcal{B}^{\Delta\nabla}$ is either of $\mathcal{B}^{\uparrow\downarrow}$, $\mathcal{B}^{\cap\cup}$, or $\mathcal{B}^{\wedge\vee}$, i.e.

$$\mathcal{B}^{\uparrow\downarrow}(X, Y, I) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\uparrow} = B,\ B^{\downarrow} = A\},$$
$$\mathcal{B}^{\cap\cup}(X, Y, I) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\cap} = B,\ B^{\cup} = A\},$$
$$\mathcal{B}^{\wedge\vee}(X, Y, I) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\wedge} = B,\ B^{\vee} = A\}.$$

The sets of fixpoints are completely lattice **L**-ordered sets with **L**-equality $\approx$ and **L**-order $\preceq$ given by

$$\langle A_1, B_1\rangle \approx \langle A_2, B_2\rangle = A_1 \approx^X A_2 = B_1 \approx^Y B_2 \tag{26}$$

$$\langle A_1, B_1\rangle \preceq \langle A_2, B_2\rangle = \begin{cases} S(A_1, A_2) = S(B_2, B_1) & \text{for } \langle\Delta, \nabla\rangle = \langle\uparrow, \downarrow\rangle \\ S(A_1, A_2) = S(B_1, B_2) & \text{for } \langle\Delta, \nabla\rangle = \langle\cap, \cup\rangle \\ S(A_2, A_1) = S(B_2, B_1) & \text{for } \langle\Delta, \nabla\rangle = \langle\wedge, \vee\rangle \end{cases} \tag{27}$$

and called the standard (resp. object-oriented, resp. property-oriented) **L**-concept lattices associated with I, and their elements are called standard (resp. object-oriented, resp. property-oriented) formal **L**-concepts. In this paper we call them just **L**-concepts as it is always clear which kind is considered.

For a concept lattice $\mathcal{B}^{\Delta\nabla}(X, Y, I)$, denote the corresponding sets of extents and intents by $\mathrm{Ext}^{\Delta\nabla}(X, Y, I)$ and $\mathrm{Int}^{\Delta\nabla}(X, Y, I)$. That is,

$$\mathrm{Ext}^{\Delta\nabla}(X, Y, I) = \{A \in L^X \mid \langle A, B\rangle \in \mathcal{B}^{\Delta\nabla}(X, Y, I) \text{ for some } B\},$$
$$\mathrm{Int}^{\Delta\nabla}(X, Y, I) = \{B \in L^Y \mid \langle A, B\rangle \in \mathcal{B}^{\Delta\nabla}(X, Y, I) \text{ for some } A\}.$$

The operators induced by an **L**-context and their sets of fixpoints have extensively been studied, see e.g. [3,5,6,18,29].

We will need the following result.

Theorem 2. *(See [10].) Consider* **L***-contexts* $\langle X, Y, I\rangle$, $\langle X, F, A\rangle$, *and* $\langle F, Y, B\rangle$.

(a) $\mathrm{Int}^{\cap\cup}(X, Y, I) \subseteq \mathrm{Int}^{\cap\cup}(F, Y, B)$ *if and only if there exists* $A' \in L^{X\times F}$ *such that* $I = A' \circ B$,
(b) $\mathrm{Ext}^{\wedge\vee}(X, Y, I) \subseteq \mathrm{Ext}^{\wedge\vee}(X, F, A)$ *if and only if there exists* $B' \in L^{F\times Y}$ *such that* $I = A \circ B'$,
(c) $\mathrm{Int}^{\uparrow\downarrow}(X, Y, I) \subseteq \mathrm{Int}^{\uparrow\downarrow}(F, Y, B)$ *if and only if there exists* $A' \in L^{X\times F}$ *such that* $I = A' \triangleleft B$,
(d) $\mathrm{Ext}^{\uparrow\downarrow}(X, Y, I) \subseteq \mathrm{Ext}^{\uparrow\downarrow}(X, F, A)$ *if and only if there exists* $B' \in L^{F\times Y}$ *such that* $I = A \triangleright B'$,
(e) $\mathrm{Ext}^{\uparrow\downarrow}(X, Y, I) \subseteq \mathrm{Ext}^{\cap\cup}(X, F, A)$ *if and only if there exists* $B' \in L^{F\times Y}$ *such that* $I = A \triangleleft B'$,
(f) $\mathrm{Int}^{\uparrow\downarrow}(X, Y, I) \subseteq \mathrm{Int}^{\wedge\vee}(F, Y, B)$ *if and only if there exists* $A' \in L^{X\times Y}$ *such that* $I = A' \triangleright B$.

In addition,

(g) $\mathrm{Ext}^{\cap\cup}(X, Y, A \circ B) \subseteq \mathrm{Ext}^{\cap\cup}(X, F, A)$.
(h) $\mathrm{Int}^{\wedge\vee}(X, Y, A \circ B) \subseteq \mathrm{Int}^{\wedge\vee}(F, Y, B)$.

2.4.1. Illustrative example (start)

Let **L** be the 6-element Łukasiewicz chain (i.e. $L = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$) and $\langle X, Y, I\rangle$ be a formal **L**-context, where $X = \{\mathrm{BV}, \mathrm{LH}, \mathrm{MD}, \mathrm{TSS}\}$, $Y = \{c1, c2, c3, c4, c5\}$, and I is depicted in Fig. 1. Elements of X are four selected movies by the director David Lynch, elements of Y are five film critics, and values of I are ratings the critics assigned to the movies, taken from www.metacritic.com and rescaled to the six-element scale. The context is our central example and we work with it throughout the paper. For now, we use it just to give an example of an **L**-concept lattice.

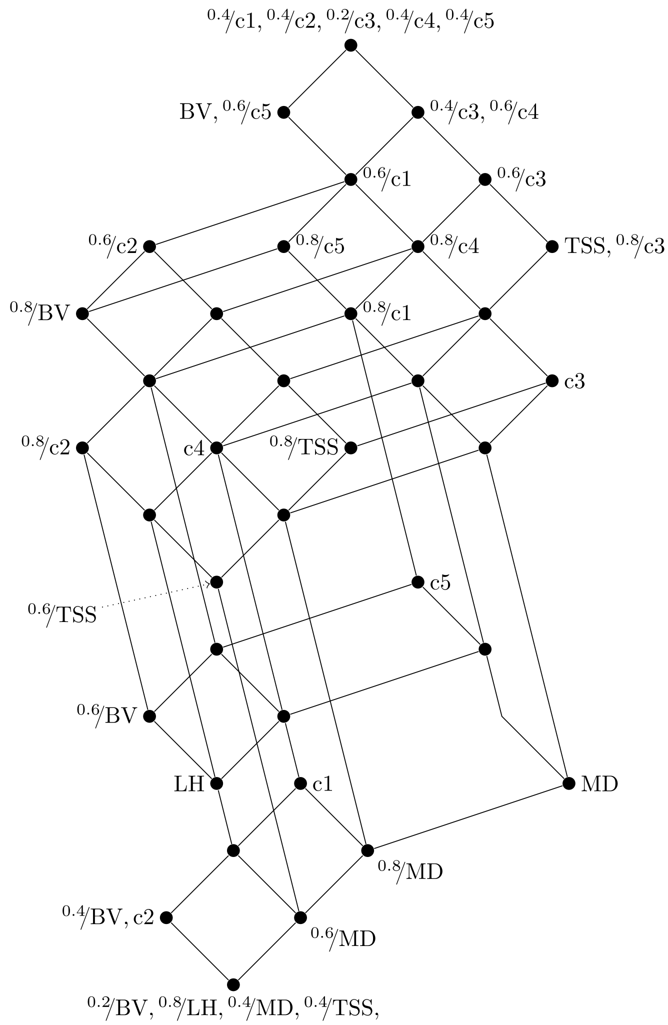

Fig. 2. **L**-concept lattice $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$ of movies, critics and ratings (from Fig. 1).

Note that our data are suitable for interpreting by means of fuzzy logic. A movie rating (usually given by a number of "stars" or by a percentage) can be interpreted as the truth degree of the proposition "Critic y likes movie x". It is clear that in this situation we need a finer scale of truth degrees than just 0 and 1. Also, there are no doubts about the truth degree assigned to the proposition (especially, if given by a film critic).

The **L**-concept lattice $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$ is depicted in Fig. 2. When displaying **L**-concept lattices, we use labeled Hasse diagrams to include all the information on the corresponding formal **L**-context as well as extents and intents of all concepts. For each $x \in X$ and $a \in L$, the formal **L**-concept $\langle \{{}^{a}\!/x\}^{\uparrow\downarrow}, \{{}^{a}\!/x\}^{\uparrow} \rangle$ is labeled by ${}^{a}\!/x$ and for each $y \in Y$ and $b \in L$, the formal **L**-concept $\langle \{{}^{b}\!/y\}^{\downarrow}, \{{}^{b}\!/y\}^{\downarrow\uparrow} \rangle$ is labeled by ${}^{b}\!/y$. We use labels x resp. y instead of ${}^{1}\!/x$ resp. ${}^{1}\!/y$ in the diagram and omit redundant labels (i.e., if a concept has labels ${}^{a_1}\!/x$ and ${}^{a_2}\!/x$ then we keep only that with the greater degree; dually for attributes).

By the crisp order version of the Basic Theorem of fuzzy concept lattices [4, Theorem 5], for each $x \in X$ and $y \in Y$, $I(x, y)$ is equal to the greatest $a \otimes b$, such that $\langle \{{}^{a}\!/x\}^{\uparrow\downarrow}, \{{}^{a}\!/x\}^{\uparrow} \rangle \leq \langle \{{}^{b}\!/y\}^{\downarrow}, \{{}^{b}\!/y\}^{\downarrow\uparrow} \rangle$. Thus, the **L**-relation I can be reconstructed from the diagram. Moreover, for any $x \in X$, $y \in Y$ and formal **L**-concept $\langle A, B \rangle$ we have $A(x) \geq a$ and $B(y) \geq b$ if and only if there is a formal **L**-concept $\langle A_1, B_1 \rangle \leq \langle A, B \rangle$, labeled by ${}^{a}\!/x$ and a formal **L**-concept $\langle A_2, B_2 \rangle \geq \langle A, B \rangle$, labeled by ${}^{b}\!/y$.

For example, if $\langle A, B \rangle$ is the **L**-concept labeled by c4 in the diagram then $A = \{{}^{0.4}\!/\mathrm{BV}, \mathrm{LH}, {}^{0.8}\!/\mathrm{MD}, {}^{0.4}\!/\mathrm{TSS}\}$ and $B = \{{}^{0.8}\!/\mathrm{c1}, {}^{0.6}\!/\mathrm{c2}, {}^{0.8}\!/\mathrm{c3}, \mathrm{c4}\}$.

In $\mathcal{B}^{\cap\cup}(X, Y, I)$ (Fig. 3), for any $x \in X$, $y \in Y$ and formal **L**-concept $\langle A, B \rangle$ we have $A(x) \leq a$ and $B(y) \geq b$ if and only if there is a formal **L**-concept $\langle A_1, B_1 \rangle \leq \langle A, B \rangle$ labeled by ${}^{a}\!/x$ and a formal **L**-concept $\langle A_2, B_2 \rangle \geq \langle A, B \rangle$ labeled by ${}^{b}\!/y$.

2.5. Fuzzy Galois connections, fuzzy closure and interior systems

An antitone **L**-Galois connection [6] between **L**-ordered sets **U** and **V** is a pair $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ of mappings ${}^{\uparrow} : U \to V$, ${}^{\downarrow} : V \to U$, satisfying

$$u \preceq v^{\downarrow} = v \preceq u^{\uparrow} \qquad (28)$$

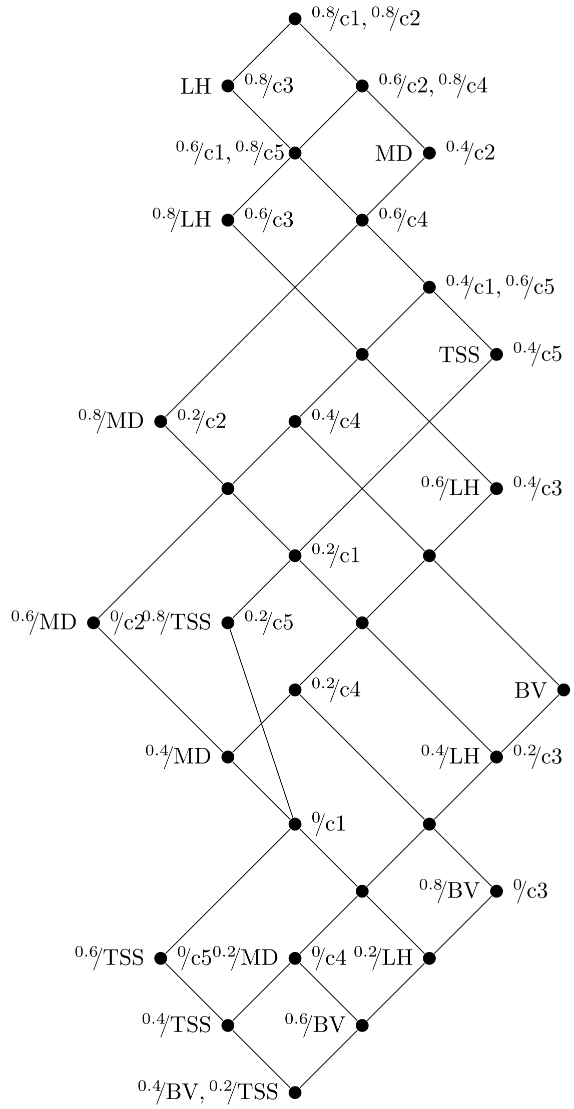

Fig. 3. **L**-concept lattice $\mathcal{B}^{\cap\cup}(X, Y, I)$ of movies, critics and ratings (from Fig. 1).

for every $u \in U$ and $v \in V$. An antitone **L**-Galois connection between sets X and Y is an antitone **L**-Galois connection between the **L**-ordered sets $\mathbf{L}^X$ and $\mathbf{L}^Y$.

An isotone **L**-Galois connection [18] between **L**-ordered sets **U** and **V** is a pair $\langle {}^\cap, {}^\cup \rangle$ of mappings ${}^\cap : U \to V$, ${}^\cup : V \to U$, satisfying

$$u \preceq v^\cup = u^\cap \preceq v \tag{29}$$

for every $u \in U$ and $v \in V$.

For $\mathbf{U} = \mathbf{V}$, the isotone **L**-Galois connection $\langle {}^\cap, {}^\cup \rangle$ is called extensive if for each $u \in U$ we have $u \preceq u^\cup$ (which is equivalent with $u^\cap \preceq u$).

An isotone **L**-Galois connection between sets X and Y is defined as an isotone **L**-Galois connection between the **L**-ordered sets $\mathbf{L}^X$ and $\mathbf{L}^Y$.

Remark 2. We purposely denote antitone (resp. isotone) **L**-Galois connection by the same symbols as concept-forming operators (23) (resp. (24)), because each pair of concept-forming operators $\langle \uparrow, \downarrow \rangle$ (resp. $\langle \cap, \cup \rangle$) on a formal **L**-context $\langle X, Y, I \rangle$ is an antitone (resp. isotone) **L**-Galois connection and for each antitone (resp. isotone) **L**-Galois connection between sets X and Y there is an **L**-relation I which induces it as a pair of concept-forming operators [6,18].

Note also that (28) and (29) become

$$S(A, B^\downarrow) = S(B, A^\uparrow) \quad \text{and} \quad S(A, B^\cup) = S(A^\cap, B),$$

respectively, for $A \in \mathbf{L}^X$, $B \in \mathbf{L}^Y$.

A system of **L**-sets $V \subseteq L^X$ is called an **L**-*interior system* [12] if

– V is closed under a-multiplication for every $a \in L$, i.e. for every $C \in V$ and $a \in L$ we have $a \otimes C \in V$;
– V is closed under union, i.e. $\bigcup_{j \in J} C_j \in V$ whenever $C_j \in V$ for all $j \in J$.

The above two conditions can be replaced with the following single condition: V is closed under unions of fuzzy subsets, i.e. $\bigcup U \in V$ for each $U \in L^X$, $U \subseteq V$.

$V \subseteq L^X$ is called an **L**-*closure system* [3] if

– V is closed under a-shifts for every $a \in L$; i.e. for every $C \in V$ and $a \in L$ we have $a \rightarrow C \in V$;
– V is closed under intersection, i.e. $\bigcap_{j \in J} C_j \in V$ whenever $C_j \in V$ for all $j \in J$.

Again, the above two conditions can be replaced with the following one: V is closed under intersections of fuzzy subsets, i.e. $\bigcap U \in V$ for each $U \in L^X$, $U \subseteq V$.

Remark 3. It is worth noting that the notions of antitone and isotone **L**-Galois connection, **L**-closure system, and **L**-interior system are proper generalizations of their crisp counterparts. That is, they become their crisp counterparts when **L** is the two-element Boolean algebra.

Remark 4. One can find examples of **L**-closure and **L**-interior systems in the framework of formal fuzzy concept analysis as follows: for an **L**-context $\langle X, Y, I\rangle$, the sets $\mathrm{Ext}^{\uparrow\downarrow}(X, Y, I)$, $\mathrm{Ext}^{\cap\cup}(X, Y, I)$, $\mathrm{Int}^{\wedge\vee}(X, Y, I)$, and $\mathrm{Int}^{\uparrow\downarrow}(X, Y, I)$ are **L**-closure systems [3,18], while $\mathrm{Ext}^{\wedge\vee}(X, Y, I)$ and $\mathrm{Int}^{\cap\cup}(X, Y, I)$ are **L**-interior systems [12,18].

3. Results

First, we provide a definition of block **L**-relation—a convenient generalization of the notion of block-relation from [32]. In Sections 3.1–3.3 we separately study three instances of block **L**-relations: with respect to antitone concept-forming operators and a weaker and a stronger version of block **L**-relations with respect to isotone concept-forming operators. Then, in Section 3.4 we show that block **L**-relations correspond to particular automorphisms on associated concept lattices. Finally, in Section 3.5, we show that they also correspond to extensive isotone **L**-Galois connections and complete **L**-tolerances on associated concept lattices.

Definition 1. Let $I \in L^{X \times Y}$ be an **L**-relation.

$J \in L^{X \times Y}$, $J \supseteq I$, is called a *block* **L**-*relation of* I *w.r.t.* $\langle\uparrow, \downarrow\rangle$ if

$$\begin{aligned} \mathrm{Ext}^{\uparrow\downarrow}(X, Y, J) &\subseteq \mathrm{Ext}^{\uparrow\downarrow}(X, Y, I), \\ \mathrm{Int}^{\uparrow\downarrow}(X, Y, J) &\subseteq \mathrm{Int}^{\uparrow\downarrow}(X, Y, I). \end{aligned} \tag{30}$$

$J \in L^{X \times Y}$, $J \subseteq I$, is called a *block* **L**-*relation of* I *w.r.t.* $\langle\cap, \cup\rangle$ if

$$\begin{aligned} \mathrm{Ext}^{\cap\cup}(X, Y, J) &\subseteq \mathrm{Ext}^{\cap\cup}(X, Y, I), \\ \mathrm{Int}^{\cap\cup}(X, Y, J) &\subseteq \mathrm{Int}^{\cap\cup}(X, Y, I). \end{aligned} \tag{31}$$

$J \in L^{X \times Y}$, $J \subseteq I$, is called a *block* **L**-*relation of* I *w.r.t.* $\langle\wedge, \vee\rangle$ if

$$\begin{aligned} \mathrm{Ext}^{\wedge\vee}(X, Y, J) &\subseteq \mathrm{Ext}^{\wedge\vee}(X, Y, I), \\ \mathrm{Int}^{\wedge\vee}(X, Y, J) &\subseteq \mathrm{Int}^{\wedge\vee}(X, Y, I). \end{aligned} \tag{32}$$

Denote the set of block relations of I w.r.t. operators $\langle\vartriangle, \triangledown\rangle$ by $\mathrm{BR}^{\vartriangle\triangledown}(X, Y, I)$.

Remark 5. Wille [32] defines block relation (w.r.t. $\langle\uparrow, \downarrow\rangle$) as a relation $J \supseteq I$ where each row is an intent of I and each column is an extent of I. We prove in Theorem 3(e) that the notion of block **L**-relation is a proper generalization of crisp block relation for the case of antitone concept-forming operators $\langle\uparrow, \downarrow\rangle$, since $\{x\}^{\uparrow_J}$s and $\{y\}^{\downarrow_J}$s correspond to rows and columns of J, respectively.

3.1. Block **L**-relations w.r.t. antitone concept-forming operators

In this section, we study only the case $\langle\vartriangle, \triangledown\rangle = \langle\uparrow, \downarrow\rangle$. The other two cases are studied in Section 3.2. The following theorem provides characterizations of block **L**-relations w.r.t. $\langle\uparrow, \downarrow\rangle$.

Theorem 3. *Let $I \in L^{X\times Y}$ be an* **L***-relation. The following statements are equivalent:*

(a) $J \in \mathrm{BR}^{\uparrow\downarrow}(X,Y,I)$.
(b) $J = I \triangleright S_Y$ *with* $S_Y \in L^{Y\times Y}$ *and for the induced mapping* ${}^{\wedge S_Y}$ *we have* $B^{\wedge S_Y} \in \mathrm{Int}^{\uparrow\downarrow}(X,Y,I)$ *and* $B \subseteq B^{\wedge S_Y}$ *for each* $B \in \mathrm{Int}^{\uparrow\downarrow}(X,Y,I)$.
(c) $J = S_X \triangleleft I$ *with* $S_X \in L^{X\times X}$ *and for the induced mapping* ${}^{\cup S_X}$ *we have* $A^{\cup S_X} \in \mathrm{Ext}^{\uparrow\downarrow}(X,Y,I)$ *and* $A \subseteq A^{\cup S_X}$ *for each* $A \in \mathrm{Ext}^{\uparrow\downarrow}(X,Y,I)$.
(d) $I \subseteq J = S_X \triangleleft I = I \triangleright S_Y$ *for some* $S_X \in L^{X\times X}$ *and* $S_Y \in L^{Y\times Y}$.
(e) $I \subseteq J$; $\{x\}^{\uparrow_J} \in \mathrm{Int}^{\uparrow\downarrow}(X,Y,I)$ *for each* $x \in X$ *and* $\{y\}^{\downarrow_J} \in \mathrm{Ext}^{\uparrow\downarrow}(X,Y,I)$ *for each* $y \in Y$.

Proof. First we show (a) ⇔ (b):

"⇒": By (30) we have $\mathrm{Ext}^{\uparrow\downarrow}(X,Y,J) \subseteq \mathrm{Ext}^{\uparrow\downarrow}(X,Y,I)$. Using Theorem 2, we have that there exists $S_Y \in L^{Y\times Y}$ s.t. $J = I \triangleright S_Y$. Now, by (30) we have $\mathrm{Int}^{\uparrow\downarrow}(X,Y,I\triangleright S_Y) \subseteq \mathrm{Int}^{\uparrow\downarrow}(X,Y,I)$, thus

$$A^{\uparrow_{I\triangleright S_Y}} = A \triangleleft (I \triangleright S_Y) = (A \triangleleft I) \triangleright S_Y \in \mathrm{Int}^{\uparrow\downarrow}(X,Y,I)$$

for each $A \in L^X$. Since $A \triangleleft I = A^{\uparrow_I}$ represents any intent B in $\mathrm{Int}^{\uparrow\downarrow}(X,Y,I)$ we obtain that $B \triangleright S_Y = B^{\wedge S_Y} \in \mathrm{Int}^{\uparrow\downarrow}(X,Y,I)$ for all $B \in \mathrm{Int}^{\uparrow\downarrow}(X,Y,I)$.

Finally, we need to show that $B \subseteq B^{\wedge S_Y}$ for each $B \in \mathrm{Int}^{\uparrow\downarrow}(X,Y,I)$. There exists $A \in L^X$ s.t. $A^{\uparrow_I} = B$. Since $I \subseteq J$, we have

$$B = A \triangleleft I \subseteq A \triangleleft J = A \triangleleft (I \triangleright S_Y) = (A \triangleleft I) \triangleright S_Y = B^{\wedge S_Y}.$$

"⇐": From Theorem 2 we have $\mathrm{Ext}^{\uparrow\downarrow}(X,Y,J) \subseteq \mathrm{Ext}^{\uparrow\downarrow}(X,Y,I)$. Directly from assumptions we have for every $A \in L^X$

$$A^{\uparrow_J} = A \triangleleft J = A \triangleleft (I \triangleright S_Y) = (A \triangleleft I) \triangleright S_Y = B^{\wedge S_Y},$$

where B is the intent $A^{\uparrow_I}$. Hence we have $\mathrm{Int}^{\uparrow\downarrow}(X,Y,J) \subseteq \mathrm{Int}^{\uparrow\downarrow}(X,Y,I)$.

Now, consider rows of **L**-relation I as **L**-sets $I_x \in L^Y$ given by

$$I_x(y) = I(x,y) = \{x\}^{\uparrow_I}(y) \quad \text{for all } y \in Y;$$

analogously for J. Since rows I_x are in $\mathrm{Int}^{\uparrow\downarrow}(X,Y,I)$, by definition of $\triangleleft$ and Remark 1 we have

$$I_x \subseteq I_x^{\wedge S_Y} = I_x \triangleleft S_Y = J_x$$

showing that $I_x \subseteq J_x$ for each $x \in X$, thus $I \subseteq J$.

Proof of (a) ⇔ (c) is similar; proof of (a) ⇔ (d) follows directly from Theorem 2.

(a) ⇔ (e): Since we can write every $A \in L^X$ as $A = \bigcup_{x\in X} A(x) \otimes \{x\}$ then for all $y \in Y$ we have

$$\begin{aligned} A^{\uparrow_J}(y) &= \bigwedge_{x'\in X} A(x') \to J(x',y) \\ &= \bigwedge_{x'\in X} \Big(\bigcup_{x\in X} A(x) \otimes \{x\}\Big)(x') \to J(x',y) \\ &= \bigwedge_{x'\in X} \Big(\bigvee_{x\in X} A(x) \otimes \{x\}(x')\Big) \to J(x',y) \\ &= \bigwedge_{x\in X} A(x) \to \bigwedge_{x'\in X} (\{x\}(x')) \to J(x',y) \\ &= \bigwedge_{x\in X} A(x) \to \{x\}^{\uparrow_J}(y) \\ &= \Big(\bigcap_{x\in X} A(x) \to \{x\}^{\uparrow_J}\Big)(y). \end{aligned}$$

Since $\{x\}^{\uparrow_J} \in \mathrm{Int}^{\uparrow\downarrow}(X,Y,I)$ and $\mathrm{Int}^{\uparrow\downarrow}(X,Y,I)$ is an **L**-interior system we obtain that $A^{\uparrow_J} \in \mathrm{Int}^{\uparrow\downarrow}(X,Y,I)$. Because every intent in $\mathrm{Int}^{\uparrow\downarrow}(X,Y,J)$ has the form $A^{\uparrow_J}$ for some $A \in L^X$ we conclude that $\mathrm{Int}^{\uparrow\downarrow}(X,Y,J) \subseteq \mathrm{Int}^{\uparrow\downarrow}(X,Y,I)$. Similarly, one can show that $\{y\}^{\downarrow_J} \in \mathrm{Ext}^{\uparrow\downarrow}(X,Y,I)$ for each $y \in Y$ implies $\mathrm{Ext}^{\uparrow\downarrow}(X,Y,J) \in \mathrm{Ext}^{\uparrow\downarrow}(X,Y,I)$.

The converse is trivial. □

Theorem 4. *Let $I \in L^{X\times Y}$ be an* **L***-relation between X and Y.*

(a) $\mathrm{BR}^{\uparrow\downarrow}(X,Y,I)$ *is an* **L***-closure system.*

J	c1	c2	c3	c4	c5
BV	0.6	0.6	0.4	0.6	0.8
LH	1	1	1	1	1
MD	1	0.6	1	1	1
TSS	0.6	0.6	1	0.8	0.6

S_X	BV	LH	MD	TSS
BV	0.8	0	0	0
LH	0	0.8	0	0
MD	0	0	0.8	0
TSS	0	0	0	0.8

S_Y	c1	c2	c3	c4	c5
c1	0.8	0	0	0	0
c2	0	0.8	0	0	0
c3	0	0	0.8	0	0
c4	0	0	0	0.8	0
c5	0	0	0	0	0.8

Fig. 4. Block **L**-relation $J = 0.8 \to I$ w.r.t. $\langle\uparrow,\downarrow\rangle$ (top) of the **L**-context in Fig. 1 and the corresponding **L**-relations S_X (middle) and S_Y (bottom).

*(b) The set of all S_X for all block relations J of I (from Theorem 3) is an **L**-interior system.*
*(c) The set of all S_Y for all block relations J of I (from Theorem 3) is an **L**-interior system.*

Proof. (a) We need to show that $\mathrm{BR}^{\uparrow\downarrow}(X,Y,I)$ is closed under intersections and a-shifts for all $a \in L$.

First, we show closedness under intersections. Let us have a collection of block **L**-relations J_i of I w.r.t. $\langle\uparrow,\downarrow\rangle$. Let $J = \bigcap_i J_i$ and let $B \in \mathrm{Int}^{\uparrow\downarrow}(X,Y,J)$, hence $B = A^{\uparrow_J}$ for some $A \in L^X$. By definition of ${}^{\uparrow_J}$ and (18) we have

$$A^{\uparrow_J} = A \lhd J = A \lhd (\bigcap_i J_i) = \bigcap_i (A \lhd J_i) = \bigcap_i A^{\uparrow_{J_i}}.$$

Thus we have $A^{\uparrow_J} = \bigcap_i A^{\uparrow_{J_i}} \in \mathrm{Int}^{\uparrow\downarrow}(X,Y,I)$ since $\mathrm{Int}^{\uparrow\downarrow}(X,Y,I)$ is an **L**-closure system. Similarly, $B^{\downarrow_J} = \bigcap_i B^{\downarrow_{J_i}} \in \mathrm{Ext}^{\uparrow\downarrow}(X,Y,I)$. From $I \subseteq J_i$ we have $I \subseteq \bigcap_i J_i = J$, whence $J \in \mathrm{BR}^{\uparrow\downarrow}(X,Y,I)$.

Now we prove that $\mathrm{BR}^{\uparrow\downarrow}(X,Y,I)$ is closed under all a-shifts. For any $A \in L^X, a \in L$, and a block **L**-relation $J \in \mathrm{BR}^{\uparrow\downarrow}(X,Y,I)$ we have

$$\begin{aligned} A^{\uparrow_{a\to J}} &= A \lhd (a \to J) = A \lhd (\mathrm{Id}_a \lhd J) = (A \circ \mathrm{Id}_a) \lhd J = \\ &= (\mathrm{Id}_a \circ A) \lhd J = \mathrm{Id}_a \lhd (A \lhd J) = a \to (A \lhd J) = a \to A^{\uparrow_J}. \end{aligned}$$

We have $A^{\uparrow_{a\to J}} = a \to A^{\uparrow_J} \in \mathrm{Int}^{\uparrow\downarrow}(X,Y,I)$ since $\mathrm{Int}^{\uparrow\downarrow}(X,Y,I)$ is an **L**-closure system. Similarly, one can show that $B^{\downarrow_{a\to J}} \in \mathrm{Ext}^{\uparrow\downarrow}(X,Y,I)$. From $I \subseteq J$ we have $I \subseteq a \to J_i$ and $a \to J \in \mathrm{BR}^{\uparrow\downarrow}(X,Y,I)$.

This proves that $\mathrm{BR}^{\uparrow\downarrow}(X,Y,I)$ of I is an **L**-closure system.

(b) Let $\langle X,Y,I\rangle$ be an **L**-context. We need to show that the system $\mathcal{S}_X$ of all **L**-relations $S_X \in \mathbf{L}^{X\times X}$ s.t. $S_X \lhd I \in \mathrm{BR}^{\uparrow\downarrow}(X,Y,I)$ is closed under unions and a-multiplications for all $a \in L$.

First, we show closedness under unions. Let us have a collection of **L**-relations $S_{Xi} \in \mathcal{S}_X$. From Theorem 4(a) and (18) we have that

$$\mathrm{BR}^{\uparrow\downarrow}(X,Y,I) \ni \bigcap_i (S_{Xi} \lhd I) = (\bigcup_i S_{Xi}) \lhd I;$$

whence $(\bigcup_i S_{Xi}) \in \mathcal{S}_X$.

Now we show closedness under a-multiplications for all $a \in L$. Consider $S_X \in \mathcal{S}_X$ and arbitrary $a \in L$. Then from Theorem 4(a), we have

$$\mathrm{BR}^{\uparrow\downarrow}(X,Y,I) \ni a \to (S_X \lhd I) = \mathrm{Id}_a \lhd (S_X \lhd I) = (\mathrm{Id}_a \circ S_X) \lhd I = (a \otimes S_X) \lhd I;$$

whence $a \otimes S_X \in \mathcal{S}_X$. We conclude that $\mathcal{S}_X$ is an **L**-interior system.

(c) can be proved similarly as (b). □

Example (cont.)

By Theorem 4, for each $a \in L$ the **L**-relation $a \to I$ is a block relation of $\langle X,Y,I\rangle$. In Fig. 4 (top), it is depicted the block relation $J = 0.8 \to I$ for our example formal context $\langle X,Y,I\rangle$. In the same figure, we can see **L**-relations S_X and S_Y from Theorem 3.

In Fig. 5, we can see a more interesting example of a block relation, together with corresponding **L**-relations S_X and S_Y. Will explain the way this block relation was constructed later.

J	c1	c2	c3	c4	c5
BV	0.6	0.4	0.4	0.6	0.6
LH	1	0.8	1	1	1
MD	1	0.6	1	1	1
TSS	0.4	0.4	0.8	0.6	0.4

S_X	BV	LH	MD	TSS
BV	0.8	1	1	0.8
LH	0.2	0.8	0.6	0.4
MD	0.2	0.8	0.8	0.4
TSS	0.4	1	1	1

S_Y	c1	c2	c3	c4	c5
c1	0.8	1	0.6	0.8	0.8
c2	0.4	0.8	0.4	0.4	0.4
c3	0.6	0.8	0.8	0.6	0.6
c4	0.8	1	0.8	0.8	0.8
c5	1	1	0.6	0.8	1

Fig. 5. Block **L**-relation J (top) w.r.t. $\langle\uparrow,\downarrow\rangle$ of the **L**-context in Fig. 1 and the corresponding **L**-relations S_X (middle) and S_Y (bottom).

3.2. Block L-relations w.r.t. isotone concept-forming operators

In this section we study the case of block **L**-relations w.r.t. $\langle\cap,\cup\rangle$; the case of $\langle\wedge,\vee\rangle$ is omitted since it is dual. First, we provide a characterization of block **L**-relations w.r.t. $\langle\cap,\cup\rangle$.

Theorem 5. *Let $I \in L^{X\times Y}$ be an* **L***-relation. The following statements are equivalent:*

(a) $J \in \mathrm{BR}^{\cap\cup}(X,Y,I)$.
(b) $J = S_X \circ I$ *with* $S_X \in L^{X\times X}$ *and for the induced mapping* ${}^{\cup_{S_X}}$ *we have* $A^{\cup_{S_X}} \in \mathrm{Ext}^{\cap\cup}(X,Y,I)$ *and* $A \subseteq A^{\cup_{S_X}}$ *for each* $A \in \mathrm{Ext}^{\cap\cup}(X,Y,I)$.

Proof. (a) $\Rightarrow$ (b): From $\mathrm{Int}^{\cap\cup}(X,Y,J) \subseteq \mathrm{Int}^{\cap\cup}(X,Y,I)$ and Theorem 2(b) we have $J = S_X \circ I$ for some $S_X \in L^{X\times X}$. Now, since $\mathrm{Ext}^{\cap\cup}(X,Y,J) \subseteq \mathrm{Ext}^{\cap\cup}(X,Y,I)$ and since each element of $\mathrm{Ext}^{\cap\cup}(X,Y,J)$ is in the form $B^{\cup_J}$ for some $B \in L^Y$, we can write

$$B^{\cup_J} = J \triangleleft B = (S_X \circ I) \triangleleft B = S_X \triangleleft (I \triangleleft B) = (B^{\cup_I})^{\cup_{S_X}}.$$

This shows that $A^{\cup_{S_X}} \in \mathrm{Ext}^{\cap\cup}(X,Y,I)$ for each $A \in \mathrm{Ext}^{\cap\cup}(X,Y,I)$.

Finally, we need to show that ${}^{\cup_{S_X}}$ is extensive. We have $J \subseteq I$ whence $J \triangleleft B \subseteq I \triangleleft B$ for each $B \in L^Y$. Thus, we have

$$A^{\cup_{S_X}} = A^{\cap_I\cup_I\cup_{S_X}} = S_X \triangleleft (I \triangleleft (A \circ I)) = S_X \triangleleft (I \triangleleft B) = (S_X \circ I) \triangleleft B = J \triangleleft B \supseteq I \triangleleft B = A.$$

(b) $\Rightarrow$ (a): From $J = S_X \circ I$ we have $\mathrm{Int}^{\cap\cup}(X,Y,J) \subseteq \mathrm{Int}^{\cap\cup}(X,Y,I)$ by Theorem 2(a).

Since each extent A in $\mathrm{Ext}^{\cap\cup}(X,Y,J)$ has the form $A = B^{\cup_J}$ we have

$$B^{\cup_J} = J \triangleleft B = (S_X \circ I) \triangleleft B = S_X \triangleleft (I \triangleleft B) = (B^{\cup_I})^{\cup_{S_X}} \in \mathrm{Ext}^{\cap\cup}(X,Y,I)$$

proving that $\mathrm{Ext}^{\cap\cup}(X,Y,J) \subseteq \mathrm{Ext}^{\cap\cup}(X,Y,I)$.

Finally, we have $A \subseteq A^{\cup_{S_X}}$ for each $A \in \mathrm{Ext}^{\cap\cup}(X,Y,I)$, thus we have

$$I \triangleleft B = B^{\cup_I} \subseteq (B^{\cup_I})^{\cup_{S_X}} = S_X \triangleleft (I \triangleleft B) = (S_X \circ I) \triangleleft B = J \triangleleft B$$

for each $B \in L^Y$. Thus $J \subseteq I$. □

Remark 6.

(a) Note that in the isotone case we cannot characterize the notion of block **L**-relation a similar way as in Theorem 3(e). While rows of J are still intents of $\mathcal{B}^{\cap\cup}(X,Y,I)$, the columns generally are not extents. In Section 3.3 we study a stronger version of isotone block **L**-relations which makes a characterization analogous to the Theorem 3(e) possible.
(b) Note, that the opposite decomposition ($J = S_X \circ I$), analogous to Theorem 5(a), does not always exist, as Example 1 shows.

Example 1. Consider **L** being a four-element chain $0 < a < b < 1$ with $\otimes$ being a drastic product [30], i.e.

$$x \otimes y = \begin{cases} x \wedge y & \text{if } x = 1 \text{ or } y = 1, \\ 0 & \text{otherwise,} \end{cases}$$

for each $x, y \in L$. One can easily see that $x \otimes \bigvee_j y_j = \bigvee_j (x \otimes y_j)$ and thus an adjoint operation $\rightarrow$ exists such that $\langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ is a complete residuated lattice. Namely, $\rightarrow$ is given as follows:

$$x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if } x = 1, \\ b & \text{otherwise,} \end{cases}$$

for each $x, y \in L$.

Now, consider the following **L**-relation I

I	y_1	y_2	y_3
x_1	0	1	a
x_2	b	1	b

One can check that the following **L**-relation J is a block **L**-relation of $\langle X, Y, I\rangle$ w.r.t. $\langle \cap, \cup\rangle$.

J	y_1	y_2	y_3
x_1	0	0	0
x_2	0	a	0

While we have $J = S_X \circ I$ for

S_X	x_1	x_2
x_1	0	0
x_2	0	a

there is no $S_Y \in L^{Y\times Y}$ such that $J = I \circ S_Y$.

Theorem 6. *Let $I \in L^{X\times Y}$ be an **L**-relation between X and Y.*

*(a) $\mathrm{BR}^{\cap\cup}(X, Y, I)$ is an **L**-interior system.*
*(b) The set of all S_X of all block **L**-relations of I (from Theorem 5) is an **L**-interior system.*

Proof. We need to show that $\mathrm{BR}^{\cap\cup}(X, Y, I)$ is closed under unions and a-multiplications for all $a \in L$.

First, we show closedness under unions. Let us have a collection of block **L**-relations J_i of I w.r.t. $\langle \cap, \cup\rangle$. Let $J = \bigcup_i J_i$ and let $B \in \mathrm{Int}^{\cap\cup}(X, Y, J)$. There exists $A \in L^X$ s.t. $B = A^{\cap_J}$. By Remark 1 and (16) we have

$$A^{\cap_J} = A \circ J = A \circ (\bigcup_i J_i) = \bigcup_i (A \circ J_i) = \bigcup_i A^{\cap_{J_i}}.$$

Hence, $A^{\cap_J} = \bigcup_i A^{\cap_{J_i}} \in \mathrm{Int}^{\cap\cup}(X, Y, I)$, since $\mathrm{Int}^{\cap\cup}(X, Y, I)$ is an **L**-interior system. Thus we have $\mathrm{Int}^{\cap\cup}(X, Y, J) \subseteq \mathrm{Int}^{\cap\cup}(X, Y, I)$.

Similarly, let $A \in \mathrm{Ext}^{\cap\cup}(X, Y, J)$. There exists $B \in L^Y$ s.t. $A = B^{\cup_J}$. By Remark 1 and (17) we have

$$B^{\cup_J} = J \lhd B = (\bigcup_i J_i) \lhd B = \bigcap_i (J_i \lhd B) = \bigcap_i B^{\cup_{J_i}},$$

thus $B^{\cup_J} = \bigcap_i B^{\cup_{J_i}} \in \mathrm{Ext}^{\cap\cup}(X, Y, I)$, since $\mathrm{Ext}^{\cap\cup}(X, Y, I)$ is an **L**-closure system. Thus we have $\mathrm{Ext}^{\cap\cup}(X, Y, J) \subseteq \mathrm{Ext}^{\cap\cup}(X, Y, I)$.

Since $I \supseteq J_i$ for all i, we also have $I \supseteq \bigcup_i J_i$. Hence $\bigcup_i J_i \in \mathrm{BR}^{\cap\cup}(X, Y, I)$.

Now we prove that $\mathrm{BR}^{\cap\cup}(X, Y, I)$ is closed under all a-multiplications. For any $a \in L$, $A \in L^X$, $B \in L^Y$ and block **L**-relation J we have

$$\begin{aligned} A^{\cap_{a\otimes J}} &= A \circ (\mathrm{Id}_a \circ J) = (A \circ \mathrm{Id}_a) \circ J = \\ &= (\mathrm{Id}_a \circ A) \circ J = \mathrm{Id}_a \circ (A \circ J) = a \otimes A^{\uparrow_J}. \end{aligned}$$

$A^{\cap_J} \in \mathrm{Int}^{\cap\cup}(X, Y, I)$ implies $a \otimes A^{\cap_J} \in \mathrm{Int}^{\cap\cup}(X, Y, I)$ because $\mathrm{Int}^{\cap\cup}(X, Y, I)$ is an **L**-interior system. Thus we have $\mathrm{Int}^{\cap\cup}(X, Y, a \otimes J) \subseteq \mathrm{Int}^{\cap\cup}(X, Y, I)$.

Similarly, let $A \in \mathrm{Ext}^{\cap\cup}(X, Y, J)$. There exists $B \in L^Y$ s.t. $B^{\cup} = A$. We have

$$B^{\cup_{a\otimes J}} = (\mathrm{Id}_a \circ J) \lhd B = \mathrm{Id}_a \lhd (J \lhd B) = a \rightarrow B^{\cup}.$$

We have $B^{\cup_{a\otimes J}} = a \rightarrow B^{\cup_J} \in \mathrm{Ext}^{\cap\cup}(X, Y, I)$ since $\mathrm{Ext}^{\cap\cup}(X, Y, I)$ is an **L**-closure system. Thus we have $\mathrm{Ext}^{\cap\cup}(X, Y, a \otimes J) \subseteq \mathrm{Ext}^{\cap\cup}(X, Y, I)$. Finally, from $I \supseteq J$ we have $I \supseteq a \otimes J$. Hence $a \otimes J \in \mathrm{BR}^{\cap\cup}(X, Y, I)$.

We conclude that $\mathrm{BR}^{\cap\cup}(X, Y, I)$ is an **L**-interior system.

(b) can be proved similarly. □

3.3. Strong block L-relations w.r.t. isotone concept-forming operators

In this section, we provide a notion of a block **L**-relation defined by means of both isotone concept-forming operators, $\langle\cap,\cup\rangle$ and $\langle\wedge,\vee\rangle$. This notion happens to be stronger than the one studied in Section 3.2.

Definition 2. Let $I \in L^{X\times Y}$ be an **L**-relation. $J \subseteq I$ is called a strong block **L**-relation of I if

$$\begin{aligned}&\mathrm{Ext}^{\wedge\vee}(X,Y,J) \subseteq \mathrm{Ext}^{\wedge\vee}(X,Y,I),\\ &\mathrm{Int}^{\cap\cup}(X,Y,J) \subseteq \mathrm{Int}^{\cap\cup}(X,Y,I).\end{aligned} \tag{33}$$

The following theorem provides characterizations of strong block **L**-relations.

Theorem 7. *The following statements are equivalent.*

(a) $J \in \mathrm{BR}^{\cap\cup}(X,Y,I) \cap \mathrm{BR}^{\wedge\vee}(X,Y,I)$.
(b) J *is a strong block* **L***-relation of* I.
(c) $I \supseteq J = S_X \circ I = I \circ S_Y$ *for some* $S_X \in L^{X\times X}$, $S_Y \in L^{Y\times Y}$.
(d) $I \supseteq J$; $\{x\}^{\cap_J} \in \mathrm{Int}^{\cap\cup}(X,Y,I)$ *for each* $x \in X$ *and* $\{y\}^{\vee_J} \in \mathrm{Ext}^{\wedge\vee}(X,Y,I)$ *for each* $y \in Y$.
(e) $J = S_X \circ I$ *and the induced mapping* ${}^{\vee_{S_X}}$ *satisfies* $A^{\vee_{S_X}} \in \mathrm{Ext}^{\wedge\vee}(X,Y,I)$ *and* $A^{\vee_{S_X}} \subseteq A$ *for each* $A \in \mathrm{Ext}^{\wedge\vee}(X,Y,I)$.
(f) $J = I \circ S_Y$ *and the induced mapping* ${}^{\cap_{S_Y}}$ *satisfies* $B^{\cap_{S_Y}} \in \mathrm{Int}^{\cap\cup}(X,Y,I)$ *and* $B^{\cap_{S_X}} \subseteq B$ *for each* $B \in \mathrm{Int}^{\cap\cup}(X,Y,I)$.

Proof. (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) follows directly from Theorem 2; (a) $\Rightarrow$ (d) is trivial.
(d) $\Rightarrow$ (a): Since we can write every $A \in L^X$ as $A = \bigcup_{x\in X} A(x) \otimes \{x\}$ we have for all $y \in Y$

$$\begin{aligned}A^{\cap_J}(y) &= \bigvee_{x'\in X} A(x') \otimes J(x',y)\\ &= \bigvee_{x'\in X} (\bigcup_{x\in X} A(x) \otimes \{x\})(x') \otimes J(x',y)\\ &= \bigvee_{x'\in X} (\bigvee_{x\in X} A(x) \otimes \{x\}(x')) \otimes J(x',y)\\ &= \bigvee_{x\in X} A(x) \otimes \bigvee_{x'\in X} \{x\}(x') \otimes J(x',y)\\ &= \bigvee_{x\in X} A(x) \otimes \{x\}^{\cap_J}(y)\\ &= (\bigcup_{x\in X} A(x) \otimes \{x\}^{\cap_J})(y).\end{aligned}$$

Since $\{x\}^{\cap_J} \in \mathrm{Int}^{\cap\cup}(X,Y,I)$ and $\mathrm{Int}^{\cap\cup}(X,Y,I)$ is an **L**-interior system we obtain that $A^{\cap_J} \in \mathrm{Int}^{\cap\cup}(X,Y,I)$. Because every intent in $\mathrm{Int}^{\cap\cup}(X,Y,J)$ has the form $A^{\cap_J}$ for some $A \in L^X$ we conclude that $\mathrm{Int}^{\cap\cup}(X,Y,J) \subseteq \mathrm{Int}^{\cap\cup}(X,Y,I)$. Similarly, one can show that $\{y\}^{\vee_J} \in \mathrm{Ext}^{\wedge\vee}(X,Y,I)$ for all $y \in Y$ implies $\mathrm{Ext}^{\wedge\vee}(X,Y,J) \subseteq \mathrm{Ext}^{\wedge\vee}(X,Y,I)$.

(a) $\Rightarrow$ (f): From (33) we have $\mathrm{Ext}^{\wedge\vee}(X,Y,J) \subseteq \mathrm{Ext}^{\wedge\vee}(X,Y,I)$. Using Theorem 2(b) we get that there exists $S_Y \in L^{Y\times Y}$ s.t. $J = I \circ S_Y$. Since $\mathrm{Int}^{\cap\cup}(X,Y,I\circ S_Y) \subseteq \mathrm{Int}^{\cap\cup}(X,Y,I)$, by (33), and each $B \in \mathrm{Int}^{\cap\cup}(X,Y,I)$ can be written as $B = A^{\cap_I}$ for some $A \in L^X$, we have

$$B^{\cap_{S_Y}} = (A^{\cap_I})^{\cap_{S_Y}} = (A\circ I)\circ S_Y = A\circ(I\circ S_Y) = A\circ J = A^{\cap_J} \in \mathrm{Int}^{\cap\cup}(X,Y,I) \tag{34}$$

for each $B \in \mathrm{Int}^{\cap\cup}(X,Y,I)$. In addition, we have that $J \subseteq I$ implies $A^{\cap_I} \supseteq A^{\cap_J}$. As $A^{\cap_J} = (A^{\cap_I})^{\cap_{S_Y}}$ we obtain that $B^{\cap_{S_Y}} \subseteq B$.

(f) $\Rightarrow$ (a): From $J = I \circ S_Y$ we have $\mathrm{Ext}^{\wedge\vee}(X,Y,J) \subseteq \mathrm{Ext}^{\wedge\vee}(X,Y,I)$ by Theorem 2(b). For each $B \in \mathrm{Int}^{\cap\cup}(X,Y,I)$ there is some $A \in L^X$ s.t. $A^{\cap_I} = B$. We have

$$\mathrm{Int}^{\cap\cup}(X,Y,I) \ni B^{\cap_{S_Y}} = (A^{\cap_I})^{\cap_{S_Y}} = (A\circ I)\circ S_Y = A\circ(I\circ S_Y) = A\circ J = A^{\cap_J}. \tag{35}$$

As all intents from $\mathrm{Int}^{\cap\cup}(X,Y,J)$ are of the form $A^{\cap_J}$, we obtain that $\mathrm{Int}^{\cap\cup}(X,Y,J) \subseteq \mathrm{Int}^{\cap\cup}(X,Y,I)$. Finally, $B^{\cap_{S_Y}} \subseteq B$ for each $B \in \mathrm{Int}^{\cap\cup}(X,Y,I)$ is equivalent to $(A^{\cap_I})^{\cap_{S_Y}} \subseteq A^{\cap_I}$ for each $A \in L^X$. As $(A^{\cap_I})^{\cap_{S_Y}} = A^{\cap_J}$, we get that $J \subseteq I$.

(a) $\Leftrightarrow$ (e) can be proved similarly as (a) $\Leftrightarrow$ (f). $\square$

Remark 7. If the Double Negation Law holds true in **L**, i.e. if for all $a \in L$ we have $\neg\neg a = a$, the notions of block **L**-relation w.r.t. $\langle\cap,\cup\rangle$, block **L**-relation w.r.t. $\langle\wedge,\vee\rangle$, and strong block **L**-relation become the same notion [9]. In the general setting, it is not the case, as one can check in Example 1.

J	c1	c2	c3	c4	c5
BV	0.2	0.2	0	0.2	0.4
LH	0.6	0.6	0.6	0.8	0.8
MD	0.6	0.2	0.8	0.6	0.8
TSS	0.2	0.2	0.6	0.4	0.2

S_X	BV	LH	MD	TSS
BV	0.8	0	0	0
LH	0	0.8	0	0
MD	0	0	0.8	0
TSS	0	0	0	0.8

S_Y	c1	c2	c3	c4	c5
c1	0.8	0	0	0	0
c2	0	0.8	0	0	0
c3	0	0	0.8	0	0
c4	0	0	0	0.8	0
c5	0	0	0	0	0.8

Fig. 6. Block **L**-relation $J = 0.8 \otimes I$ w.r.t. $\langle \cap, \cup \rangle$ (top) of the **L**-context in Fig. 1 and the corresponding **L**-relations S_X (middle) and S_Y (bottom).

J	c1	c2	c3	c4	c5
BV	0.2	0.2	0.2	0.2	0.4
LH	0.6	0.6	0.8	0.8	0.8
MD	0.6	0.4	0.8	0.6	0.8
TSS	0.2	0.2	0.6	0.4	0.4

S_X	BV	LH	MD	TSS
BV	0.8	0.2	0.2	0.4
LH	1	0.8	0.8	1
MD	1	0.6	0.8	1
TSS	0.8	0.4	0.4	0.8

S_Y	c1	c2	c3	c4	c5
c1	0.4	0.4	0.6	0.6	0.6
c2	0.4	0.4	0.6	0.6	0.6
c3	0.6	0.4	0.8	0.6	0.8
c4	0.6	0.6	0.8	0.8	0.8
c5	0.6	0.6	0.8	0.8	0.8

Fig. 7. Block **L**-relation J (top) w.r.t. $\langle \cap, \cup \rangle$ of the **L**-context in Fig. 1 and the corresponding **L**-relations S_X (middle) and S_Y (bottom).

Theorem 8.

*(a) The system of all strong block **L**-relations of I is an **L**-interior system.*
*(b) The system of all **L**-relations S_X (from Theorem 7) is an **L**-interior system.*
*(c) The system of all **L**-relations S_Y (from Theorem 7) is an **L**-interior system.*

Proof. (a) From Theorem 7 we have that each strong block **L**-relation is a block **L**-relation w.r.t. both $\langle \cap, \cup \rangle$ and $\langle \wedge, \vee \rangle$. Thus, the system of all strong block **L**-relations is an intersection of two **L**-interior systems—$\mathrm{BR}^{\cap\cup}(X, Y, I)$ and $\mathrm{BR}^{\wedge\vee}(X, Y, I)$. It is therefore an **L**-interior system. Similarly for (b) and (c). □

Example (cont.)
We present two examples, analogous to the examples from Section 3.1. Theorem 8(a) yields that for each $a \in L$ the product $a \otimes I$ is a strong block relation of the formal **L**-context w.r.t. both isotone concept-forming operators. Fig. 6 shows the strong block relation $0.8 \otimes I$ of our example formal context $\langle X, Y, I \rangle$, together with **L**-relations S_X and S_Y.
In Fig. 7, we can see a more interesting example of a strong block relation, together with relations S_X and S_Y.

*3.4. Block **L**-relations and automorphisms of closure and interior systems*

In this section we show that block **L**-relations are in correspondence with particular automorphisms of **L**-closure and **L**-interior systems.

Definition 3. (See [10].)

(a) A mapping $h : V \to W$ from an **L**-interior system $V \subseteq L^X$ into an **L**-interior system $W \subseteq L^Y$ is called an *i-morphism* if it is a $\otimes$- and $\bigvee$-morphism, i.e.

- $h(a \otimes C) = a \otimes h(C)$ for each $a \in L$ and $C \in V$;
- $h(\bigvee_{k \in K} C_k) = \bigvee_{k \in K} h(C_k)$ for every collection of $C_k \in V$ $(k \in K)$.

An i-morphism $h : V \to W$ is said to be an *extendable i-morphism* if h can be extended to an i-morphism of L^X to L^Y, i.e. if there exists an i-morphism $h' : L^X \to L^Y$ such that for every $C \in V$ we have $h'(C) = h(C)$.

(b) A mapping $h : V \to W$ from an **L**-closure system $V \subseteq L^X$ into an **L**-closure system $W \subseteq L^Y$ is called a *c-morphism* if it is a $\to$- and $\bigwedge$-morphism and it preserves a-complements, i.e. if
- $h(a \to C) = a \to h(C)$ for each $a \in L$ and $C \in V$;
- $h(\bigwedge_{k \in K} C_k) = \bigwedge_{k \in K} h(C_k)$ for every collection of $C_k \in V$ $(k \in K)$;
- if C is an a-complement then $h(C)$ is an a-complement.

A c-morphism $h : V \to W$ is called an *extendable c-morphism* if h can be extended to a c-morphism of L^X to L^Y, i.e. if there exists a c-morphism $h' : L^X \to L^Y$ such that for every $C \in V$ we have $h'(C) = h(C)$.

An $\{i, c\}$-morphism $h : V \to V$ is called *$\{i, c\}$-automorphism* on V.

In this paper we will consider only extendable {i, c}-morphisms. The following results will be used in the proof of Theorem 9.

Lemma 1. *(See [10].) For $V \subseteq L^X$,*

*(a) if $h : V \to L^Y$ is an i-morphism then there exists an **L**-relation $R \in L^{X \times Y}$ such that $h'(C) = C \circ R$ for every $C \in L^X$.*
*(b) if $h : V \to L^Y$ is a c-morphism then there exists an **L**-relation $R \in L^{X \times Y}$ such that $h'(C) = C \triangleright R$ for every $C \in L^X$.*

Lemma 2. *(See [10].) Let $R \in L^{Y \times X}$,*

(a) the mapping $h_R : L^X \to L^Y$ defined by $h_R(C) = R \circ C$ for all $C \in L^X$ and the mapping $g_R : L^Y \to L^X$ defined by $g_R(D) = D \circ R$ for all $D \in L^Y$ are i-morphisms.
(b) the mapping $h_R : L^X \to L^Y$ defined by $h_R(C) = R \triangleleft C$ for all $C \in L^X$ and the mapping $g_R : L^Y \to L^X$ defined by $g_R(D) = D \triangleright R$ for all $D \in L^Y$ are c-morphisms.

Now, we can prove that there is a bijection between above defined morphisms and block relations.

Theorem 9.

(a) There is a bijection between $\mathrm{BR}^{\uparrow\downarrow}(X, Y, I)$ and the set of all extensive c-automorphisms on $\mathrm{Int}^{\uparrow\downarrow}(X, Y, I)$. There is a bijection between $\mathrm{BR}^{\uparrow\downarrow}(X, Y, I)$ and the set of all c-automorphisms on $\mathrm{Ext}^{\uparrow\downarrow}(X, Y, I)$.
(b) There is a bijection between $\mathrm{BR}^{\cap\cup}(X, Y, I)$ and the set of all intensive c-automorphisms on $\mathrm{Ext}^{\cap\cup}(X, Y, I)$. There is a bijection between $\mathrm{BR}^{\wedge\vee}(X, Y, I)$ and the set of all intensive c-automorphisms on $\mathrm{Int}^{\wedge\vee}(X, Y, I)$.
*(c) There is a bijection between the set of all strong block **L**-relations of $I \subseteq L^{X \times Y}$ and intensive i-automorphisms on $\mathrm{Ext}^{\wedge\vee}(X, Y, I)$ and with intensive i-automorphisms on $\mathrm{Int}^{\cap\cup}(X, Y, I)$.*

Proof. (a) First, we show that for each block **L**-relation of I w.r.t. $\langle\uparrow, \downarrow\rangle$ we can obtain an extensive c-automorphism on $\mathrm{Int}^{\uparrow\downarrow}(X, Y, I)$: Let $J \supseteq I \in L^{X \times Y}$ be a block **L**-relation of I w.r.t. $\langle\uparrow, \downarrow\rangle$. By Theorem 3 we have $J = I \triangleright S_Y$, s.t. $\wedge_{S_Y}$ is extensive mapping from $\mathrm{Int}^{\uparrow\downarrow}(X, Y, I) \to \mathrm{Int}^{\uparrow\downarrow}(X, Y, I)$. By Lemma 2(b) $^{\wedge S_Y}$ is a c-morphism. To sum up, $^{\wedge S_Y}$ is an extensive c-automorphism on $\mathrm{Int}^{\uparrow\downarrow}(X, Y, I)$.

Second, we show that for each extensive c-automorphism on $\mathrm{Int}^{\uparrow\downarrow}(X, Y, I)$ we can obtain a block **L**-relation of I w.r.t. $\langle\uparrow, \downarrow\rangle$: Let $f : \mathrm{Int}^{\uparrow\downarrow}(X, Y, I) \to \mathrm{Int}^{\uparrow\downarrow}(X, Y, I)$ be an extensive c-automorphism. By Lemma 1 there is $S_Y \in L^{Y \times Y}$ s.t. $B^{\wedge S_Y} = f(B)$ for each $B \in \mathrm{Int}^{\uparrow\downarrow}(X, Y, I)$. The **L**-relation $I \subseteq J = I \triangleright S_Y$ is a block **L**-relation of I w.r.t. $\langle\uparrow, \downarrow\rangle$ by Theorem 3.

One can check that the two procedures are mutually inverse. The second statement in (a) can be proved similarly.

(b) First we show that for each **L**-relation of I w.r.t. $\langle\cap, \cup\rangle$ we can obtain an extensive c-automorphism on $\mathrm{Ext}^{\cap\cup}(X, Y, I)$: Let $J \subseteq I$ be a block **L**-relation of $I \in L^{X \times Y}$ w.r.t. $\langle\cap, \cup\rangle$. By Theorem 5 we have $J = S_X \circ I$, s.t. $\cup_{S_X}$ is extensive mapping from $\mathrm{Ext}^{\cap\cup}(X, Y, I) \to \mathrm{Ext}^{\cap\cup}(X, Y, I)$. By Lemma 2(b), $^{\cup S_X}$ is a c-morphism. To sum up, $^{\cup S_X}$ is an extensive c-automorphism on $\mathrm{Ext}^{\cap\cup}(X, Y, I)$.

Second, we show that for each extensive c-automorphism on $\mathrm{Ext}^{\cap\cup}(X, Y, I)$ we can obtain a block **L**-relation of I w.r.t. $\langle\cap, \cup\rangle$: Let $f : \mathrm{Ext}^{\cap\cup}(X, Y, I) \to \mathrm{Ext}^{\cap\cup}(X, Y, I)$ be an extensive c-automorphism. By Lemma 1 there is $S_X \in L^{X \times X}$ s.t. $A^{\cup S_X} = f(A)$ for each $A \in \mathrm{Ext}^{\cap\cup}(X, Y, I)$. By Theorem 5 the **L**-relation $J = S_X \circ I$ is a block **L**-relation of I w.r.t. $\langle\cap, \cup\rangle$.

One can check that the two procedures are mutually inverse. The second statement in (b) can be proved similarly.

(c) First we show that for each strong **L**-relation of I we can obtain an extensive i-automorphism on $\mathrm{Ext}^{\wedge\vee}(X, Y, I)$: Let $J \subseteq I$ be a strong block **L**-relation of $I \in L^{X \times Y}$. By Theorem 7 we have $J = S_X \circ I$, s.t. $\vee_{S_X}$ is intensive mapping from $\mathrm{Ext}^{\wedge\vee}(X, Y, I) \to \mathrm{Ext}^{\wedge\vee}(X, Y, I)$. By Lemma 2(b) $\vee_{S_X}$ is an i-morphism. To sum up, $\vee_{S_X}$ is an intensive i-automorphism of $\mathrm{Ext}^{\wedge\vee}(X, Y, I)$.

Second, we show that for each extensive i-automorphism on $\mathrm{Ext}^{\cap\cup}(X,Y,I)$ we can obtain a strong block **L**-relation of I: Let $f:\mathrm{Ext}^{\wedge\vee}(X,Y,I)\to\mathrm{Ext}^{\wedge\vee}(X,Y,I)$ be an intensive i-automorphism. By Lemma 1 there is $S_X\in L^{X\times X}$ s.t. $A^{\vee S_X}=f(A)$ for each $A\in\mathrm{Ext}^{\wedge\vee}(X,Y,I)$. By Theorem 7 the **L**-relation $J=S_X\circ I$ is a strong block **L**-relation of I.

One can check that the two procedures are mutually inverse. □

3.5. Factorizations of concept lattices

As mentioned in the introduction, one of vital problems in formal concept analysis, esp. in fuzzy setting, is the problem of reduction of the size of a concept lattice. In this section, we present some results on reducing the size of concept lattices by means of factorization. We generalize the results from the crisp case [31] (see also [16]) to our fuzzy setting. Our results also cover results from [28] as a special case (see example below).

The basic idea is as follows. Sometimes we do not need to distinguish between concepts we consider in some sense similar or indistinguishable. The exact notion of indistinguishability between concepts depends on our needs. If it is expressed by a complete **L**-tolerance relation (see below for definition) then it can be used for factorization of the concept lattice, resulting in a smaller completely lattice **L**-ordered set [24]. As we show in this section, this completely lattice **L**-ordered set is, in fact, isomorphic to the **L**-concept lattice of a block relation, which can be easily computed based on the used complete **L**-tolerance.

In the crisp case, a complete tolerance on a complete lattice is a tolerance $\sim$ (i.e. a reflexive and symmetric relation) that preserves arbitrary suprema and infima: If $u_i\sim v_i$ for each $i\in\mathcal{I}$ then $\bigwedge_{i\in\mathcal{I}}u_i\sim\bigwedge_{i\in\mathcal{I}}v_i$ and $\bigvee_{i\in\mathcal{I}}u_i\sim\bigvee_{i\in\mathcal{I}}v_i$.

A quotient (factor) set $X/_\sim$ of a set X by a tolerance $\sim$ on X is the set of all maximal blocks of $\sim$, i.e. all maximal (w.r.t. set inclusion) sets, containing only pairwise tolerant elements. In the case X is a complete lattice and $\sim$ is complete, a natural extension of the ordering on X to $X/_\sim$ leads to a complete lattice ordering [15,31,16].

There is a bijection between the set of block relations and the set of complete tolerances on associated concept lattices. Moreover, it is known that a concept lattice, factorized by a complete tolerance, is isomorphic to the concept lattice of the corresponding block relation [31,16].

Now we turn to the fuzzy case. We start by introducing a few necessary notions and results. See [24] for details.

For any binary **L**-relation R on a set X and **L**-sets $A_1,A_2\in L^X$ we set [17]

$$R^+(A_1,A_2)=S(A_1,R\circ A_2)\wedge S(A_2,A_1\circ R),\tag{36}$$

obtaining a binary **L**-relation on L^X.

For an **L**-tolerance $\sim$ on X, an **L**-set $B\in L^X$ is called a block of $\sim$ if for each $x_1,x_2\in X$ it holds $B(x_1)\otimes B(x_2)\le(x_1\sim x_2)$ [8]. A block B is called maximal if for each block B', $B\subseteq B'$ implies $B=B'$. The set of all maximal blocks of $\sim$, which always exists by Zorn's lemma, is called the quotient (factor) set of X by $\sim$ and denoted $X/\sim$.

We set for each $x\in X$, $[\![x]\!]_\sim(y)=x\sim y$, obtaining an **L**-set $[\![x]\!]_\sim\in L^X$ called the class of $\sim$ determined by x.

An **L**-tolerance $\sim$ on a completely lattice **L**-ordered set $\mathbf{U}=\langle\langle U,\approx\rangle,\preceq\rangle$ is called complete [24] if it is compatible with $\approx$ and for any two **L**-sets $V_1,V_2\in L^U$ it holds

$$V_1\sim^+V_2\le\inf V_1\sim\inf V_2,\qquad V_1\sim^+V_2\le\sup V_1\sim\sup V_2.\tag{37}$$

In crisp case, the conditions (37) become the above standard condition of completeness of $\sim$.

We set for each $u\in U$

$$u_\sim=\inf[\![u]\!]_\sim,\qquad u^\sim=\sup[\![u]\!]_\sim.\tag{38}$$

(Note that as $[\![u]\!]_\sim$ is an **L**-set in U, its infimum is an element of U: $u_\sim\in U$. Similarly, $u^\sim\in U$.) It holds

$$[\![u]\!]_\sim=[\![u_\sim,u^\sim]\!].\tag{39}$$

The pair $\langle_\sim,{}^\sim\rangle$ is an extensive isotone **L**-Galois connection on **U**.

In [24] we generalized the above results from [15,32,16] on complete tolerances on complete lattices to complete **L**-tolerances on completely lattice **L**-ordered sets. The results we will need are the following.

Theorem 10. *(See [24].) Let $\sim$ be a complete **L**-tolerance on a completely lattice **L**-ordered set $\mathbf{U}=\langle\langle U,\approx\rangle,\preceq\rangle$. Maximal blocks of $\sim$ are exactly intervals $[\![u,v]\!]$ where $\langle v,u\rangle$ are fixpoints of $\langle_\sim,{}^\sim\rangle$. The **L**-relations $\approx^+$ and $\preceq^+$ on the quotient set $\mathbf{U}/_\sim$ satisfy for each $u_1,u_2,v_1,v_2\in U$*

$$[\![u_1,v_1]\!]\approx^+[\![u_2,v_2]\!]=u_1\approx u_2=v_1\approx v_2,$$

$$[\![u_1,v_1]\!]\preceq^+[\![u_2,v_2]\!]=u_1\preceq u_2=v_1\preceq v_2.$$

*$\langle\langle U/_\sim,\approx^+\rangle,\preceq^+\rangle$ is a completely lattice **L**-ordered set.*

By [24], the system CTol **U** of complete **L**-tolerances on a completely lattice **L**-ordered set **U** is an **L**-closure system. Consequently $\langle\langle \text{CTol}\,\mathbf{U}, \approx^{U\times U}\rangle, S\rangle$ is a completely lattice **L**-ordered set with infima given by intersections (2).

In [24] we also introduced a structure of **L**-ordered set on the set IGal(**U**, **V**) of isotone **L**-Galois connections between **U** and **V** by

$$\langle f_1, g_1\rangle \approx_{\mathrm{IGal}(\mathbf{U},\mathbf{V})} \langle f_2, g_2\rangle = \bigwedge_{u\in U}(f_2(u) \approx f_1(u)) \wedge \bigwedge_{v\in V}(g_1(v) \approx g_2(v)), \tag{40}$$

$$\langle f_1, g_1\rangle \preceq_{\mathrm{IGal}(\mathbf{U},\mathbf{V})} \langle f_2, g_2\rangle = \bigwedge_{u\in U}(f_2(u) \preceq f_1(u)) \wedge \bigwedge_{v\in V}(g_1(v) \preceq g_2(v)). \tag{41}$$

This definition can be simplified using the following lemma.

Lemma 3. *For each* $\langle f_1, g_1\rangle, \langle f_2, g_2\rangle \in \mathrm{IGal}(\mathbf{U},\mathbf{V})$ *it holds*

$$\langle f_1, g_1\rangle \approx_{\mathrm{IGal}(\mathbf{U},\mathbf{V})} \langle f_2, g_2\rangle = \bigwedge_{u\in U}(f_2(u) \approx f_1(u)) = \bigwedge_{v\in V}(g_1(v) \approx g_2(v)), \tag{42}$$

$$\langle f_1, g_1\rangle \preceq_{\mathrm{IGal}(\mathbf{U},\mathbf{V})} \langle f_2, g_2\rangle = \bigwedge_{u\in U}(f_2(u) \preceq f_1(u)) = \bigwedge_{v\in V}(g_1(v) \preceq g_2(v)). \tag{43}$$

Proof. We will prove the second equality of (43) by finding for each $u \in U$ an element $v \in V$ such that $g_1(v) \preceq g_2(v) \leq f_2(u) \preceq f_1(u)$ and for each $v \in V$ an element $u \in U$ such that $f_2(u) \preceq f_1(u) \leq g_1(v) \preceq g_2(v)$. The first equality will then follow by (41).

First, for $u \in U$ we set $v = f_1(u)$. The definition of isotone **L**-Galois connections immediately yields $u \leq g_1(f_1(u))$. Thus, by definition and transitivity of $\preceq$,

$$g_1(v) \preceq g_2(v) = f_2(g_1(v)) \preceq v = f_2(g_1(f_1(u))) \preceq f_1(u) \leq f_2(u) \preceq f_1(u).$$

For $v \in V$ we set $u = g_1(v)$ and prove the second inequality similarly.

(42) now follows by (19). □

By [24], IGal(**U**, **V**), together with $\approx_{\mathrm{IGal}(\mathbf{U},\mathbf{V})}$ and $\preceq_{\mathrm{IGal}(\mathbf{U},\mathbf{V})}$, is an **L**-ordered set.

For a completely lattice **L**-ordered set **U**, we denote by EIGal(**U**) the subset of IGal(**U**, **U**) consisting of extensive isotone **L**-Galois connections. This set inherits the structure of **L**-ordered set from IGal(**U**, **U**) (40), (41).

Theorem 11. *(See [24].) The mapping* $\sim\ \mapsto \langle \underset{\sim}{}, \overset{\sim}{}\rangle$ *is an isomorphism between* CTol **U** *and* EIGal(**U**)*. Its inverse is* $\langle f, g\rangle \mapsto \sim_{\langle f,g\rangle}$, *where* $\sim_{\langle f,g\rangle}$ *is given by*

$$u \sim_{\langle f,g\rangle} v = (f(u) \preceq v) \wedge (f(v) \preceq u). \tag{44}$$

As we already mentioned, $\langle\langle \text{CTol}\,\mathbf{U}, \approx^{U\times U}\rangle, S\rangle$ is a completely lattice **L**-ordered set. Thus, by the above theorem, EIGal(**U**) is also a completely lattice **L**-ordered set. In the next theorem, we show an explicit expression for infima in EIGal(**U**). We need a result on intersection of a system of intervals, which is proved in the following lemma.

Lemma 4. *Let* **U** *be a completely lattice* **L***-ordered set,* $H \in L^{L^U}$ *an* **L***-set of* **L***-intervals in* U, $H = \{{}^{a_k}\!/[\![u_k, v_k]\!] \mid k \in K\}$*. Denote by* $H_{\min}$*, resp.* $H_{\max}$*, the associated* **L***-set in* **U** *of lower, resp. upper, bounds of the intervals:*

$$H_{\min} = \{{}^{a_k}\!/u_k \mid k \in K\}, \qquad H_{\max} = \{{}^{a_k}\!/v_k \mid k \in K\}. \tag{45}$$

Finally, denote $u = \sup H_{\min}$ *and* $v = \inf H_{\max}$*. Then if* $u \leq v$ *then* $\bigcap H = [\![u, v]\!]$.

Proof. By definition of interval, $[\![u, v]\!] = \mathcal{U}\{u\} \cap \mathcal{L}\{v\}$. It can be shown by definition of supremum that $\mathcal{U}\{u\} = \mathcal{U}\{\sup H_{\min}\} = \mathcal{U}H_{\min}$ (the upper cone of a set is equal to the upper cone of its supremum) and similarly, $\mathcal{L}\{v\} = \mathcal{L}H_{\max}$. For the upper cone of $H_{\min}$ we have

$$\mathcal{U}H_{\min}(w) = \bigwedge_{w'\in U} H_{\min}(w') \to (w' \preceq w) = \bigwedge_{k\in K} a_k \to (u_k \preceq w)$$

and, similarly, $\mathcal{L}H_{\max}(w) = \bigwedge_{k\in K} a_k \to (w \preceq v_k)$. Thus by (45), (21), and (2),

$$\begin{aligned}
[\![u, v]\!](w) &= \mathcal{U}\{u\}(w) \wedge \mathcal{L}\{v\}(w) = \mathcal{U}H_{\min}(w) \wedge \mathcal{L}H_{\max}(w) \\
&= \Big(\bigwedge_{k\in K} a_k \to (u_k \preceq w)\Big) \wedge \Big(\bigwedge_{k\in K} a_k \to (w \preceq v_k)\Big) \\
&= \bigwedge_{k\in K} a_k \to (u_k \preceq w) \wedge (w \preceq v_k) = \bigwedge_{k\in K} a_k \to [\![u_k, v_k]\!](w) \\
&= \bigcap H(w). \qquad \square
\end{aligned}$$

For classical sets, the above lemma tells the well-known fact that if the intersection of a system of intervals in a complete lattice is non-empty then it is equal to an interval. The lower bound of this interval is the supremum of lower bounds of the intervals from the system and the upper bound is the infimum of upper bounds of the intervals from the system.

Now we give the promised theorem on infima of **L**-sets of extensive isotone **L**-Galois connections.

Theorem 12. *Let $M \in L^{\mathrm{EIGal}(\mathbf{U})}$ be an **L**-set of extensive isotone **L**-Galois connections on a completely lattice **L**-ordered set **U**, $M = \{{}^{a_k}/\langle f_k, g_k\rangle \mid k \in K\}$. Then $\inf M = \langle f, g\rangle$, where f and g are given by*

$$f(u) = \sup\{{}^{a_k}/f_k(u) \mid k \in K\}, \qquad g(v) = \inf\{{}^{a_k}/g_k(v) \mid k \in K\}. \tag{46}$$

Proof. We will use the isomorphism of CTol(**U**) and EIGal(**U**) from Theorem 11. Set $\sim_k = \sim_{\langle f_k, g_k\rangle}$. We have $u_{\sim_k} = f_k(u)$ and $u^{\sim_k} = g_k(u)$ for each $u \in U$, $k \in K$. For $T = \{{}^{a_k}/\sim_k \mid k \in K\}$ and $\sim = \bigcap T$ it holds $\inf M = \langle \sim, {}^{\sim}\rangle$ (Theorem 11).

For the class $[\![u]\!]_\sim$ of an element $u \in U$ we have

$$\begin{aligned}
[\![u]\!]_\sim(v) &= [\![u]\!]_{\bigcap T}(v) = \Big(\bigcap T\Big)(\langle u, v\rangle) = \bigwedge_{k\in K} a_k \to (u \sim_k v) \\
&= \bigwedge_{k\in K} a_k \to [\![u]\!]_{\sim_k}(v) = \Big(\bigcap\{{}^{a_k}/[\![u]\!]_{\sim_k} \mid k \in K\}\Big)(v).
\end{aligned}$$

Thus, $[\![u]\!]_\sim = \bigcap\{{}^{a_k}/[\![u]\!]_{\sim_k} \mid k \in K\}$. As $[\![u]\!]_\sim = [\![u_\sim, u^\sim]\!]$ and $[\![u]\!]_{\sim_k} = [\![u_{\sim_k}, u^{\sim_k}]\!] = [\![f_k(u), g_k(u)]\!]$ (39), we can use Lemma 4 and obtain (46). $\square$

*3.5.1. The correspondence with block **L**-relations*

Now we are ready to introduce the correspondence between block relations on a formal **L**-context and extensive isotone **L**-Galois connections on the associated concept lattice. In the following lemma, int(C) (resp. ext(C)) denotes the intent (resp. extent) of a formal **L**-concept C. As before, symbols $\triangle$, $\triangledown$ denote one of the pairs $\langle\uparrow,\downarrow\rangle$, $\langle\cap,\cup\rangle$, $\langle\wedge,\vee\rangle$.

Lemma 5. *Let $J \in \mathrm{BR}^{\triangle\triangledown}(X, Y, I)$. Denote by $\langle f_J, g_J\rangle$ a pair of mappings on $\mathcal{B}^{\triangle\triangledown}(X, Y, I)$ defined by*

$$f_J(\langle A, B\rangle) = \langle A^{\triangle_J \triangledown_I}, A^{\triangle_J}\rangle, \qquad g_J(\langle A, B\rangle) = \langle B^{\triangledown_J}, B^{\triangledown_J \triangle_I}\rangle$$

for each $\langle A, B\rangle \in \mathcal{B}^{\triangle\triangledown}(X, Y, I)$. The mapping $J \mapsto \langle f_J, g_J\rangle$ is a bijection between $\mathrm{BR}^{\triangle\triangledown}(X, Y, I)$ and $\mathrm{EIGal}(\mathcal{B}^{\triangle\triangledown}(X, Y, I))$. Its inverse $\langle f, g\rangle \mapsto J_{\langle f, g\rangle}$ is given by

- *Case $\langle\triangle,\triangledown\rangle = \langle\uparrow,\downarrow\rangle$:*

$$\begin{aligned}
J_{\langle f,g\rangle}(x, y) &= \mathrm{int}(f(\langle\{x\}^{\uparrow_I\downarrow_I}, \{x\}^{\uparrow_I}\rangle))(y) \\
&= \mathrm{ext}(g(\langle\{y\}^{\downarrow_I}, \{y\}^{\downarrow_I\uparrow_I}\rangle))(x),
\end{aligned}$$

- *Case $\langle\triangle,\triangledown\rangle = \langle\cap,\cup\rangle$:*

$$J_{\langle f,g\rangle}(x, y) = \mathrm{int}(f(\langle\{x\}^{\cap_I\cup_I}, \{x\}^{\cap_I}\rangle))(y),$$

- *Case $\langle\triangle,\triangledown\rangle = \langle\wedge,\vee\rangle$:*

$$J_{\langle f,g\rangle}(x, y) = \mathrm{ext}(g(\langle\{y\}^{\vee_I}, \{y\}^{\vee_I\wedge_I}\rangle))(x).$$

Proof. We prove the lemma only for the case $\langle\triangle,\triangledown\rangle = \langle\uparrow,\downarrow\rangle$; the proof for the other two cases is similar.

First, we show that for a block **L**-relation J of $I \in \mathbf{L}^{X\times Y}$ w.r.t. $\langle\uparrow,\downarrow\rangle$ the pair of mappings $\langle f_J, g_J\rangle$ is an isotone **L**-Galois connection on $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$. This follows from

$$\begin{aligned} f_J(\langle A,B\rangle) \preceq \langle A',B'\rangle &= S(A^{\uparrow_J\downarrow_I}, A') \\ &= S(A^{\uparrow_J\downarrow_I}, B'^{\downarrow_I}) \\ &= S(B', A^{\uparrow_J\downarrow_I\uparrow_I}) \\ &= S(B', A^{\uparrow_J}) \\ &= S(A, B'^{\downarrow_J}) \\ &= \langle A,B\rangle \preceq g_J(\langle A',B'\rangle) \end{aligned}$$

(the third and fifth equality follow by basic properties of $\preceq$ (27), and $A^{\uparrow_J\downarrow_I\uparrow_I} = A^{\uparrow_J}$ since $A^{\uparrow_J}$ is an intent of I). We need to show that g_J is extensive; i.e. $\langle A,B\rangle \preceq g_J(\langle A,B\rangle)$; that is $A \subseteq B^{\downarrow_J}$ for each $A \in \mathbf{L}^X$. Since A is an extent of $\mathcal{B}^{\uparrow\downarrow}(X,Y,I)$, the latter inclusion is equivalent to $B^{\downarrow_I} \subseteq B^{\downarrow_J}$, which is true by $I \subseteq J$.

To prove the converse: Let $\langle f,g\rangle$ be an extensive isotone **L**-Galois connection on $\mathcal{B}^{\uparrow\downarrow}(X,Y,I)$. Consider the following operators $\Uparrow\colon L^X \to L^Y$ and $\Downarrow\colon L^Y \to L^X$ defined as

$$\begin{aligned} A^{\Uparrow} &= \mathrm{int}(f(\langle A^{\uparrow_I\downarrow_I}, A^{\uparrow_I}\rangle)) \\ B^{\Downarrow} &= \mathrm{ext}(g(\langle B^{\downarrow_I}, B^{\downarrow_I\uparrow_I}\rangle)) \end{aligned} \tag{47}$$

for all $A \in L^X$, $B \in L^Y$. Now we show that $\langle \Uparrow, \Downarrow\rangle$ forms an antitone **L**-Galois connection. For all $A \in \mathbf{L}^X$, $B \in \mathbf{L}^Y$ we have (27)

$$\begin{aligned} S(A, B^{\Downarrow}) &= S(A, \mathrm{ext}(g(\langle B^{\downarrow_I}, B^{\downarrow_I\uparrow_I}\rangle))) \\ &= S(A, \mathrm{int}(g(\langle B^{\downarrow_I}, B^{\downarrow_I\uparrow_I}\rangle))^{\downarrow_I}) \\ &= S(\mathrm{int}(g(\langle B^{\downarrow_I}, B^{\downarrow_I\uparrow_I}\rangle)), A^{\uparrow_I}) \\ &= S(\mathrm{int}(g(\langle B^{\downarrow_I}, B^{\downarrow_I\uparrow_I}\rangle)), A^{\uparrow_I\downarrow_I\uparrow_I}) \\ &= S(A^{\uparrow_I\downarrow_I}, \mathrm{int}(g(\langle B^{\downarrow_I}, B^{\downarrow_I\uparrow_I}\rangle))^{\downarrow_I}) \\ &= S(A^{\uparrow_I\downarrow_I}, \mathrm{ext}(g(\langle B^{\downarrow_I}, B^{\downarrow_I\uparrow_I}\rangle))) \\ &= \langle A^{\uparrow_I\downarrow_I}, A^{\uparrow_I}\rangle \preceq g(\langle B^{\uparrow_I\downarrow_I}, B^{\uparrow_I}\rangle). \end{aligned}$$

Similarly, we have

$$\begin{aligned} S(B, A^{\Uparrow}) &= S(B, \mathrm{int}(f(\langle A^{\uparrow_I\downarrow_I}, A^{\uparrow_I}\rangle))) \\ &= S(B, \mathrm{ext}(f(\langle A^{\uparrow_I\downarrow_I}, A^{\uparrow_I}\rangle))^{\uparrow_I}) \\ &= S(\mathrm{ext}(f(\langle A^{\uparrow_I\downarrow_I}, A^{\uparrow_I}\rangle)), B^{\downarrow_I}) \\ &= f(\langle A^{\uparrow_I\downarrow_I}, A^{\uparrow_I}\rangle) \preceq \langle B^{\downarrow_I}, B^{\downarrow_I\uparrow_I}\rangle; \end{aligned}$$

whence we have $S(A, B^{\Downarrow}) = S(B, A^{\Uparrow})$, proving that $\langle \Uparrow, \Downarrow\rangle$ is an antitone **L**-Galois connection between L^X and L^Y. By [2] there is an **L**-relation J such that $\langle \Uparrow, \Downarrow\rangle = \langle \uparrow_J, \downarrow_J\rangle$. J is given by $J(x,y) = \{x\}^{\Uparrow}(y) = \{y\}^{\Downarrow}(x)$.

Directly from (47) we have $\mathrm{Ext}^{\uparrow\downarrow}(X,Y,J) \subseteq \mathrm{Ext}^{\uparrow\downarrow}(X,Y,I)$ and $\mathrm{Int}^{\uparrow\downarrow}(X,Y,J) \subseteq \mathrm{Int}^{\uparrow\downarrow}(X,Y,I)$. From extensivity of g we have

$$\langle B^{\downarrow_I}, B^{\downarrow_I\uparrow_I}\rangle \preceq g(\langle B^{\downarrow_I}, B^{\downarrow_I\uparrow_I}\rangle)$$

for each $B \in L^Y$. From that we get $B^{\downarrow_I} \subseteq \mathrm{ext}(g(\langle B^{\downarrow_I}, B^{\downarrow_I\uparrow_I}\rangle)) = B^{\downarrow_J}$ for each $B \in L^Y$, proving that $J \supseteq I$.

It is easy to observe that these procedures are mutually inverse. □

By Theorem 4, the set $\mathrm{BR}^{\uparrow\downarrow}(X,Y,I)$ of all block relations of I w.r.t. $\langle\uparrow,\downarrow\rangle$ is an **L**-closure system. Together with the **L**-equality $\approx^{X\times Y}$ and the graded subsethood relation S (both restricted accordingly), it is a completely lattice **L**-ordered set.

Analogously, $\mathrm{BR}^{\cap\cup}(X,Y,I)$ and $\mathrm{BR}^{\wedge\vee}(X,Y,I)$ are **L**-interior systems by Theorem 6. Together with the **L**-equality $\approx^{X\times Y}$ and the graded subsethood relation S (both restricted accordingly), they are completely lattice **L**-ordered sets.

Lemma 6. *The mapping $J \mapsto \langle f_J, g_J\rangle$ from Lemma 5 is an isomorphism of* **L***-ordered sets.*

Proof. We prove the lemma only for the case $\langle \triangle, \triangledown\rangle = \langle\uparrow,\downarrow\rangle$; the proof for the isotone cases is similar. It suffices to show that for any two block relations J_1, J_2 it holds

$$\langle f_{J_1}, g_{J_1}\rangle \preceq \langle f_{J_2}, g_{J_2}\rangle = S(J_1, J_2), \tag{48}$$

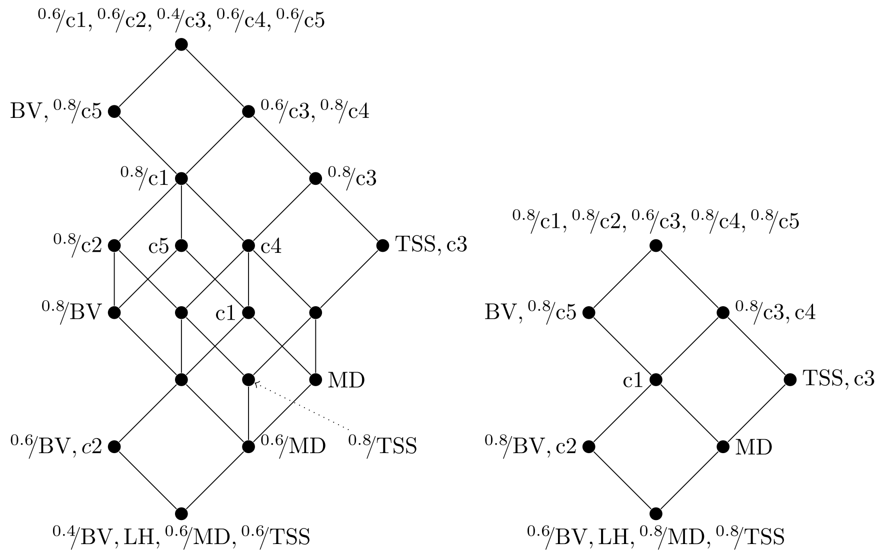

Fig. 8. **L**-concept lattice of movies, critics and ratings (from Fig. 1) factorized with respect to the threshold $a = 0.8$ (left) and $a = 0.6$ (right).

where $\preceq$ denotes $\preceq_{\mathrm{IGal}(\mathcal{B}^{\uparrow\downarrow}(X,Y,I))}$. The left-hand side is equal to (by (43))

$$\bigwedge_{C \in \mathcal{B}^{\uparrow\downarrow}(X,Y,I)} (f_{J_2}(C) \preceq f_{J_1}(C))$$

which by Lemma 5 and (27) equals

$$\bigwedge_{A \in \mathrm{Ext}^{\uparrow\downarrow}(X,Y,I)} S(A^{\uparrow_{J_1}}, A^{\uparrow_{J_2}}).$$

For the right-hand side of (48) we have

$$\begin{aligned} S(J_1, J_2) &= \bigwedge_{x \in X} \bigwedge_{y \in Y} J_1(x,y) \to J_2(x,y) = \bigwedge_{x \in X} \bigwedge_{y \in Y} \{x\}^{\uparrow_{J_1}}(y) \to \{x\}^{\uparrow_{J_2}}(y) \\ &= \bigwedge_{x \in X} S(\{x\}^{\uparrow_{J_1}}, \{x\}^{\uparrow_{J_2}}) = \bigwedge_{x \in X} S(\{x\}^{\uparrow_I \downarrow_I \uparrow_{J_1}}, \{x\}^{\uparrow_I \downarrow_I \uparrow_{J_2}}) \\ &\geq \bigwedge_{A \in \mathrm{Ext}^{\uparrow\downarrow}(X,Y,I)} S(A^{\uparrow_{J_1}}, A^{\uparrow_{J_2}}), \end{aligned}$$

proving the "$\leq$" part of (48).

Now for each extent A we have

$$S(A^{\uparrow_{J_1}}, A^{\uparrow_{J_2}}) = S(A \triangleleft J_1, A \triangleleft J_2) \geq S(J_1, J_2)$$

by properties of compositions operators. That proves the converse inequality. $\square$

Theorem 13. *There is an isomorphism between the* **L***-set of block* **L***-relations of* I *w.r.t.* $\langle \vartriangle, \triangledown \rangle$ *and the* **L***-set of complete* **L***-tolerances on* $\mathcal{B}^{\vartriangle\triangledown}(X,Y,I)$*. The isomorphism and its inverse are given by*

$$J \mapsto \sim_J \quad \text{and} \quad \sim \mapsto J_\sim,$$

where

$$\langle A_1, B_1 \rangle \sim_J \langle A_2, B_2 \rangle = S(A_1, B_2^{\triangledown_J}) \wedge S(A_2, B_1^{\triangledown_J}), \tag{49}$$

- *Case* $\langle \vartriangle, \triangledown \rangle = \langle \uparrow, \downarrow \rangle$

$$\begin{aligned} J_\sim(x,y) &= \mathrm{int}(\langle \{x\}^{\uparrow_I \downarrow_I}, \{x\}^{\uparrow_I} \rangle_\sim)(y) \\ &= \mathrm{ext}(\langle \{y\}^{\downarrow_I}, \{y\}^{\downarrow_I \uparrow_I} \rangle^\sim)(x), \end{aligned} \tag{50}$$

- *Case* $\langle \vartriangle, \triangledown \rangle = \langle \cap, \cup \rangle$

$$J_\sim(x,y) = \mathrm{int}(\langle \{x\}^{\cap_I \cup_I}, \{x\}^{\cap_I} \rangle_\sim)(y), \tag{51}$$

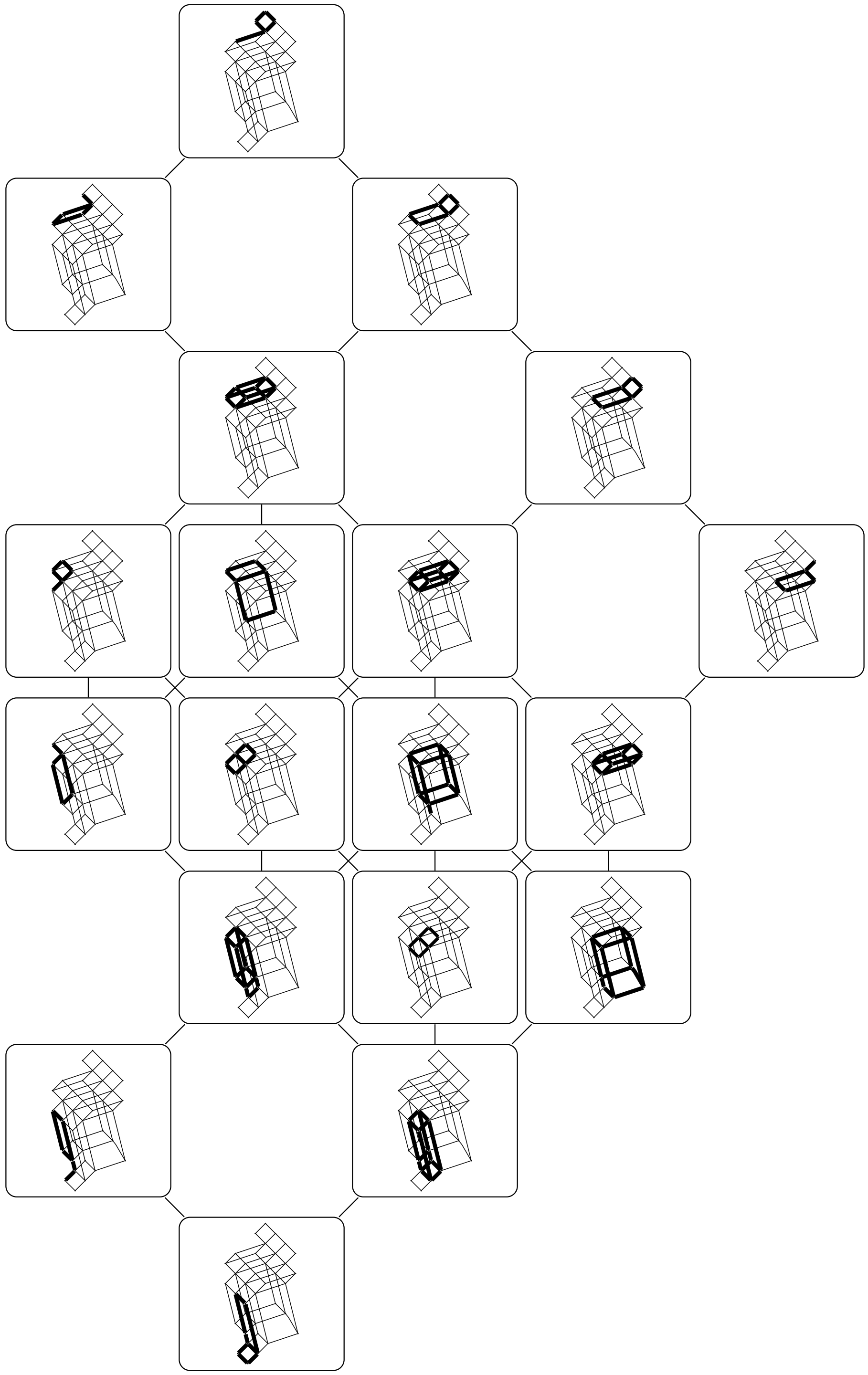

Fig. 9. **L**-concept lattices from Fig. 8 (left) depicted as a lattice of intervals (crisp parts of maximal blocks) of the original **L**-concept lattice.

- *Case* $\langle \triangle, \triangledown \rangle = \langle \wedge, \vee \rangle$

$$J_{\sim}(x, y) = \mathrm{ext}(\langle \{y\}^{\vee_I}, \{y\}^{\vee_I \wedge_I} \rangle^{\sim})(x). \tag{52}$$

Proof. It follows from Lemma 5, Lemma 6, and Theorem 11. □

The following theorem shows a simple way of computing the quotient completely lattice **L**-ordered set $\mathcal{B}^{\triangle\triangledown}(X, Y, I)/_{\sim}$ for given complete **L**-tolerance $\sim$: it is isomorphic to the **L**-concept lattice $\mathcal{B}^{\triangle\triangledown}(X, Y, J)$, where J is the block relation associated with $\sim$ by means of Theorem 13.

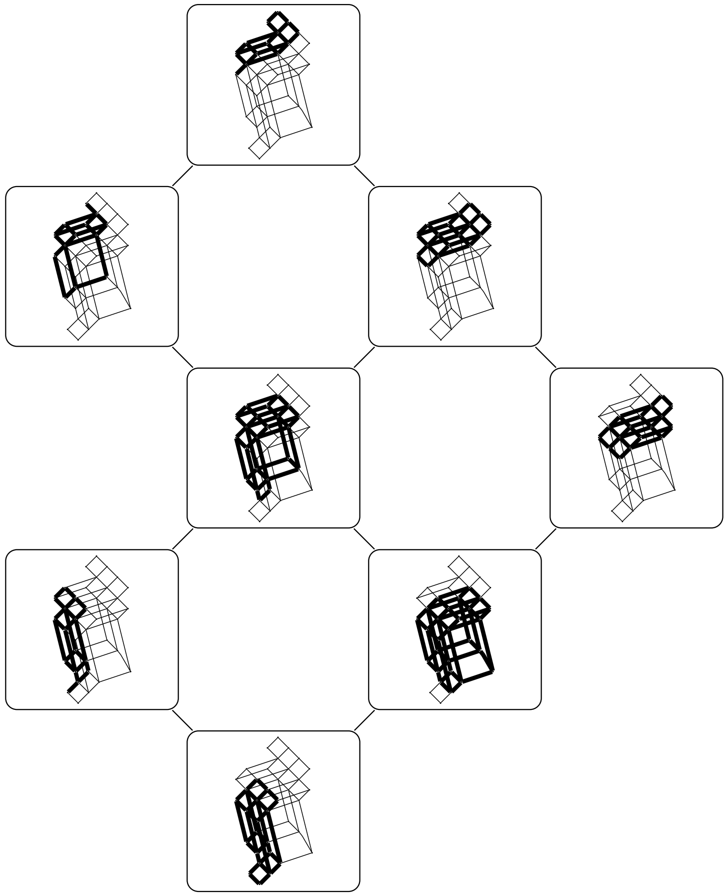

Fig. 10. **L**-concept lattices from Fig. 8 (right) depicted as a lattice of intervals (crisp parts of maximal blocks) of the original **L**-concept lattice.

Theorem 14. *Let $\sim$ be a complete* **L***-tolerance on $\mathcal{B}^{\Delta\nabla}(X, Y, I)$, J be the block* **L***-relation of I w.r.t. $\langle\triangle, \triangledown\rangle$ associated with $\sim$ by means of Theorem 13. Then the completely lattice* **L***-ordered sets $\mathcal{B}^{\Delta\nabla}(X, Y, I)/_\sim$ and $\mathcal{B}^{\Delta\nabla}(X, Y, J)$ are isomorphic.*

Proof. By Lemma 5, if $\langle C_1, C_0\rangle$ is a fixpoint of $\langle f_J, g_J\rangle$ then the extent of C_1 is an extent of $\langle X, Y, J\rangle$ and vice versa, each extent of $\langle X, Y, J\rangle$ is also the extent of some $C_1 \in \mathcal{B}^{\Delta\nabla}(X, Y, I)$ such that $\langle C_1, C_0\rangle$ is a fixpoint of $\langle f_J, g_J\rangle$ for some C_0. This way we obtain a bijection between the set of all fixpoints of $\langle f_J, g_J\rangle$ and $\mathrm{Ext}^{\Delta\nabla}(X, Y, J)$. Using Theorem 10 and (27) we obtain a bijection between $\mathcal{B}^{\Delta\nabla}(X, Y, I)/_\sim$ and $\mathcal{B}^{\Delta\nabla}(X, Y, J)$ which is evidently an isomorphism of **L**-ordered sets. $\square$

3.6. Factorization by similarity

By similarity of concepts we mean any a-shift of the **L**-equality $\approx$ (27), where $a \in L$ is arbitrary threshold. The **L**-relation $\approx$ itself is a complete **L**-tolerance on $\mathcal{B}^{\Delta\nabla}(X, Y, I)$, which follows easily from (44) where we set $\langle f, g\rangle$ to be the trivial isotone Galois connection $f(u) = g(u) = u$. The a-shift $\sim_a = a \to \approx$ of $\approx$ is also a complete **L**-tolerance because the system of all complete **L**-tolerances on $\mathcal{B}^{\Delta\nabla}(X, Y, I)$ is an **L**-closure system [24].

For two **L**-concepts $C_1 = \langle A_1, B_1\rangle$, $C_2 = \langle A_2, B_2\rangle$ from $\mathcal{B}^{\Delta\nabla}(X, Y, I)$ we have $C_1 \sim_a C_2 = 1$ iff $C_1 \approx C_2 \geq a$. Thus, the concepts are fully indistinguishable by $\sim_a$ iff the degree to which they are similar w.r.t. $\approx$ is at least a.

Now let $\langle\triangle, \triangledown\rangle = \langle\uparrow, \downarrow\rangle$ and $J = a \to I$. We already know that J is a block relation. By Theorem 13 (49), the associated complete **L**-tolerance is given by

$$\begin{aligned} C_1 \sim_J C_2 &= S(A_1, B_2^{\downarrow_J}) \wedge S(A_2, B_1^{\downarrow_J}) \\ &= S(A_1, a \to B_2^{\downarrow_I}) \wedge S(A_2, a \to B_1^{\downarrow_I}) \\ &= (a \to S(A_1, A_2)) \wedge (a \to S(A_2, A_1)) = a \to (A_1 \approx^X A_2) \\ &= a \to (C_1 \approx C_2) = C_1 \sim_a C_2. \end{aligned}$$

Thus, $\sim_J$ is equal to $\sim_a$ and by Theorem 14, the factor completely lattice **L**-ordered set $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)/_{a\to\approx}$ is isomorphic to $\mathcal{B}^{\uparrow\downarrow}(X, Y, a \to I)$.

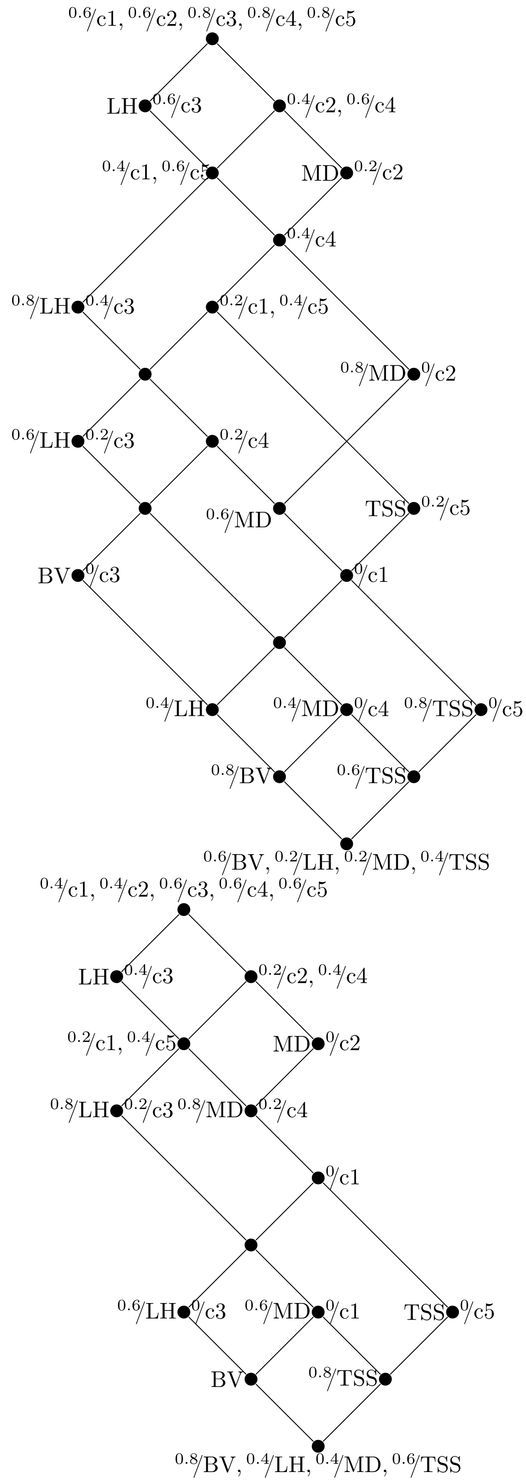

Fig. 11. **L**-concept lattice $\mathcal{B}^{\cap\cup}(X, Y, a \otimes I)$ of movies, critics and ratings (from Fig. 1), factorized with respect to the threshold $a = 0.8$ (left) and $a = 0.6$ (right).

This approach corresponds to results from [7]. The isomorphism of the crisp part of the factor concept lattice with the crisp part of $\mathcal{B}^{\uparrow\downarrow}(X, Y, a \to I)$ has been noted in [11].

Similar results can be proved for the isotone cases: For $\langle \triangle, \triangledown \rangle = \langle \cap, \cup \rangle$ we know that $J = a \otimes I$ is a block relation. By Theorem 13 (49), the associated complete **L**-tolerance is given by

$$\begin{aligned} C_1 \sim_J C_2 &= S(A_1, B_2^{\cup_J}) \wedge S(A_2, B_1^{\cup_J}) \\ &= S(A_1, a \to B_2^{\cup_I}) \wedge S(A_2, a \to B_1^{\cup_I}) \end{aligned}$$

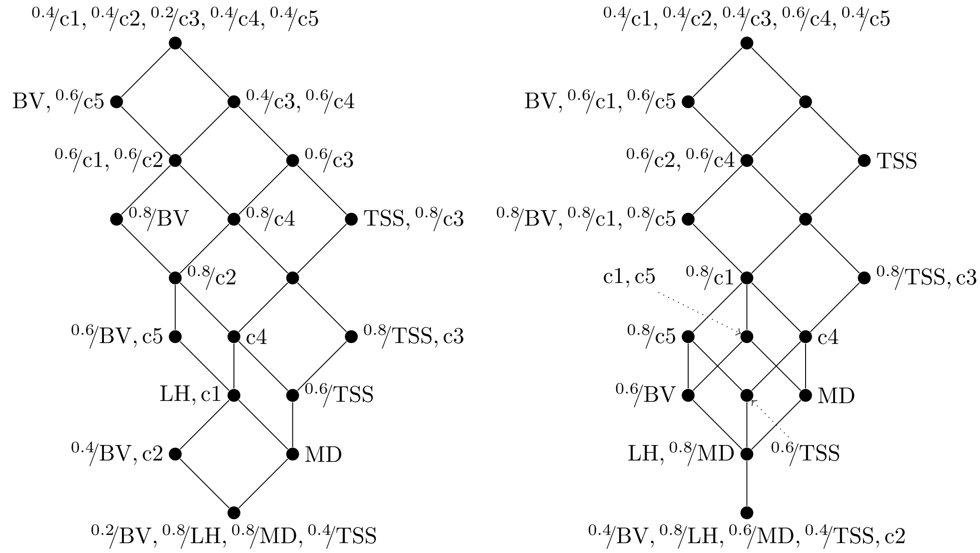

Fig. 12. **L**-concept lattice of movies, critics and ratings (form Fig. 1), factorized with respect to an **L**-set of important objects $X' = \{{}^{0.8}/\text{BV}, \text{LH}, \text{MD}, \text{TSS}\}$ and **L**-set of important attributes $Y' = \{\text{c1}, {}^{0.8}/\text{c2}, \text{c3}, \text{c4}, \text{c5}\}$ (left) and with respect to an **L**-set of important objects $X' = \{\text{BV}, \text{LH}, \text{MD}, \text{TSS}\}$ and **L**-set of important attributes $Y' = \{\text{c1}, {}^{0.6}/\text{c2}, \text{c3}, \text{c4}, \text{c5}\}$ (right).

$$= (a \to S(A_1, A_2)) \wedge (a \to S(A_2, A_1)) = a \to (A_1 \approx^X A_2)$$
$$= a \to (C_1 \approx C_2) = C_1 \sim_a C_2.$$

Thus, $\sim_J$ is equal to $\sim_a$ and by Theorem 14, the factor completely lattice **L**-ordered set $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)/_{a\to\approx}$ is isomorphic to $\mathcal{B}^{\cap\cup}(X, Y, a \otimes I)$.

The case $\langle \triangle, \triangledown \rangle = \langle \wedge, \vee \rangle$ is similar.

Illustrative example (cont.)

Consider again our formal **L**-context $\langle X, Y, I\rangle$ of movies, critics and ratings from Fig. 1. In Fig. 4 we showed the block relation $0.8 \to I$. In Fig. 8 (top), it is depicted the **L**-concept lattice $\mathcal{B}^{\uparrow\downarrow}(X, Y, 0.8 \to I)$. In addition, in the same figure (right) we can see the **L**-concept lattice $\mathcal{B}^{\uparrow\downarrow}(X, Y, a \to I)$ for the threshold $a = 0.6$. These concept lattices are isomorphic to the factor completely lattice **L**-ordered sets $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)/_{\sim_{0.8}}$ and $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)/_{\sim_{0.6}}$, respectively. In Fig. 9 and Fig. 10, we can see crisp parts of these factor completely lattice **L**-ordered sets depicted as a lattice of the crisp parts of corresponding maximal blocks.

In the isotone case, factorization of the **L**-concept lattice $\mathcal{B}^{\cap\cup}(X, Y, I)$ by $\sim_a$ produces a lattice isomorphic to $\mathcal{B}^{\cap\cup}(X, Y, a \otimes I)$. By Theorem 8, the **L**-relation $a \otimes I$ is a strong block **L**-relation of I. Fig. 11 displays associated concept lattices $\mathcal{B}^{\cap\cup}(X, Y, a \otimes I)$ for $a = 0.8$ and $a = 0.6$.

3.7. Approximations in fuzzy concept lattices

To show another, more general application of Theorem 13 and Theorem 14, we first generalize some results from [28] to our fuzzy setting.

Let i and c be an **L**-interior and **L**-closure operator on a completely lattice **L**-ordered set **U**, respectively, with the respective sets of fixpoints denoted by $\operatorname{Fix} i$ and $\operatorname{Fix} c$. For $u \in U$, $i(u)$ (resp. $c(u)$) can be regarded lower (resp. upper) approximation of u.

We call an extensive isotone **L**-Galois connection $\langle f, g\rangle$ on **U** (i, c)-*compatible* if for each $u \in U$, $f(u) \in \operatorname{Fix} i$ and $g(u) \in \operatorname{Fix} c$. A complete **L**-tolerance $\sim$ on **U** is (i, c)-*compatible* if the extensive isotone **L**-Galois connection $\langle \sim, {}^{\sim} \rangle$ is (i, c)-compatible.

Lemma 7.

(a) The set of all (i, c)-compatible complete **L**-*tolerances on* **U** *is an* **L**-*closure system in* CTol **U**.
(b) The set of all (i, c)-compatible extensive isotone **L**-*Galois connections on* **U** *is an* **L**-*closure system in* EIGal(**U**).

Proof. We will prove (b), (a) will then follow by definition and isomorphism between CTol **U** and EIGal(**U**) (Theorem 11). Let $M = \{{}^{a_k}/\langle f_k, g_k\rangle \mid k \in K\}$ be an **L**-set in EIGal(**U**) such that $\langle f_k, g_k\rangle$ is (i, c)-compatible for each $k \in K$. As EIGal(**U**) is a completely lattice **L**-ordered set, the **L**-set M has infimum. By Theorem 12, $\inf M = \langle f, g\rangle$ where $f(u) = \sup F_u$ and $g(v) = \inf G_v$ for $F_u = \{{}^{a_k}/f_k(u) \mid k \in K\}$ and $G_v = \{{}^{a_k}/g_k(v) \mid k \in K\}$.

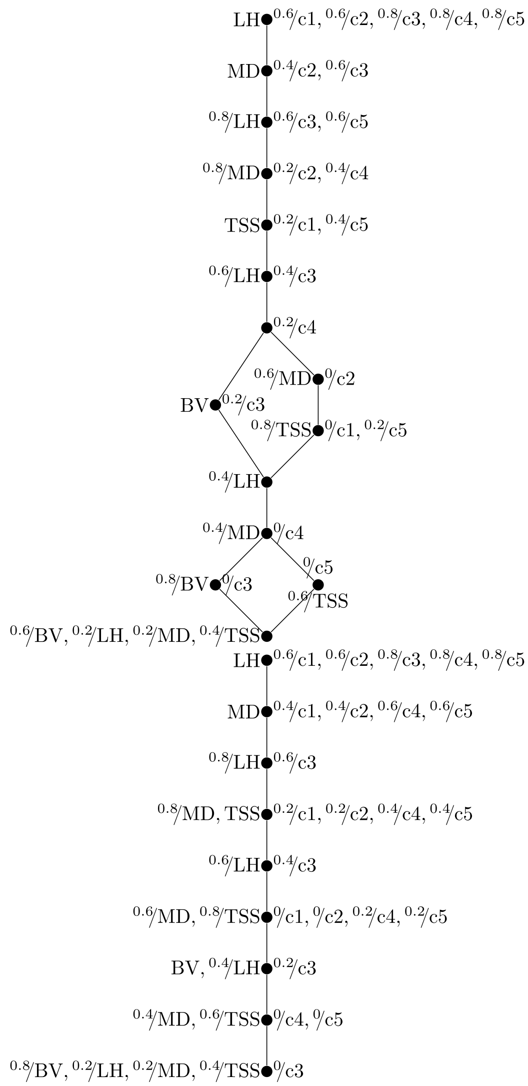

Fig. 13. **L**-concept lattice $\mathcal{B}^{\cap\cup}(X, Y, J_{i_{X'}, c_{Y'}})$ of movies, critics and ratings (form Fig. 1), factorized with respect to an **L**-set of important objects $X' = \{^{0.8}/\text{BV}, \text{LH}, \text{MD}, \text{TSS}\}$ and **L**-set of important attributes $Y' = \{\text{c1}, {}^{0.8}/\text{c2}, \text{c3}, \text{c4}, \text{c5}\}$ (left) and with respect to an **L**-set of important objects $X' = \{\text{BV}, \text{LH}, \text{MD}, \text{TSS}\}$ and **L**-set of important attributes $Y' = \{\text{c1}, {}^{0.6}/\text{c2}, \text{c3}, \text{c4}, \text{c5}\}$ (right).

Since $f_k(u) \in \text{Fix}\, i$ for each $k \in K$, $F_u \subseteq \text{Fix}\, i$. As $\text{Fix}\, i$ is an interior system, $\sup F_u = f(u) \in \text{Fix}\, i$. Similarly we obtain $\inf G_v = g(v) \in \text{Fix}\, c$ and conclude $\langle f, g\rangle$ is (i, c)-compatible. □

Denote by $\sim_{i,c}$ the smallest (w.r.t. **L**-set inclusion) (i, c)-compatible complete **L**-tolerance on **U**. According to the previous lemma, $\sim_{i,c}$ always exists. Further denote by $\langle f_{i,c}, g_{i,c}\rangle$ the smallest (w.r.t. the **L**-order (41)) (i, c)-compatible element of $\text{ElGal}(\mathbf{U})$. By the same lemma, $\langle f_{i,c}, g_{i,c}\rangle$ exists and by Theorem 11, it holds $\langle f_{i,c}, g_{i,c}\rangle = \langle \sim_{i,c}, {}^{\sim_{i,c}}\rangle$.

Now let $\mathbf{U} = \mathcal{B}^{\Delta\nabla}(X, Y, I)$ for some formal **L**-context $\langle X, Y, I\rangle$ and let J be a block relation of I w.r.t. $\langle \Delta, \nabla\rangle$. J is called *(i, c)-compatible*, if for each $A \in L^X$ and $B \in L^Y$, A^{Δ_J} is the intent of a concept from $\text{Fix}\, i$ and B^{∇_J} is the extent of a concept from $\text{Fix}\, c$.

Lemma 8.

*(a) J is (i, c)-compatible iff the extensive isotone **L**-Galois connection $\langle f_J, g_J\rangle$ from Lemma 5 is (i, c)-compatible.*

*(b) The set of all (i, c)-compatible block relations of I w.r.t. $\langle \uparrow, \downarrow\rangle$ is an **L**-closure system in $\text{BR}^{\uparrow\downarrow}(X, Y, I)$. The set of all (i, c)-compatible block relations of I w.r.t. $\langle \cap, \cup\rangle$ ($\langle \wedge, \vee\rangle$) is an **L**-interior system in $\text{BR}^{\cap\cup}(X, Y, I)$ ($\text{BR}^{\wedge\vee}(X, Y, I)$).*

Proof. (a) follows directly from definition of $\langle f_J, g_J\rangle$. (b) follows from (a). □

Denote by $J_{i,c}$ the smallest (w.r.t. **L**-set inclusion) (i, c)-compatible block relation of I. $J_{i,c}$ always exists by the above lemma. By Lemma 6, $\langle f_{i,c}, g_{i,c}\rangle = \langle f_{J_{i,c}}, g_{J_{i,c}}\rangle$.

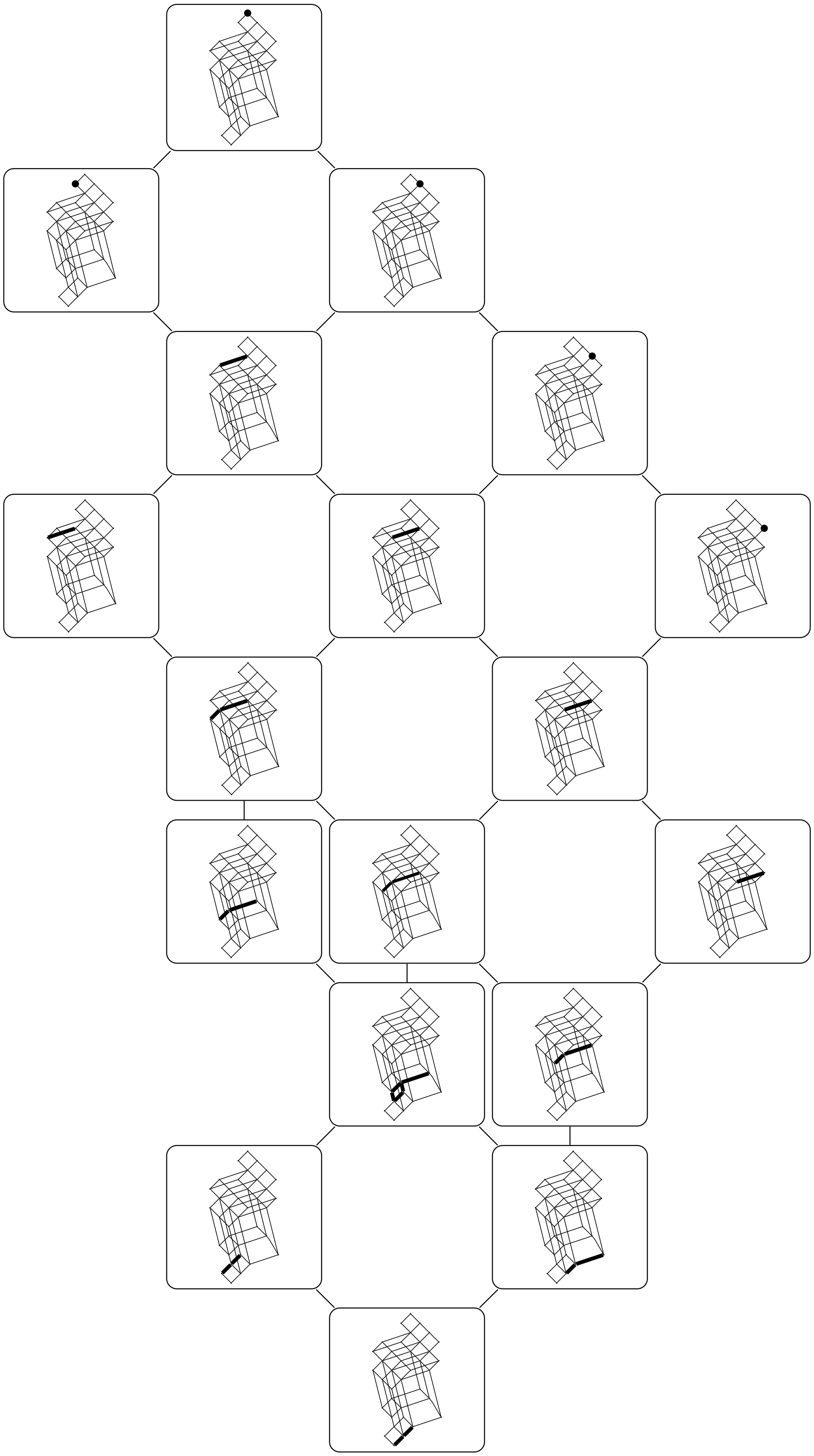

Fig. 14. **L**-concept lattice from Fig. 12 (left) depicted as a lattice of intervals (crisp parts of maximal blocks) of the original **L**-concept lattice.

Note that our results cover results from Sec. 3.6 as a special case. Namely, if we set $i_a(u) = (u^{a\rightarrow\approx})_{a\rightarrow\approx}$ and $c_a(u) = (u_{a\rightarrow\approx})^{a\rightarrow\approx}$ we obtain $a \rightarrow \approx = \sim_{i_a,c_a}$ and $a \rightarrow I = J_{i_a,c_a}$.

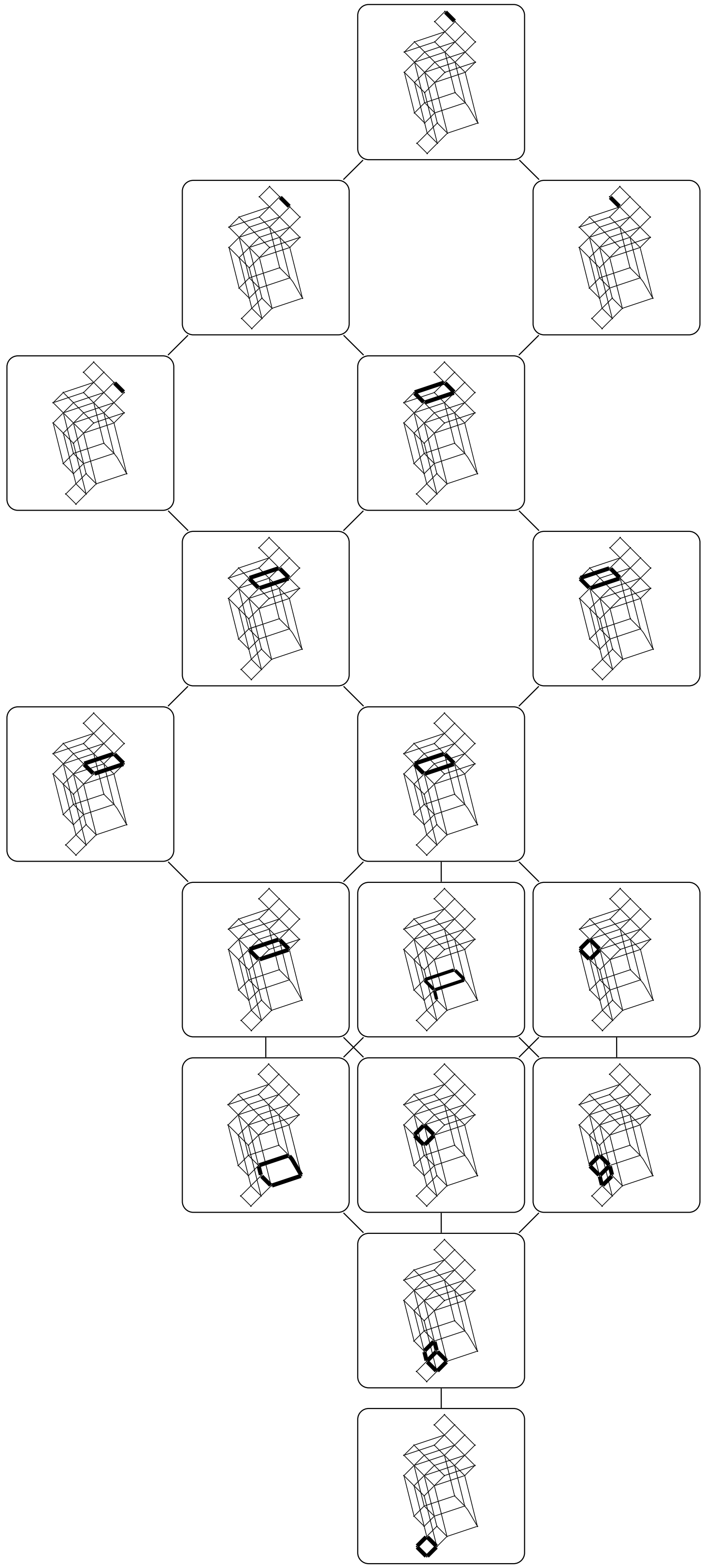

Fig. 15. **L**-concept lattice from Fig. 12 (right) depicted as a lattice of intervals (crisp parts of maximal blocks) of the original **L**-concept lattice.

Illustrative example (cont.)

In the last part of our running example, we show how the above considerations can be used for reducing the size of an **L**-concept lattice by assigning degrees of importance to objects and attributes. Again, we follow the approach from [28].

For a formal **L**-context $\langle X, Y, I\rangle$ we select two **L**-sets $X' \in L^X$ and $Y' \in L^Y$ and interpret them as **L**-sets of "important objects" and "important attributes", respectively. Thus, for an object $x \in X$, the value $X'(x)$ is the degree to which x is important and similarly for attributes. Intents of the form $A^{\uparrow_I}$ where $A \subseteq X'$ are considered important and similarly for intents.

Now, the idea is to approximate from below any concept by the greatest smaller concept with important intent and from above by the smallest greater concept with important extent. The approximations are realized by an interior operator $i_{X'}$ and a closure operator $c_{Y'}$, defined as follows.

$$i_{X'}(\langle A, B\rangle) = \langle (A \cap X')^{\uparrow_I\downarrow_I}, (A \cap X')^{\uparrow_I}\rangle,$$
$$c_{Y'}(\langle A, B\rangle) = \langle (B \cap Y')^{\downarrow_I}, (B \cap Y')^{\downarrow_I\uparrow_I}\rangle.$$

We apply the above considerations to our example. Suppose we consider the film BV less important than the other films (perhaps because we have not seen BV) and the critic c2 less important than the other critics (because we do not like his opinion on MD). More precisely, set $X' = \{{}^a\!/\mathrm{BV}, \mathrm{LH}, \mathrm{MD}, \mathrm{TSS}\}$ and $Y' = \{\mathrm{c1}, {}^b\!/\mathrm{c2}, \mathrm{c3}, \mathrm{c4}, \mathrm{c5}\}$, where $a, b \in L$. The associated block relation $J_{i_{X'},c_{Y'}}$ w.r.t. $\langle\uparrow,\downarrow\rangle$ for $a = 0.8$ is depicted in Fig. 5. In Fig. 12 we can see the resulting concept lattices $\mathcal{B}^{\uparrow\downarrow}(X, Y, J_{i_{X'},c_{Y'}})$ in two cases: first $a = b = 0.8$ and second $a = 1$ and $b = 0.6$.

In the isotone case, the approximations are realized by an interior operator $i_{X'}$ and a closure operator $c_{Y'}$, defined as follows.

$$i_{X'}(\langle A, B\rangle) = \langle (A \cap X')^{\cap_I\cup_I}, (A \cap X')^{\cap_I}\rangle,$$
$$c_{Y'}(\langle A, B\rangle) = \langle (B \cup Y')^{\cup_I}, (B \cup Y')^{\cup_I\cap_I}\rangle.$$

The associated block relation $J_{i_{X'},c_{Y'}}$ w.r.t. $\langle\cap,\cup\rangle$ for $a = b = 0.8$ is depicted in Fig. 7. In Fig. 13 we can see the resulting concept lattices for $a = b = 0.8$ and for $a = 1$ and $b = 0.6$.

In Fig. 14 and Fig. 15 we can see lattices of the crisp part of each of the maximal blocks.

4. Conclusions

We have provided a proper generalization of block relations to fuzzy setting. Our future research in this area includes study and generalization of other interesting cases of morphisms in FCA, especially infomorphisms and scale measures. Also, we find interesting the idea to investigate 'heterogeneous' block **L**-relations whose conditions combine the cases $\langle\uparrow,\downarrow\rangle$ and $\langle\cap,\cup\rangle$.

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The paper is an extended version of a part of [25].

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H On Homogeneous L-bonds and Heterogeneous L-bonds

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In this paper, we deal with suitable generalizations of the notion of bond between contexts. We study different generalizations of the notion of bond within the L-fuzzy setting. Specifically, given a formal context, there are three prototypical pairs of concept-forming operators, and this immediately leads to three possible versions of the notion of bond (so-called homogeneous bond w.r.t. a certain pair of concept-forming operators). The first results show a close correspondence between a homogeneous bond between two contexts and certain special types of mappings between the sets of extents (or intents) of the corresponding concept lattices. Later, we introduce the so-called heterogeneous bonds (considering simultaneously two types of concept-forming operators) and generalize the previous relationship to mappings between the sets of extents (or intents) of the corresponding concept lattices.

For all the defined bonds we provide their characterization, description of a structure they form and their relationship to direct products of relations. Finally, we explain the relationship of the bonds to the morphisms of **L**-closure systems and **L**-interior systems.

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On homogeneous *L*-bonds and heterogeneous *L*-bonds

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ABSTRACT

In this paper, we deal with suitable generalizations of the notion of bond between contexts, as part of the research area of Formal Concept Analysis. We study different generalizations of the notion of bond within the *L*-fuzzy setting. Specifically, given a formal context, there are three prototypical pairs of concept-forming operators, and this immediately leads to three possible versions of the notion of bond (so-called *homogeneous bond* wrt certain pair of concept-forming operators). The first results show a close correspondence between a homogeneous bond between two contexts and certain special types of mappings between the sets of extents (or intents) of the corresponding concept lattices. Later, we introduce the so-called *heterogeneous bonds* (considering simultaneously two types of concept-forming operators) and generalize the previous relationship to mappings between the sets of extents (or intents) of the corresponding concept lattices.

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Formal concept analysis; residuated lattice; fuzzy logic; bond; morphism; direct product

1. Introduction

Formal concept analysis (FCA) has become a very active research topic, both theoretical and practical; its wide applicability justifies the need of a deeper knowledge of its underlying mechanisms, and one important way to obtain this extra knowledge turns out to be via generalization.

Since the seminal paper (Burusco and Fuentes-González 1994), several fuzzy variants of generalized FCA have been introduced and developed both from the theoretical and the practical side. The consideration of the adjointness property in residuated lattices as the main building blocks of fuzzy concept lattices was an important milestone simultaneously developed by Pollandt (1997) and Belohlavek (1998).

More recently, a number of new generalizations have been introduced, either based on fuzzy set theory (Alcalde, Burusco, and Fuentes-González 2010; Alcalde et al. 2011), or the multi-adjoint framework (Medina, Ojeda-Aciego, and Ruiz-Calviño 2009; Medina and Ojeda-Aciego 2010, 2013) or heterogeneous approaches (Butka, Pócsová, and Pócs 2012; Medina and Ojeda-Aciego 2012; Díaz, Medina, and Ojeda-Aciego 2014).

FCA has been extended as well by considering alternative paradigms, for instance, one can find generalizations of the framework and scope of FCA based on from possibility

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theory (Dubois and Prade 2012) or rough set theory (Wu and Liu 2009; Lei and Luo 2009; Lai and Zhang 2009; Medina 2012; Kang et al. 2013).

Concerning applications of techniques of generalized FCA, one can see papers ranging from ontology merging (Chen, Bau, and Yeh 2011) and resolution of fuzzy or multi-adjoint relational equations (Alcalde, Burusco, and Fuentes-González 2012; Díaz and Medina 2013) to applications to the semantic web using the notion of concept similarity or rough sets (Formica 2012), and from noise control in document classification (Li and Tsai 2011) to ontology-based sentiment analysis (Kontopoulos et al. 2013), or the study of fuzzy databases, in areas such as functional dependencies (Mora et al. 2012), or even linguistics (Falk and Gardent 2014).

All the generalizations stated above focused on the development of a general framework of FCA including extra features (fuzzy, possibilistic, rough, etc.) and some of its possible applications. However, not much has been published on the suitable general version of certain specific notions, such as the bonds between formal contexts.

One of the motivations for introducing the notion of bond was to provide a tool for studying mappings between formal contexts, somehow mimicking the behaviour of Galois connections between their corresponding concept lattices. In this paper, we deal with generalizations of the notion of bond for which, to the best of our knowledge, only one general version has been introduced, see (Krídlo, Krajči, and Ojeda-Aciego 2012), wrt the standard concept-forming operators used in Belohlavek (1998).

The notions of bonds, scale measures and informorphisms were studied by Krötzsch, Hitzler, and Zhang (2005), aiming at a thorough study of the theory of morphisms in FCA; in areas related to ontology research, just infomorphisms are used, whereas more general approaches, namely more general heterogenous bonds, could be utilized. Krídlo et al. (2013) use bonds to include background knowledge into data; the heterogeneous bonds described in this paper enable us to give an alternative semantics, the background knowledge. Another application of bonds can be seen in Meschke (2010) where bonds are used to approximate concepts, allowing to focus on just a sub context without losing implicational knowledge and, hence, reducing the size of a concept lattice.

We study generalizations of the notion of bond within the L-fuzzy setting. Specifically, given a formal context, there are three prototypical pairs of concept-forming operators, and this immediately leads to three possible versions of the notion of bond (so-called *homogeneous bond* wrt certain pair of concept-forming operators). The first results show a close correspondence between a homogeneous bond between two contexts and certain special types of mappings between the sets of extents (or intents) of the corresponding concept lattices. Later, we introduce the so-called *heterogeneous bonds* (considering simultaneously two types of concept-forming operators) and generalize the previous relationship to mappings between the sets of extents (or intents) of the corresponding concept lattices.

2. Preliminaries

2.1. Residuated lattices, fuzzy sets and fuzzy relations

We use complete residuated lattices as basic structures of truth degrees. A complete residuated lattice is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$, such that

(i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, i.e. a partially ordered set in which arbitrary infima and suprema exist;
(ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation which is commutative, associative and $a \otimes 1 = a$ for each $a \in L$;
(iii) $\otimes$ and $\rightarrow$ satisfy adjointness, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$.

Recall that the partial order of $\mathbf{L}$ is denoted by $\leq$, elements 0 and 1 denote the least and greatest elements and, note that throughout this work, $\mathbf{L}$ denotes an arbitrary complete residuated lattice whose multiplicative unit is also its greatest element in the spirit of Goguen (1967).

Elements a of L are called truth degrees. Operations $\otimes$ (multiplication) and $\rightarrow$ (residuum) play the role of (truth functions of) "fuzzy conjunction" and "fuzzy implication". Furthermore, we define the complement of $a \in L$ as

$$\neg a = a \rightarrow 0 \tag{1}$$

An $\mathbf{L}$-set (or $\mathbf{L}$-fuzzy set) A in a universe set X is a mapping assigning to each $x \in X$ some truth degree $A(x) \in L$. The set of all $\mathbf{L}$-sets in a universe X is denoted $\mathbf{L}^X$.

The operations with $\mathbf{L}$-sets are defined componentwise. For instance, the intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^X$ is an $\mathbf{L}$-set $A \cap B$ in X, such that $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in X$, etc. An $\mathbf{L}$-set $A \in \mathbf{L}^X$ is also denoted $\{{}^{A(x)}/x \mid x \in X\}$. If, for all $y \in X$ distinct from $x_1, x_2, \dots, x_n$, we have $A(y) = 0$, we also write

$$\left\{{}^{A(x_1)}/x_1, {}^{A(x_2)}/x_2, \dots, {}^{A(x_n)}/x_n\right\}. \tag{2}$$

Furthermore, in (2), we write just x instead of ${}^1/x$.

An $\mathbf{L}$-set $A \in \mathbf{L}^X$ is called crisp if $A(x) \in \{0, 1\}$ for each $x \in X$. Crisp $\mathbf{L}$-sets can be identified with ordinary sets. For a crisp A, we also write $x \in A$ for $A(x) = 1$ and $x \notin A$ for $A(x) = 0$. An $\mathbf{L}$-set $A \in \mathbf{L}^X$ is called empty (denoted by $\emptyset$) if $A(x) = 0$ for each $x \in X$. For $a \in L$ and $A \in \mathbf{L}^X$, the $\mathbf{L}$-sets $a \otimes A$, $a \rightarrow A$, $A \rightarrow a$ and $\neg A$ in X are defined by

$$(a \otimes A)(x) = a \otimes A(x), \tag{3}$$
$$(a \rightarrow A)(x) = a \rightarrow A(x), \tag{4}$$
$$(A \rightarrow a)(x) = A(x) \rightarrow a, \tag{5}$$
$$\neg A(x) = A(x) \rightarrow 0. \tag{6}$$

For $A \in \mathbf{L}^X$, the $\mathbf{L}$-sets $a \otimes A, a \rightarrow A, A \rightarrow a$ are called a-multiplication, a-shift and a-complement, respectively.

Binary $\mathbf{L}$-relations (binary $\mathbf{L}$-fuzzy relations) between X and Y can be thought of as $\mathbf{L}$-sets in the universe $X \times Y$. That is, a binary $\mathbf{L}$-relation $I \in \mathbf{L}^{X \times Y}$ between a set X and a set Y is a mapping, assigning to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$ (a degree to which x and y are related by I). By I^{T} we denote the transpose of I; i.e. $I^{\mathrm{T}} \in \mathbf{L}^{Y \times X}$ with $I^{\mathrm{T}}(y, x) = I(x, y)$ for all $x \in X, y \in Y$.

Various composition operators for binary $\mathbf{L}$-relations were extensively studied by Kohout and Bandler (1985); we will use the following three composition operators, defined for relations $A \in \mathbf{L}^{X \times F}$ and $B \in \mathbf{L}^{F \times Y}$:

$$(A \circ B)(x,y) = \bigvee_{f \in F} A(x,f) \otimes B(f,y), \tag{7}$$

$$(A \triangleleft B)(x,y) = \bigwedge_{f \in F} A(x,f) \rightarrow B(f,y), \tag{8}$$

$$(A \triangleright B)(x,y) = \bigwedge_{f \in F} B(f,y) \rightarrow A(x,f). \tag{9}$$

All of them have natural verbal descriptions. For instance, $(A \circ B)(x,y)$ is the truth degree of the proposition "*there is factor f, such that f applies to object x and attribute y is a manifestation of f*"; $(A \triangleleft B)(x,y)$ is the truth degree of "*for every factor f, if f applies to object x then attribute y is a manifestation of f*". Note also that for $L = \{0,1\}$, $A \circ B$ coincides with the well-known composition of binary relations.

We will occasionally use some of the following properties concerning the associativity of several composition operators (see Belohlavek 2002).

Theorem 1: *The operators above have the following properties concerning composition.*

- *Associativity:*

$$R \circ (S \circ T) = (R \circ S) \circ T, \tag{10}$$

$$R \triangleleft (S \triangleright T) = (R \triangleleft S) \triangleright T, \tag{11}$$

$$R \triangleleft (S \triangleleft T) = (R \circ S) \triangleleft T, \tag{12}$$

$$R \triangleright (S \circ T) = (R \triangleright S) \triangleright T. \tag{13}$$

- *Distributivity:*

$$\left(\bigcup_i R_i\right) \circ S = \bigcup_i \left(R_i \circ S\right), \quad \textit{and} \quad R \circ \left(\bigcup_i S_i\right) = \bigcup_i \left(R \circ S_i\right), \tag{14}$$

$$\left(\bigcap_i R_i\right) \triangleright S = \bigcap_i \left(R_i \triangleright S\right), \quad \textit{and} \quad R \triangleright \left(\bigcup_i S_i\right) = \bigcap_i \left(R \triangleright S_i\right), \tag{15}$$

$$\left(\bigcup_i R_i\right) \triangleleft S = \bigcap_i \left(R_i \triangleleft S\right), \quad \textit{and} \quad R \triangleleft \left(\bigcap_i S_i\right) = \bigcap_i \left(R \triangleleft S_i\right). \tag{16}$$

2.2. Formal fuzzy concept analysis

An $\mathbf{L}$-context is a triplet $\langle X, Y, I \rangle$, where X and Y are (ordinary non-empty) sets and $I \in \mathbf{L}^{X \times Y}$ is an $\mathbf{L}$-relation between X and Y. Elements of X are called objects, elements of Y are called attributes and I is called an incidence relation. $I(x,y) = a$ is read: "The object x has the attribute y to degree a".

Consider the following pairs of operators induced by an $\mathbf{L}$-context $\langle X, Y, I \rangle$. First, the pair $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ of operators ${}^{\uparrow} : \mathbf{L}^X \rightarrow \mathbf{L}^Y$ and ${}^{\downarrow} : \mathbf{L}^Y \rightarrow \mathbf{L}^X$ is defined, for all $A \in \mathbf{L}^X$ and $B \in \mathbf{L}^Y$, by

$$A^{\uparrow}(y) = \bigwedge_{x \in X} A(x) \rightarrow I(x,y), \quad B^{\downarrow}(x) = \bigwedge_{y \in Y} B(y) \rightarrow I(x,y). \tag{17}$$

Second, the pair $\langle ^{\cap}, ^{\cup} \rangle$ of operators $^{\cap}: \mathbf{L}^X \to \mathbf{L}^Y$ and $^{\cup}: \mathbf{L}^Y \to \mathbf{L}^X$ is defined by

$$A^{\cap}(y) = \bigvee_{x \in X} A(x) \otimes I(x,y), \quad B^{\cup}(x) = \bigwedge_{y \in Y} I(x,y) \to B(y), \tag{18}$$

Third, the pair $\langle ^{\wedge}, ^{\vee} \rangle$ of operators $^{\wedge}: \mathbf{L}^X \to \mathbf{L}^Y$ and $^{\vee}: \mathbf{L}^Y \to \mathbf{L}^X$ is defined by

$$A^{\wedge}(y) = \bigwedge_{x \in X} I(x,y) \to A(x), \quad B^{\vee}(x) = \bigvee_{y \in Y} B(y) \otimes I(x,y), \tag{19}$$

The three previous pairs are those more commonly used in the literature related to residuated lattice-based generalizations of FCA. In this respect, it is worth to note that there exists a fourth pair of concept-forming operators not considered in the present work which can be viewed as a double dualization on the first pair.

Remark 1: Notice that the three different pairs of concept-forming operators can be interpreted as instances of the composition operators between relations. Applying the isomorphisms $\mathbf{L}^{1 \times X} \cong \mathbf{L}^X$ and $\mathbf{L}^{Y \times 1} \cong \mathbf{L}^Y$ whenever necessary, one could write them, alternatively, as follows:

$$A^{\uparrow} = A \triangleleft I \quad A^{\cap} = A \circ I \quad A^{\wedge} = A \triangleright I$$

$$B^{\downarrow} = I \triangleright B \quad B^{\cup} = I \triangleleft B \quad B^{\vee} = I \circ B$$

Furthermore, denote the corresponding sets of fixed points by $\mathcal{B}^{\uparrow\downarrow}(X,Y,I)$, $\mathcal{B}^{\cap\cup}(X,Y,I)$ and $\mathcal{B}^{\wedge\vee}(X,Y,I)$, i.e.

$$\mathcal{B}^{\uparrow\downarrow}(X,Y,I) = \{\langle A,B \rangle \in \mathbf{L}^X \times \mathbf{L}^Y \mid A^{\uparrow} = B,\ B^{\downarrow} = A\},$$
$$\mathcal{B}^{\cap\cup}(X,Y,I) = \{\langle A,B \rangle \in \mathbf{L}^X \times \mathbf{L}^Y \mid A^{\cap} = B,\ B^{\cup} = A\},$$
$$\mathcal{B}^{\wedge\vee}(X,Y,I) = \{\langle A,B \rangle \in \mathbf{L}^X \times \mathbf{L}^Y \mid A^{\wedge} = B,\ B^{\vee} = A\}.$$

The sets of fixpoints are complete lattices (Pollandt 1997; Belohlavek 1999; Georgescu and Popescu 2004), called the standard (resp. object-oriented and property-oriented) **L**-concept lattices associated with I, and their elements are called formal concepts.

For a concept lattice $\mathcal{B}^{\Delta\nabla}(X,Y,I)$, where $\mathcal{B}^{\Delta\nabla}$ is either of $\mathcal{B}^{\uparrow\downarrow}$, $\mathcal{B}^{\cap\cup}$ or $\mathcal{B}^{\wedge\vee}$, denote the corresponding sets of extents and intents by $\mathrm{Ext}^{\Delta\nabla}(X,Y,I)$ and $\mathrm{Int}^{\Delta\nabla}(X,Y,I)$. That is,

$$\mathrm{Ext}^{\Delta\nabla}(X,Y,I) = \{A \in \mathbf{L}^X \mid \langle A,B \rangle \in \mathcal{B}^{\Delta\nabla}(X,Y,I) \text{ for some } B\},$$
$$\mathrm{Int}^{\Delta\nabla}(X,Y,I) = \{B \in \mathbf{L}^Y \mid \langle A,B \rangle \in \mathcal{B}^{\Delta\nabla}(X,Y,I) \text{ for some } A\},$$

The operators induced by an **L**-context and their sets of fixpoints have extensively been studied (see e.g. Pollandt 1997; Belohlavek 1999, 2001, 2004; Georgescu and Popescu 2004).

We will need the following result by Belohlavek and Konecny (2012b).

Theorem 2: *Consider* **L**-*contexts* $\langle X,Y,I \rangle$, $\langle X,F,A \rangle$ *and* $\langle F,Y,B \rangle$.

(a) $\mathrm{Int}^{\cap\cup}(X,Y,I) \subseteq \mathrm{Int}^{\cap\cup}(F,Y,B)$ *if and only if there exists* $A' \in \mathbf{L}^{X \times F}$, *such that* $I = A' \circ B$,

(b) $\mathrm{Ext}^{\wedge\vee}(X,Y,I) \subseteq \mathrm{Ext}^{\wedge\vee}(X,F,A)$ *if and only if there exists* $B' \in \mathbf{L}^{F\times Y}$, *such that* $I = A \circ B'$,
(c) $\mathrm{Int}^{\uparrow\downarrow}(X,Y,I) \subseteq \mathrm{Int}^{\uparrow\downarrow}(F,Y,B)$ *if and only if there exists* $A' \in \mathbf{L}^{X\times F}$, *such that* $I = A' \triangleleft B$,
(d) $\mathrm{Ext}^{\uparrow\downarrow}(X,Y,I) \subseteq \mathrm{Ext}^{\uparrow\downarrow}(X,F,A)$ *if and only if there exists* $B' \in \mathbf{L}^{F\times Y}$, *such that* $I = A \triangleright B'$.
(e) $\mathrm{Ext}^{\uparrow\downarrow}(X,Y,I) \subseteq \mathrm{Ext}^{\cap\cup}(X,F,A)$ *if and only if there exists* $B' \in \mathbf{L}^{F\times Y}$, *such that* $I = A \triangleleft B'$,
(f) $\mathrm{Int}^{\uparrow\downarrow}(X,Y,I) \subseteq \mathrm{Int}^{\wedge\vee}(F,Y,B)$ *if and only if there exists* $A' \in \mathbf{L}^{X\times F}$, *such that* $I = A' \triangleright B$.

In addition, we also have

(g) $\mathrm{Ext}^{\cap\cup}(X,Y,A \circ B) \subseteq \mathrm{Ext}^{\cap\cup}(X,F,A)$.
(h) $\mathrm{Int}^{\wedge\vee}(X,Y,A \circ B) \subseteq \mathrm{Int}^{\wedge\vee}(F,Y,B)$.

We will also utilize the following lemma by Belohlavek and Konecny (2011).

Lemma 1: *Let* $I,J \in L^{X\times Y}$. *We have* $B^{\cup_I} = B^{\cup_J}$ *for each* $B \in L^Y$ *iff* $I = J$.

2.3. Morphisms of closure and interior systems

A system of $\mathbf{L}$-sets $V \subseteq \mathbf{L}^X$ is called an $\mathbf{L}$-*interior system* if

- V is closed under $\otimes$-multiplication, i.e. for every $a \in L$ and $C \in V$, we have $a \otimes C \in V$;
- V is closed under union, i.e. $\bigcup_{j\in J} C_j \in V$ whenever $C_j \in V$ for all $j \in J$.

$V \subseteq \mathbf{L}^X$ is called an $\mathbf{L}$-*closure system* if

- V is closed under left $\rightarrow$-multiplication (or $\rightarrow$-shift), i.e. for every $a \in L$ and $C \in V$, we have $a \rightarrow C \in V$;
- V is closed under intersection, i.e. $\bigcap_{j\in J} C_j \in V$ whenever $C_j \in V$ for all $j \in J$.

One can find examples of $\mathbf{L}$-closure and $\mathbf{L}$-interior systems in the framework of formal fuzzy concept analysis as follows: for an $\mathbf{L}$-context $\langle X,Y,I\rangle$, the sets $\mathrm{Ext}^{\uparrow\downarrow}(X,Y,I)$, $\mathrm{Ext}^{\cap\cup}(X,Y,I)$, $\mathrm{Int}^{\wedge\vee}(X,Y,I)$, and $\mathrm{Int}^{\uparrow\downarrow}(X,Y,I)$ are $\mathbf{L}$-closure systems, while $\mathrm{Ext}^{\wedge\vee}(X,Y,I)$ and $\mathrm{Int}^{\cap\cup}(X,Y,I)$ are $\mathbf{L}$-interior systems (see Belohlavek and Konecny 2011, 2012b; Konecny 2012).

Definition 1:

(a) A mapping $h : V \rightarrow W$ from an $\mathbf{L}$-interior system $V \subseteq \mathbf{L}^X$ into an $\mathbf{L}$-interior system $W \subseteq \mathbf{L}^Y$ is called an *i-morphism* if it is a $\otimes$- and $\bigvee$-morphism, i.e.

- $h(a \otimes C) = a \otimes h(C)$ for each $a \in L$ and $C \in V$;
- $h(\bigvee_{k\in K} C_k) = \bigvee_{k\in K} h(C_k)$ for every collection of $C_k \in V$ $(k \in K)$.

An i-morphism $h : V \rightarrow W$ is said to be an *extendable i-morphism* if h can be extended to an i-morphism of $\mathbf{L}^X$ to $\mathbf{L}^Y$, i.e. if there exists an i-morphism $h' : \mathbf{L}^X \rightarrow \mathbf{L}^Y$, such that for every $C \in V$, we have $h'(C) = h(C)$.

(b) A mapping $h : V \to W$ from an $\mathbf{L}$-closure system $V \subseteq \mathbf{L}^X$ into an $\mathbf{L}$-closure system $W \subseteq \mathbf{L}^Y$ is called a *c-morphism* if it is a $\to$- and $\bigwedge$-morphism and it preserves a-complements, i.e. if

- $h(a \to C) = a \to h(C)$ for each $a \in L$ and $C \in V$;
- $h(\bigwedge_{k \in K} C_k) = \bigwedge_{k \in K} h(C_k)$ for every collection of $C_k \in V$ $(k \in K)$;
- if C is an a-complement, then $h(C)$ is an a-complement.

A c-morphism $h : V \to W$ is called an *extendable c-morphism* if h can be extended to a c-morphism of $\mathbf{L}^X$ to $\mathbf{L}^Y$, i.e. if there exists a c-morphism $h' : \mathbf{L}^X \to \mathbf{L}^Y$, such that for every $C \in V$, we have $h'(C) = h(C)$.

(c) A mapping $h : V \to W$ from an $\mathbf{L}$-interior system $V \subseteq \mathbf{L}^X$ into an $\mathbf{L}$-closure system $W \subseteq \mathbf{L}^Y$ is called an *a-morphism* if

- $h(a \otimes C) = a \to h(C)$ for each $a \in L$ and $C \in V$;
- $h(\bigvee_{k \in K} C_k) = \bigwedge_{k \in K} h(C_k)$ for every collection of $C_k \in V$.

An a-morphism $h : V \to W$ is called an *extendable a-morphism* if h can be extended to an a-morphism of $\mathbf{L}^X$ to $\mathbf{L}^Y$, i.e. if there exists an a-morphism $h' : \mathbf{L}^X \to \mathbf{L}^Y$, such that for every $C \in V$, we have $h'(C) = h(C)$.

In this paper, we will consider only extendable morphisms, for which the following results will be used hereafter (see Belohlavek and Konecny 2011, 2012b; Konecny 2012).

Lemma 2: *For $V \subseteq \mathbf{L}^X$,*

(a) if $h : V \to \mathbf{L}^Y$ is an i-morphism, then there exists an $\mathbf{L}$-relation $R \in \mathbf{L}^{X \times Y}$, such that $h(C) = C \circ R$ for every $C \in V$.
(b) if $h : V \to \mathbf{L}^Y$ is a c-morphism, then there exists an $\mathbf{L}$-relation $R \in \mathbf{L}^{X \times Y}$, such that $h(C) = C \triangleright R$ for every $C \in V$.
(c) if $h : V \to \mathbf{L}^Y$ is an a-morphism, then there exists an $\mathbf{L}$-relation $R \in \mathbf{L}^{X \times Y}$, such that $h(C) = C \triangleleft R$ for every $C \in V$.

Lemma 3: *Let $R \in \mathbf{L}^{Y \times X}$,*

(a) the mapping $h_R : \mathbf{L}^X \to \mathbf{L}^Y$ defined by $h_R(C) = R \circ C$ and the mapping $g_R : \mathbf{L}^Y \to \mathbf{L}^X$ defined by $g_R(C) = C \circ R$ are i-morphisms.
(b) the mapping $h_R : \mathbf{L}^X \to \mathbf{L}^Y$ defined by $h_R(C) = R \triangleleft C$ and the mapping $g_R : \mathbf{L}^Y \to \mathbf{L}^X$ defined by $g_R(C) = C \triangleright R$ are c-morphisms.
(c) the mapping $g_R : \mathbf{L}^X \to \mathbf{L}^Y$ defined by $h_R(C) = R \triangleright C$ and the mapping $g_R : \mathbf{L}^Y \to \mathbf{L}^X$ defined by $g_R(C) = C \triangleleft R$ are a-morphisms.

The previous lemmas together with Remark 1 allow for establishing a link between {i, c, a}-morphisms with formal fuzzy concept analysis in that, for instance, $h_R(C)$ in (a) coincides with $C^{\vee}$ just using R as incidence relation (hence, we will denote the corresponding concept-forming operator as $\vee_R$). Similarly, we will use $\downarrow_R$, $\cup_R$ and so on.

3. Homogeneous $\mathbb{L}$-bonds

This section introduces some new notions studied in this work. To begin with, we introduce the notion of *homogeneous* $\mathbf{L}$-bond as a convenient generalization of bond. Firstly, it will be convenient to recall the classical notion of bond.

A bond between two contexts $\mathbb{K}_1 = \langle X_1, Y_1, I_1 \rangle$ and $\mathbb{K}_2 = \langle X_2, Y_2, I_2 \rangle$ is a relation $\beta \subseteq X_1 \times Y_2$, such that

(B1) For all $x \in X_1$, the set $\beta(x)$ is an intent of $\langle X_2, Y_2, I_2 \rangle$;
(B2) For all $y \in Y_2$, the set $\beta^{-1}(y)$ is an extent of $\langle X_1, Y_1, I_1 \rangle$;

where $\beta(x) \subseteq Y_2, \beta^{-1}(y) \subseteq X_1$ s.t. $(\beta(x))(y) = \beta(x, y) = \beta^{-1}(y)(x)$.

Note that, in the classical case, these conditions are equivalent to

(B1′) Each extent of $\langle X_1, Y_2, \beta \rangle$ is an extent of $\langle X_1, Y_1, I_1 \rangle$.
(B2′) Each intent of $\langle X_1, Y_2, \beta \rangle$ is an intent of $\langle X_2, Y_2, I_2 \rangle$.

These conditions lead us to the following generalization to the **L**-fuzzy case.

Definition 2: A homogeneous bond wrt $\langle \triangle, \triangledown \rangle$ between two **L**-contexts $\mathbb{K}_1 = \langle X_1, Y_1, I_1 \rangle$ and $\mathbb{K}_2 = \langle X_2, Y_2, I_2 \rangle$ is an **L**-relation $\beta \in L^{X_1 \times Y_2}$ s.t.

$$\mathrm{Ext}^{\triangle\triangledown}(X_1, Y_2, \beta) \subseteq \mathrm{Ext}^{\triangle\triangledown}(X_1, Y_1, I_1) \text{ and } \mathrm{Int}^{\triangle\triangledown}(X_1, Y_2, \beta) \subseteq \mathrm{Int}^{\triangle\triangledown}(X_2, Y_2, I_2).$$

Now, we can explain the use of the term *homogeneous* in that the same pair of concept-forming operators is used in both inclusions in the definition above. Later, in Section 4, we will consider *heterogeneous* bonds in which the concept-forming operators appear mixed in the inclusions above.

In this section, we study homogeneous bonds with respect to $\langle \cap, \cup \rangle$ and homogeneous bonds with respect to $\langle \wedge, \vee \rangle$.

Remark 2:

(a) Note that homogeneous bonds with respect to $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ were studied in Krídlo, Krajči, and Ojeda-Aciego (2012). In Section 4.2, we will provide a comparison of our results with those in the previous reference. See also Remark 5.
(b) One can observe that homogeneous bonds wrt $\langle \wedge, \vee \rangle$ from $\langle X_1, Y_1, I_1 \rangle$ to $\langle X_2, Y_2, I_2 \rangle$ are transposes of homogeneous bonds wrt $\langle \cap, \cup \rangle$ from $\langle Y_2, X_2, I_2^{\mathrm{T}} \rangle$ to $\langle Y_1, X_1, I_1^{\mathrm{T}} \rangle$.

Homogeneous bonds can be put in relation to that of c-morphism.

Theorem 3:

(a) The homogeneous bonds from $\langle X_1, Y_1, I_1 \rangle$ to $\langle X_2, Y_2, I_2 \rangle$ wrt $\langle \cap, \cup \rangle$ are in one-to-one correspondence with the c-morphisms from $\mathrm{Ext}^{\cap\cup}(X_2, Y_2, I_2)$ *to* $\mathrm{Ext}^{\cap\cup}(X_1, Y_1, I_1)$.
(b) The homogeneous bonds from $\langle X_1, Y_1, I_1 \rangle$ to $\langle X_2, Y_2, I_2 \rangle$ wrt $\langle \wedge, \vee \rangle$ are in one-to-one correspondence with the c-morphisms from $\mathrm{Int}^{\wedge\vee}(X_1, Y_1, I_1)$ *to* $\mathrm{Int}^{\wedge\vee}(X_2, Y_2, I_2)$.

Proof:

(a) We show procedures to obtain the c-morphism from a homogeneous bond and vice versa.
"$\Rightarrow$": Let β be a homogeneous bond from $\langle X_1, Y_1, I_1 \rangle$ to $\langle X_2, Y_2, I_2 \rangle$ wrt $\langle \cap, \cup \rangle$. By Definition 2, we have $\mathrm{Int}^{\cap\cup}(X_1, Y_2, \beta) \subseteq \mathrm{Int}^{\cap\cup}(X_2, Y_2, I_2)$; thus by Theorem 2, there exists $R \in L^{X_1 \times X_2}$, such that $\beta = R \circ I_2$. Now, by Lemma 3, the induced operator of this type $\cup_R \colon L^{X_2} \to L^{X_1}$, such that $C^{\cup_R} = R \triangleleft C$, is a c-morphism.

It only remains to check that when C is an extent of $\mathbb{K}_2$, its image $R \triangleleft C$ is an extent of $\mathbb{K}_1$. Assume $C \in \text{Ext}^{\cap\cup}(X_2, Y_2, I_2)$, then we have that $C = D^{\cup_{I_2}} = I_2 \triangleleft D$ for some $D \in L^{Y_2}$; now using this expression in $R \triangleleft C$, we have

$$R \triangleleft C = R \triangleleft (I_2 \triangleleft D) = (R \circ I_2) \triangleleft D = \beta \triangleleft D = D^{\cup_\beta}$$

and, as a result, we obtain that $R \triangleleft C$ is in $\text{Ext}^{\cap\cup}(X_1, Y_2, \beta)$ and, therefore, as β is a homogeneous bond, it is also an extent of $\mathbb{K}_1$.
Now, let us show that the previous construction, given a homogeneous bond β from $\langle X_1, Y_1, I_1\rangle$ to $\langle X_2, Y_2, I_2\rangle$ wrt $\langle\cap, \cup\rangle$, produces a unique c-morphism $f_\beta : \text{Ext}^{\cap\cup}(X_2, Y_2, I_2) \to \text{Ext}^{\cap\cup}(X_1, Y_1, I_1)$.

It is enough to check that the construction does not depend on the relation used to factorize β, i.e. for any $R, S \in L^{X_1\times X_2}$ satisfying $\beta = R \circ I_2 = S \circ I_2$, we have that the equality

$$C^{\cup_R} = C^{\cup_S} \tag{20}$$

holds for all $C \in \text{Ext}^{\cap\cup}(X_2, Y_2, I_2)$. Now, by definition, $\text{Ext}^{\cap\cup}(X_2, Y_2, I_2) = \{D^{\cup_{I_2}} \mid D \in L^{Y_2}\}$ the equality (20) is equivalent to

$$D^{\cup_{I_2}\cup_R} = D^{\cup_{I_2}\cup_S} \quad \text{for all } D \in L^{Y_2}. \tag{21}$$

but we have that

$$D^{\cup_{I_2}\cup_R} = R \triangleleft (I_2 \triangleleft D) = (R \circ I_2) \triangleleft D = \beta \triangleleft D$$
$$D^{\cup_{I_2}\cup_S} = S \triangleleft (I_2 \triangleleft D) = (S \circ I_2) \triangleleft D = \beta \triangleleft D$$

Thus, equality (21) holds true, and both relations R and S induce the same c-morphism $f_\beta : \text{Ext}^{\cap\cup}(X_2, Y_2, I_2) \to \text{Ext}^{\cap\cup}(X_1, Y_1, I_1)$.

"$\Leftarrow$": For a c-morphism $f : \text{Ext}^{\cap\cup}(X_2, Y_2, I_2) \to \text{Ext}^{\cap\cup}(X_1, Y_1, I_1)$, by Lemma 2, there is an **L**-relation $S \in L^{X_2\times X_1}$ s.t. $f(C) = C^{\cup_S} = S^{\mathrm{T}} \triangleleft C = C \triangleright S$ for each $C \in \text{Ext}^{\cap\cup}(X_2, Y_2, I_2)$.

By considering $\beta = S^T \circ I_2$, and using Theorem 2(a) one obtains that $\text{Int}^{\cap\cup}(X_1, Y_2, \beta) \subseteq \text{Int}^{\cap\cup}(X_2, Y_2, I_2)$. For the inclusion between the extents, it is sufficient to show that $\text{Ext}^{\cap\cup}(X_1, Y_2, \beta) \subseteq \text{Im}(f)$: assume $C \in \text{Ext}^{\cap\cup}(X_1, Y_2, \beta)$, then there exists a D, such that $C = \beta \triangleleft D$. Unfolding the definition of β and applying some relational equalities, we obtain the following:

$$C = \beta \triangleleft D = (S^T \circ I_2) \triangleleft D = S^T \triangleleft (I_2 \triangleleft D) = (I_2 \triangleleft D)^T \triangleright S = f((I_2 \triangleleft D)^T)$$

As, by assumption, $\text{Im}(f) \subseteq \text{Ext}^{\cap\cup}(X_1, Y_1, I_1)$, we have that β is a homogeneous bond from $\langle X_1, Y_1, I_1\rangle$ to $\langle X_2, Y_2, I_2\rangle$ wrt $\langle\cap, \cup\rangle$.

Let us prove now that this construction produces a unique homogeneous bond β_f from $\langle X_1, Y_1, I_1\rangle$ to $\langle X_2, Y_2, I_2\rangle$ for a given c-morphism $f : \text{Ext}^{\cap\cup}(X_2, Y_2, I_2) \to \text{Ext}^{\cap\cup}(X_1, Y_1, I_1)$ wrt $\langle\cap, \cup\rangle$. It is enough to show

$$R^{\mathrm{T}} \circ I_2 = S^{\mathrm{T}} \circ I_2. \tag{22}$$

for all relations $R, S \in L^{X_2 \times X_1}$ satisfying

$$f(C) = C \triangleright R = C \triangleright S \quad \text{for all } C \in \text{Ext}^{\cap\cup}(X_2, Y_2, I_2) \tag{23}$$

Since $\text{Ext}^{\cap\cup}(X_2, Y_2, I_2) = \{D^{\cup I_2} \mid D \in L^{Y_2}\}$, the condition (23) is equivalent to

$$f(D^{\cup I_2}) = D^{\cup I_2} \triangleright R = D^{\cup I_2} \triangleright S \quad \text{for all } D \in L^{Y_2}. \tag{24}$$

We have $D^{\cup I_2} \triangleright R = (I_2 \triangleleft D) \triangleright R = R^{\mathrm{T}} \triangleleft (I_2 \triangleleft D) = (R^{\mathrm{T}} \circ I_2) \triangleleft D$ and similarly $D^{\cup I_2} \triangleright S = (S^{\mathrm{T}} \circ I_2) \triangleleft D$, hence the condition (23) is equivalent to

$$\begin{aligned} f(D^{\cup I_2}) &= (R^{\mathrm{T}} \circ I_2) \triangleleft D = (S^{\mathrm{T}} \circ I_2) \triangleleft D \\ &= D^{\downarrow_{R^{\mathrm{T}} \circ I_2}} = D^{\downarrow_{S^{\mathrm{T}} \circ I_2}} \quad \text{for all } D \in L^{Y_2}. \end{aligned}$$

By Lemma 1, we have $R^{\mathrm{T}} \circ I_2 = S^{\mathrm{T}} \circ I_2$, and (22) is satisfied and β_f is well defined.

Finally, the one-to-one correspondence stated by the theorem will be completely proved if $\beta_{f_\beta} = \beta$ and $f_{\beta_f} = f$. For this, it is worth to recall both directions of the correspondence in purely relational terms:

- Given β, if $\beta = R \circ I_2$, then $f_\beta(C) = R \triangleleft C = C \triangleright R^{\mathrm{T}}$
- Given f, if $f(C) = C \triangleright S$, then $\beta_f = S^{\mathrm{T}} \circ I_2$

Assume that $\beta = R \circ I_2$, then $\beta_{f_\beta} = S^{\mathrm{T}} \circ I_2$ for some relation S which is a right factor of f_β wrt $\triangleright$; by definition of f_β, it is possible to consider $S^{\mathrm{T}} = R$. As a result, we obtain $\beta_{f_\beta} = \beta$.

Now, assume f can be written as $f(C) = C \triangleright S$, then $\beta_f = S^{\mathrm{T}} \circ I_2$ which, in its turn, implies that $f_{\beta_f} = S^{\mathrm{T}} \triangleleft C = C \triangleright S = f$.

(b) Follows from (a) and Remark 2(b). □

The previous remark and theorem show that the homogeneous bonds wrt $\langle \cap, \cup \rangle$ are different from homogeneous bonds wrt $\langle \wedge, \vee \rangle$

Theorem 4: *The system of all homogeneous bonds wrt $\langle \cap, \cup \rangle$ (resp. wrt $\langle \wedge, \vee \rangle$) from $\mathbb{K}_1$ to $\mathbb{K}_2$ is an* **L***-interior system.*

Proof: We prove the result only for $\langle \cap, \cup \rangle$; the other part then follows from Remark 2(b).

Consider a family $\{\beta_j \in L^{X_1 \times X_2} \mid j \in J\}$ of homogeneous bonds from $\mathbb{K}_1$ to $\mathbb{K}_2$, and let us show that $\beta = \bigcup_j \beta_j$ is a homogeneous bond, i.e. that $A^{\cap\beta} \in \text{Int}^{\cap\cup}(X_2, Y_2, I_2)$ and $B^{\cup\beta} \in \text{Ext}^{\cap\cup}(X_1, Y_1, I_1)$.

$$A^{\cap\beta} = A \circ \beta = A \circ \left(\bigcup_j \beta_j \right) = \bigcup_j (A \circ \beta_j) = \bigcup_j A^{\cap\beta_j}$$

Thus, we have that $A^{\cap\beta} = \bigcup_{j \in J} A^{\cap\beta_j}$, proving that $A^{\cap\beta} \in \text{Int}^{\cap\cup}(X_2, Y_2, I_2)$ since $\text{Int}^{\cap\cup}(X_2, Y_2, I_2)$ is an **L**-interior system.

Similarly, we have

$$B^{\cup\beta} = \beta \triangleleft B = \left(\bigcup_j \beta_j \right) \triangleleft B = \bigcap_j (\beta_j \triangleleft B) = \bigcap_j B^{\cup\beta_j}$$

Thus, we have that $B^{\cup_\beta} = \bigcap_{j\in J} B^{\cup_{\beta_j}}$, proving that $B^{\cup_\beta} \in \mathrm{Ext}^{\cap\cup}(X_2, Y_2, I_2)$ since $\mathrm{Ext}^{\cap\cup}(X_2, Y_2, I_2)$ is an **L**-closure system.

Second, we show that if β is a homogeneous bond from $\langle X_1, Y_1, I_1\rangle$ to $\langle X_2, Y_2, I_2\rangle$, then $a \otimes \beta$ is a homogeneous bond as well. For every $A \in L^{X_1}$, we have

$$A^{\cap_{a\otimes\beta}} = A \circ (\mathrm{Id}_a \circ \beta) = (A \circ \mathrm{Id}_a) \circ \beta = (a \otimes A)^{\cap_\beta},$$

where Id_a the identity relation on X_1 multiplied by $a \in L$; i.e. $\mathrm{Id}_a = a \otimes \mathrm{Id}$. Thus, $A^{\cap_{a\otimes\beta}} = (a \otimes A)^{\cap_\beta} \in \mathrm{Int}^{\cap\cup}(X_2, Y_2, I_2)$.

For every $B \in L^{Y_2}$, we have

$$B^{\cup_{a\otimes\beta}} = (\mathrm{Id}_a \circ \beta) \triangleleft B = \mathrm{Id}_a \triangleleft (\beta \triangleleft B) = a \to B^{\cup_\beta}.$$

Thus, $B^{\cup_{a\otimes\beta}} = a \to B^{\cup_\beta}$, proving that $B^{\cup_{a\otimes\beta}} \in \mathrm{Ext}^{\cap\cup}(X_2, Y_2, I_2)$ since $\mathrm{Ext}^{\cap\cup}(X_2, Y_2, I_2)$ is an **L**-closure system.

The system of all homogeneous bonds is closed under union and multiplication, whence it is an **L**-interior system. □

3.1. Strong homogeneous bonds

In this section, we will consider homogeneous bonds wrt both pairs of isotone concept-forming operators simultaneously; the antitone pair $\langle{}^\uparrow, {}^\downarrow\rangle$ will be considered in Section 4.2. Formally, we introduce the notion of *strong homogeneous bond* as follows:

Definition 3: A strong homogeneous bond from **L**-context $\mathbb{K}_1 = \langle X_1, Y_1, I_1\rangle$ to **L**-context $\mathbb{K}_2 = \langle X_2, Y_2, I_2\rangle$ is an **L**-relation $\beta \in L^{X_1\times Y_2}$ s.t. β is a homogeneous bond wrt both $\langle\cap,\cup\rangle$ and $\langle\wedge,\vee\rangle$.

The following shows that there exist homogeneous bonds which are not strong homogeneous bonds, as the following example shows.

Example 1: Consider L a finite chain $0 < a < b < 1$ with $\otimes$ defined as follows:

$$x \otimes y = \begin{cases} x \wedge y & \text{if } x = 1 \text{ or } y = 1, \\ 0 & \text{otherwise,} \end{cases}$$

for each $x, y \in L$. One can easily see that $x \otimes \bigvee_j y_j = \bigvee_j (x \otimes y_j)$, and thus an adjoint operation $\to$ exists, such that $\langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ is a complete residuated lattice. Namely, $\to$ is given as follows for all $x, y \in L$:

$$x \to y = \begin{cases} 1 & \text{if } x \le y, \\ y & \text{if } x = 1, \\ b & \text{otherwise,} \end{cases}$$

Consider the sets $X_1 = X_2 = \{x\}$, $Y_1 = Y_2 = \{y\}$ and the relations $I_1 = \{{}^a/\langle x, y\rangle\}$ and $I_2 = \{{}^b/\langle x, y\rangle\}$. One can check that we have $\mathrm{Ext}^{\cap\cup}(\{x\}, \{y\}, I_1) = \mathrm{Ext}^{\cap\cup}(\{x\}, \{y\}, I_2) = \{\{{}^b/x\}, \{x\}\}$ and, trivially, $\mathrm{Int}^{\cap\cup}(\{x\}, \{y\}, I_2) = \mathrm{Int}^{\cap\cup}(\{x\}, \{y\}, I_2)$. Thus, I_2 is a homogeneous bond between I_1 and I_2 wrt $\langle\cap,\cup\rangle$. On the other hand, I_2 is not a homogeneous bond between I_1 and I_2 wrt $\langle \wedge, \vee \rangle$ since $\mathrm{Ext}^{\wedge\vee}(\{x\},\{y\},I_1) = \{\emptyset, \{^a/x\}\} \not\supseteq \{\emptyset, \{^b/x\}\} = \mathrm{Ext}^{\wedge\vee}(\{x\},\{y\},I_2)$.

The following lemma introduces alternative characterizations of the notion of strong homogeneous bond.

Lemma 4: *The following statements are equivalent:*

(1) *β is a strong homogeneous bond from $\mathbb{K}_1 = \langle X_1, Y_1, I_1 \rangle$ to $\mathbb{K}_2 = \langle X_2, Y_2, I_2 \rangle$.*
(2) *β satisfies both $\mathrm{Ext}^{\wedge\vee}(X_1, Y_2, \beta) \subseteq \mathrm{Ext}^{\wedge\vee}(X_1, Y_1, I_1)$ and $\mathrm{Int}^{\cap\cup}(X_1, Y_2, \beta) \subseteq \mathrm{Int}^{\cap\cup}(X_2, Y_2, I_2)$.*
(3) *β satisfies both $\{y\}^{\vee_\beta} \in \mathrm{Ext}^{\wedge\vee}(X_1, Y_1, I_1)$ and $\{x\}^{\cap_\beta} \in \mathrm{Int}^{\cap\cup}(X_2, Y_2, I_2)$ for each $x \in X_1, y \in Y_2$.*
(4) *$\beta = S_\mathrm{e} \circ I_2 = I_1 \circ S_\mathrm{i}$ for some $S_\mathrm{e} \in L^{X_1 \times X_2}$ and $S_\mathrm{i} \in L^{Y_1 \times Y_2}$.*

Proof: (1) ⇒ (2): directly from definition of strong homogeneous bond.

(2) ⇒ (3): trivial since $\{y\}^{\vee_\beta} \in \mathrm{Ext}^{\wedge\vee}(X_1, Y_2, \beta)$ and $\{x\}^{\cap_\beta} \in \mathrm{Int}^{\cap\cup}(X_1, Y_2, \beta)$.

(3 ⇒ (4): each **L**-set A in $\mathbf{L}^{X_1}$ can be written in the following form $\bigcup_{x\in X_1} A(x) \otimes \{x\}$. Then, we have:

$$
\begin{aligned}
A^{\cap_\beta}(y) &= \bigvee_{x'\in X_1} \left(\bigcup_{x\in X_1} A(x) \otimes \{x\} \right)(x') \otimes \beta(x', y) \\
&= \bigvee_{x'\in X_1} \left(\bigvee_{x\in X_1} A(x) \otimes \{x\}(x') \right) \otimes \beta(x', y) \\
&= \bigvee_{x\in X_1} \bigvee_{x'\in X_1} A(x) \otimes \{x\}(x') \otimes \beta(x', y) \\
&= \bigvee_{x\in X_1} A(x) \otimes \bigvee_{x'\in X} \{x\}(x') \otimes \beta(x', y) \\
&= \bigvee_{x\in X_1} A(x) \otimes \{x\}^{\cap_\beta}(y) \\
&= \left(\bigcup_{x\in X_1} A(x) \otimes \{x\}^{\cap_\beta} \right)(y).
\end{aligned}
$$

As a result, we obtain $A^{\cap_\beta} \in \mathrm{Int}^{\cap\cup}(X_2, Y_2, I_2)$ since $\{x\}^{\cap_\beta} \in \mathrm{Int}^{\cap\cup}(X_2, Y_2, I_2)$ for each $x \in X_1$ and $\mathrm{Int}^{\cap\cup}(X_2, Y_2, I_2)$ is an **L**-interior system. Because each intent in $\mathrm{Int}^{\cap\cup}(X_1, Y_2, \beta)$ has the form $A^{\cap_\beta}$, we get $\mathrm{Int}^{\cap\cup}(X_1, Y_2, \beta) \subseteq \mathrm{Int}^{\cap\cup}(X_2, Y_2, I_2)$. The existence of S_e now follows from Theorem 2. The existence of S_i can be proved similarly.

(4) ⇒ (1): by Theorem 2 items (a),(b),(g),(h). □

Remark 3: It is worth noting that although conditions (B1′)–(B2′) are equivalent to (B1)–(B2) for the concept-forming operators $\langle ^\uparrow, ^\downarrow \rangle$, they are no longer equivalent for other concept-forming operators, i.e. $\langle \cap, \cup \rangle$ and $\langle \wedge, \vee \rangle$; instead the conditions (B1′)–(B2′) are weaker. Definition 3 corresponds to conditions (B1)–(B2) as Lemma 4 (3) shows.

Strong homogeneous bonds can be related to the i-morphisms.

Theorem 5: *The strong homogeneous bonds from $\mathbb{K}_1 = \langle X_1, Y_1, I_1 \rangle$ to $\mathbb{K}_2 = \langle X_2, Y_2, I_2 \rangle$ are in one-to-one correspondence with*

(a) *i-morphisms from* $\mathrm{Int}^{\cap\cup}(X_1, Y_1, I_1)$ *to* $\mathrm{Int}^{\cap\cup}(X_2, Y_2, I_2)$.
(b) *i-morphisms from* $\mathrm{Ext}^{\wedge\vee}(X_2, Y_2, I_2)$ *to* $\mathrm{Ext}^{\wedge\vee}(X_1, Y_1, I_1)$.

Proof: We prove only (a); the proof of (b) is dual. We show procedures to obtain the i-morphism from $\mathrm{Int}^{\cap\cup}(X_1, Y_1, I_1)$ to $\mathrm{Int}^{\cap\cup}(X_2, Y_2, I_2)$ from a strong homogeneous bond and vice versa.

"$\Rightarrow$": Let β be a strong homogeneous bond from $\mathbb{K}_1 = \langle X_1, Y_1, I_1\rangle$ to $\mathbb{K}_2 = \langle X_2, Y_2, I_2\rangle$. By Lemma 4, there is $S_{\mathrm{i}} \in L^{Y_1\times Y_2}$, such that $\beta = I_1 \circ S_{\mathrm{i}}$. The induced operator $\cap_{S_{\mathrm{i}}}$ is an i-morphism from $\mathrm{Int}^{\cap\cup}(X_1, Y_1, I_1)$ to $\mathrm{Int}^{\cap\cup}(X_2, Y_2, I_2)$ by Lemma 3(a).

"$\Leftarrow$": For i-morphism f, from $\mathrm{Int}^{\cap\cup}(X_1, Y_1, I_1)$ to $\mathrm{Int}^{\cap\cup}(X_2, Y_2, I_2)$, there is an $\mathbf{L}$-relation S_{i} s.t. $f(B) = B^{\cap_{S_{\mathrm{i}}}}$ for each $B \in \mathrm{Int}^{\cap\cup}(X_1, Y_1, I_1)$ by Lemma 2(a). Denote $\beta = I_1 \circ S_{\mathrm{i}}$.

Each $C \in \mathrm{Int}^{\cap\cup}(X_1, Y_2, \beta)$ is equal to $A^{\cap_\beta}$ for some $A \in \mathbf{L}^{X_1}$ and $A^{\cap_\beta} = A \circ \beta = A \circ (I_1 \circ S_{\mathrm{i}}) = (A \circ I_1) \circ S_{\mathrm{i}} = (A \circ I_1)^{\cap_{S_{\mathrm{i}}}} = f(A \circ I_1) \in \mathrm{Im}(f)$.

Thus, we have $\mathrm{Int}^{\cap\cup}(X_1, Y_2, \beta) \subseteq \mathrm{Im}(f) \subseteq \mathrm{Int}^{\cap\cup}(X_2, Y_2, I_2)$; furthermore, we have $\mathrm{Ext}^{\wedge\vee}(X_1, Y_2, \beta) \subseteq \mathrm{Ext}^{\wedge\vee}(X_1, Y_1, I_1)$ by Theorem 2(b). Hence, β is a strong homogeneous bond by Lemma 4.

The fact that the two mappings between bonds and i-morphisms are mutually inverse can be checked as in the proof of Theorem 3. □

Theorem 6: *The system of all strong homogeneous bonds is an* $\mathbf{L}$*-interior system.*

Proof: Using Lemma 4(2), it is an intersection of the $\mathbf{L}$-interior systems from Theorem 4. □

3.2. Direct ∘-products

In the previous section, we have studied the properties of homogeneous bonds, in particular its one-to-one correspondence with c-morphisms and i-morphisms. In this section, somehow paraphrasing (Ganter 2007), we introduce the parameterized direct product of contexts in order to elegantly describe the different families of generalized bonds between two given contexts.

Definition 4: Let $\mathbb{K}_1 = \langle X_1, Y_1, I_1\rangle$ and $\mathbb{K}_2 = \langle X_2, Y_2, I_2\rangle$ be $\mathbf{L}$-contexts. The direct ∘-product of $\mathbb{K}_1$ and $\mathbb{K}_2$ is defined as the $\mathbf{L}$-context

$$\mathbb{K}_1 \boxtimes \mathbb{K}_2 = \langle X_2 \times Y_1, X_1 \times Y_2, \Delta\rangle$$

with $\Delta(\langle x_2, y_1\rangle, \langle x_1, y_2\rangle) = I_1(x_1, y_1) \otimes I_2(x_2, y_2)$.

Theorem 7:

(a) *The* $\langle\cap, \cup\rangle$*-intents of* $\mathbb{K}_1 \boxtimes \mathbb{K}_2$ *are strong homogeneous bonds from* $\mathbb{K}_1$ *to* $\mathbb{K}_2$.
(b) *The* $\langle\wedge, \vee\rangle$*-extents of* $\mathbb{K}_1 \boxtimes \mathbb{K}_2$ *are strong homogeneous bonds from* $\mathbb{K}_2$ *to* $\mathbb{K}_1$.

Proof: We prove only (a); the (b) part is dual. We have

$$\begin{aligned}
\phi^{\cap\Delta}(x_1,y_2) &= \bigvee_{\langle x_2,y_1\rangle\in X_2\times Y_1} \phi(x_2,y_1)\otimes\Delta(\langle x_2,y_1\rangle,\langle x_1,y_2\rangle)\\
&= \bigvee_{\langle x_2,y_1\rangle\in X_2\times Y_1} \phi(x_2,y_1)\otimes I_1(x_1,y_1)\otimes I_2(x_2,y_2)\\
&= \bigvee_{y_1\in Y_1}\bigvee_{x_2\in X_2} \phi(x_2,y_1)\otimes I_1(x_1,y_1)\otimes I_2(x_2,y_2)\\
&= \bigvee_{y_1\in Y_1} I_1(x_1,y_1)\otimes \bigvee_{x_2\in X_2} \phi(x_2,y_1)\otimes I_2(x_2,y_2)\\
&= \bigvee_{y_1\in Y_1} I_1(x_1,y_1)\otimes(\phi^T\circ I_2)(y_1,y_2)\\
&= (I_1\circ\phi^T\circ I_2)(x_1,y_2).
\end{aligned}$$

Now, notice that $(I_1\circ\phi^T)\circ I_2 = I_1\circ(\phi^T\circ I_2)=\beta$ is a strong homogeneous bond by Lemma 4. □

Remark 4: It is worth mentioning that not every strong homogeneous bond is included in $\mathrm{Int}^{\cap\cup}(X_1\times Y_2, X_2\times Y_1,\Delta)$ since there are strong homogeneous bonds which are not of the form of $I_1\circ\phi^T\circ I_2$. For instance, using the same structure of truth degrees and I_1 as in Example 1, obviously I_1 is a strong homogeneous bond on $\mathbb{K}_1$ (i.e. from $\mathbb{K}_1$ to $\mathbb{K}_1$), but $\mathrm{Int}^{\cap\cup}(X_1\times Y_2, X_2\times Y_1,\Delta)$ contains only an empty set.

Corollary 1: The intents of $\mathbb{K}_1\boxtimes\mathbb{K}_2$ are exactly those strong homogeneous bonds from $\mathbb{K}_1$ to $\mathbb{K}_2$ which can be decomposed as $I_1\circ\phi^T\circ I_2$ for some $\phi\in L^{X_2\times Y_1}$.

Proof: The final line of the proof of Theorem 7 explains which strong homogeneous bonds are intents of $\mathbb{K}_1\boxtimes\mathbb{K}_2$. □

Remark 5: The relationship with the homogeneous bonds wrt $\langle{}^{\uparrow},{}^{\downarrow}\rangle$, introduced in Krídlo, Krajči, and Ojeda-Aciego (2012), is the following: If the double negation law holds true in **L**, we have the equality $\mathrm{Ext}^{\uparrow\downarrow}(X,Y,I) = \mathrm{Ext}^{\cap\cup}(X,Y,\neg I)$. Thus, for a strong homogeneous bond $\beta\in L^{X_1\times Y_2} = S_{\mathrm{e}}\circ I_2 = I_1\circ S_{\mathrm{i}}$ from $\mathbb{K}_1$ to $\mathbb{K}_2$, we have

$$\begin{aligned}
(\neg\beta)(x_1,y_2) &= \neg(S_{\mathrm{e}}\circ I_2)(x_1,y_2)\\
&= \left(\bigvee_{x_2\in X_2}\big(S_{\mathrm{e}}(x_1,x_2)\otimes I_2(x_2,y_2)\big)\right)\to 0\\
&= \bigwedge_{x_2\in X_2} (S_{\mathrm{e}}(x_1,x_2)\otimes I_2(x_2,y_2))\to 0)\\
&= \bigwedge_{x_2\in X_2} (S_{\mathrm{e}}(x_1,x_2)\to(I_2(x_2,y_2)\to 0))\\
&= (S_{\mathrm{e}}\triangleleft\neg I_2)(x_1,y_2)
\end{aligned}$$

for each $x_1\in X_1, y_2\in Y_2$. Similarly, we can show that $\neg\beta = \neg I_1\triangleright S_{\mathrm{i}}$. Thus, $\neg\beta$ is a homogeneous bond wrt $\langle{}^{\uparrow},{}^{\downarrow}\rangle$ from $\neg\mathbb{K}_1$ to $\neg\mathbb{K}_2$.

Some papers (Ganter 2007; Krídlo, Krajči, and Ojeda-Aciego 2012) have considered direct products in the crisp and the fuzzy settings, respectively, for the concept-forming operators $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$. In Krídlo, Krajči, and Ojeda-Aciego (2012), conditions are specified under which a homogeneous bond wrt $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ is present in the concept lattice of the direct product. Corollary 1 and Remark 5 provide a simplification of these conditions.

A different direct product of contexts $\mathbb{K}_1 \boxplus \mathbb{K}_2 = \langle X_2 \times Y_1, X_1 \times Y_2, \Delta \rangle$ was defined in Krídlo, Krajči, and Ojeda-Aciego (2012), with the incidence relation given by

$$\Delta(\langle x_2, y_1 \rangle, \langle x_1, y_2 \rangle) = \neg I_1(x_1, y_1) \to I_2(x_2, y_2) \\ (= \neg I_2(x_2, y_2) \to I_1(x_1, y_1)). \tag{25}$$

For the concept-forming operator ${}^{\uparrow_\Delta}$, we have

$$\begin{aligned}
\phi^{\uparrow_\Delta}(x_1, y_2) &= \bigwedge_{\langle x_2, y_1 \rangle \in X_2 \times Y_1} \phi(x_2, y_1) \to (\neg I_1(x_1, y_1) \to I_2(x_2, y_2)) \\
&= \bigwedge_{\langle x_2, y_1 \rangle \in X_2 \times Y_1} \neg I_1(x_1, y_1) \to (\phi(x_2, y_1) \to I_2(x_2, y_2)) \\
&= \bigwedge_{x_2 \in X_2} \bigwedge_{y_1 \in Y_1} \neg I_1(x_1, y_1) \to (\phi(x_2, y_1) \to I_2(x_2, y_2)) \\
&= \bigwedge_{y_1 \in Y_1} \neg I_1(x_1, y_1) \to \bigwedge_{x_2 \in X_2} (\phi(x_2, y_1) \to I_2(x_2, y_2)) \\
&= \bigwedge_{y_1 \in Y_1} \neg I_1(x_1, y_1) \to (\phi^T \triangleleft I_2)(y_1, y_2) \\
&= [\neg I_1 \triangleleft (\phi^T \triangleleft I_2)](x_1, y_2) \\
&= [(\neg I_1 \circ \phi^T) \triangleleft I_2)](x_1, y_2) \\
&= [\neg(\neg I_1 \circ \phi^T \circ \neg I_2)](x_1, y_2).
\end{aligned}$$

Whence a strong homogeneous bond wrt $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ is an intent of the concept lattice of $\mathbb{K}_1 \boxplus \mathbb{K}_2$, iff it is possible to write it as $\neg(\neg I_1 \circ \phi^T \circ \neg I_2)$, i.e. if its complement is an intent of $\neg\mathbb{K}_1 \boxtimes \neg\mathbb{K}_2$.

Example 2: Consider formal **L**-context

$$\mathbb{K} = \begin{array}{|ccc} \frac{1}{3} & \frac{1}{3} & 1 \\ 0 & \frac{2}{3} & 1 \end{array}.$$

Figure 1 depicts a lattice of all homogeneous bonds from $\mathbb{K}$ to $\mathbb{K}$ wrt $\langle \cap, \cup \rangle$ and $\langle \wedge, \vee \rangle$.

4. Heterogeneous $\mathbb{L}$-bonds

This section introduces heterogeneous **L**-bonds in the sense that conditions generalizing (B1′) and (B2′) relate different pairs of concept-forming operators. Particularly, we study the so-called a-bonds and c-bonds defined as follows.

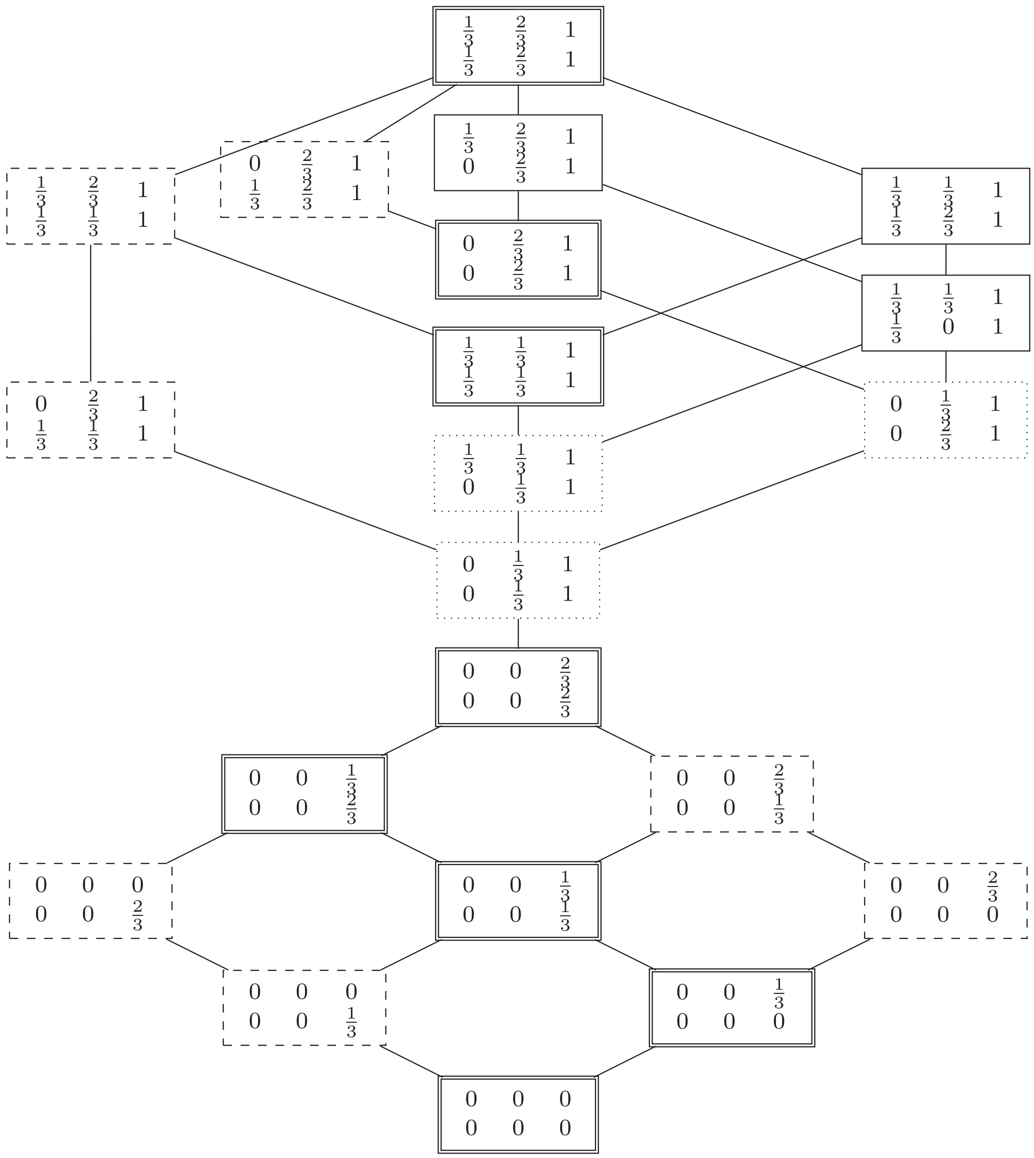

Figure 1. Lattice of all homogeneous bonds wrt isotone concept-forming operators on $\mathbb{K}$ from Example 2. Homogeneous bonds wrt $\langle\cap,\cup\rangle$ are drawn with dashed border; homogeneous bonds wrt $\langle\wedge,\vee\rangle$ are drawn with dotted border; strong homogeneous bonds are drawn with solid border; and intents of $\mathbb{K} \boxtimes \mathbb{K}$ are drawn with double border.

Definition 5:

(a) An *a-bond* from $\mathbb{K}_1 = \langle X_1, Y_1, I_1\rangle$ to $\mathbb{K}_2 = \langle X_2, Y_2, I_2\rangle$ is an **L**-relation $\beta \in L^{X_1\times Y_2}$, such that

$$\begin{aligned}\mathrm{Ext}^{\uparrow\downarrow}(X_1, Y_2, \beta) &\subseteq \mathrm{Ext}^{\cap\cup}(X_1, Y_1, I_1)\\ \mathrm{Int}^{\uparrow\downarrow}(X_1, Y_2, \beta) &\subseteq \mathrm{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2).\end{aligned} \tag{26}$$

(b) A *c-bond* from $\mathbb{K}_1 = \langle X_1, Y_1, I_1\rangle$ to $\mathbb{K}_2 = \langle X_2, Y_2, I_2\rangle$ is an **L**-relation $\beta \in L^{X_1\times Y_2}$ s.t.

$$\begin{aligned}\mathrm{Ext}^{\uparrow\downarrow}(X_1, Y_2, \beta) &\subseteq \mathrm{Ext}^{\uparrow\downarrow}(X_1, Y_1, I_1)\\ \mathrm{Int}^{\uparrow\downarrow}(X_1, Y_2, \beta) &\subseteq \mathrm{Int}^{\wedge\vee}(X_2, Y_2, I_2).\end{aligned} \tag{27}$$

Remark 6:

(a) The terms *a-bond* and *c-bond* have been chosen to match with the notions of a-morphism and c-morphism (Belohlavek and Konecny 2011, 2012b; Konecny 2012). Later, in Theorem 9, we will show that a-bonds (resp. c-bonds) are in one-to-one correspondence with a-morphisms (resp. c-morphisms) on associated sets of intents.
(b) Notice that both the sets of extents and intents in (26) and (27) are **L**-closure systems. From this point of view, the condition of subsethood is natural.
(c) Notice also that a-bonds from $\langle X_1, Y_1, I_1\rangle$ to $\langle X_2, Y_2, I_2\rangle$ are transposed versions of c-bonds from $\langle Y_2, X_2, I_2^{\mathrm{T}}\rangle$ to $\langle Y_1, X_1, I_1^{\mathrm{T}}\rangle$.

The following theorem brings a characterization of a-bonds (resp. c-bonds) in terms of relational compositions.

Theorem 8:

(a) $\beta \in L^{X_1\times Y_2}$ *is an a-bond from* $\mathbb{K}_1 = \langle X_1, Y_1, I_1\rangle$ *to* $\mathbb{K}_2 = \langle X_2, Y_2, I_2\rangle$, *iff there exist* **L**-*relations* $S_{\mathrm{i}} \in L^{Y_1\times Y_2}$ *and* $S_{\mathrm{e}} \in L^{X_1\times X_2}$, *such that*

$$\beta = I_1 \triangleleft S_{\mathrm{i}} = S_{\mathrm{e}} \triangleleft I_2.$$

(b) $\beta \in L^{X_1\times Y_2}$ *is a c-bond from* $\mathbb{K}_1 = \langle X_1, Y_1, I_1\rangle$ *to* $\mathbb{K}_2 = \langle X_2, Y_2, I_2\rangle$, *iff there exist* **L**-*relations* $S_{\mathrm{i}} \in L^{Y_1\times Y_2}$ *and* $S_{\mathrm{e}} \in L^{X_1\times X_2}$, *such that*

$$\beta = I_1 \triangleright S_{\mathrm{i}} = S_{\mathrm{e}} \triangleright I_2.$$

Proof:

(a) Follows from Definition 5 and Theorem 2, items (c) and (e).
(b) Follows from Definition 5 and Theorem 2, items (d) and (f). □

Theorem 9:

(a) *The a-bonds from* $\mathbb{K}_1 = \langle X_1, Y_1, I_1\rangle$ *to* $\mathbb{K}_2 = \langle X_2, Y_2, I_2\rangle$ *are in one-to-one correspondence with*
- *a-morphisms from* $\mathrm{Int}^{\cap\cup}(X_1, Y_1, I_1)$ *to* $\mathrm{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2)$;
- *c-morphisms from* $\mathrm{Ext}^{\uparrow\downarrow}(X_2, Y_2, I_2)$ *to* $\mathrm{Ext}^{\cap\cup}(X_1, Y_1, I_1)$.

(b) *The c-bonds from* $\mathbb{K}_1 = \langle X_1, Y_1, I_1\rangle$ *to* $\mathbb{K}_2 = \langle X_2, Y_2, I_2\rangle$ *are in one-to-one correspondence with*
- *c-morphisms from* $\mathrm{Int}^{\uparrow\downarrow}(X_1, Y_1, I_1)$ *to* $\mathrm{Int}^{\wedge\vee}(X_2, Y_2, I_2)$;
- *a-morphisms from* $\mathrm{Ext}^{\wedge\vee}(X_2, Y_2, I_2)$ *to* $\mathrm{Ext}^{\uparrow\downarrow}(X_1, Y_1, I_1)$.

Proof: (a) Let β be an a-bond from $\mathbb{K}_1$ to $\mathbb{K}_2$. By Theorem 8(a), we have $\beta = I_1 \triangleleft S_{\mathrm{i}}$. By Lemma 3 (c), $f : \mathbf{L}^{X_2} \to \mathbf{L}^{X_1}$ defined by

$$f(B) = B \triangleleft S_{\mathrm{i}} \quad (= B^{\uparrow_{S_{\mathrm{i}}}})$$

is an a-morphism. We need to show that it maps intents in $\mathrm{Int}^{\cap\cup}(X_1, Y_1, I_1)$ to intents in $\mathrm{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2)$.

For each $\langle A, B\rangle \in \mathcal{B}^{\cap\cup}(X_1, Y_1, I_1)$, we have $B = A^{\cap_{I_1}}$, which is equivalent to $B = A \circ I_1$ by Remark 1. Then, we have

$$f(B) = B \lhd S_{\mathrm{i}} = (A \circ I_1) \lhd S_{\mathrm{i}} = A \lhd (I_1 \lhd S_{\mathrm{i}}) = \\ = A \lhd \beta = A^{\uparrow_\beta} \in \mathrm{Int}^{\uparrow\downarrow}(X_1, Y_2, \beta) \subseteq \mathrm{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2).$$

For the c-morphism, by Theorem 8(a), we have $\beta = S_{\mathrm{e}} \lhd I_2$. By Lemma 3 (b), $f : \mathbf{L}^{X_2} \to \mathbf{L}^{X_1}$ defined by

$$f(A) = S_{\mathrm{e}} \lhd A \quad (= A^{\cup_{S_{\mathrm{e}}}})$$

is a c-morphism. We need to show that it maps extents in $\mathrm{Ext}^{\uparrow\downarrow}(X_2, Y_2, I_2)$ to extents in $\mathrm{Ext}^{\cap\cup}(X_1, Y_1, I_1)$. For each $\langle A, B\rangle \in \mathcal{B}^{\uparrow\downarrow}(X_2, Y_2, I_2)$, we have $A = B^{\downarrow_{I_2}}$, which is equivalent to $A = I_2 \rhd B$ by Remark 1. Then, we have

$$f(A) = S_{\mathrm{e}} \lhd A = S_{\mathrm{e}} \lhd (I_2 \rhd B) = (S_{\mathrm{e}} \lhd I_2) \rhd B = \\ = \beta \rhd B = B^{\downarrow_\beta} \in \mathrm{Ext}^{\uparrow\downarrow}(X_1, Y_2, \beta) \subseteq \mathrm{Ext}^{\cap\cup}(X_1, Y_1, I_1).$$

We have just shown how to construct the associated c-morphism and the associated a-morphism for a given a-bond. Now, we show the inverse procedures.

Given an a-morphism f from $\mathrm{Int}^{\cap\cup}(X_1, Y_1, I_1)$ to $\mathrm{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2)$, using Lemma 2 (c), there is an $\mathbf{L}$-relation $S_{\mathrm{i}} \in \mathbf{L}^{Y_1 \times Y_2}$, such that $f(B) = B \lhd S_{\mathrm{i}}$ for each $B \in \mathbf{L}^{Y_1}$. Now, we consider $\beta_f = I_1 \lhd S_{\mathrm{i}}$, and we need to show that it is an a-bond from $\mathbb{K}_1$ to $\mathbb{K}_2$.

Firstly, by Theorem 2 (e), we have $\mathrm{Ext}^{\uparrow\downarrow}(X_1, Y_2, \beta_f) \subseteq \mathrm{Ext}^{\cap\cup}(X_1, Y_1, I_1)$.

Now, all the elements in $\mathrm{Int}^{\uparrow\downarrow}(X_1, Y_2, \beta_f)$ have the form $A^{\uparrow_{\beta_f}}$ for some $A \in \mathbf{L}^{X_1}$. Thus, we can write

$$A^{\uparrow_{\beta_f}} = A \lhd \beta_f = A \lhd (I_1 \lhd S_{\mathrm{i}}) = (A \circ I_1) \lhd S_{\mathrm{i}}$$

and since $A \circ I_1 = A^{\cap_{I_1}} \in \mathrm{Int}^{\cap\cup}(X_1, Y_1, I_1)$,

$$(A \circ I_1) \lhd S_{\mathrm{i}} = A^{\cap_{I_1}} \lhd S_{\mathrm{i}} = f(A^{\cap_{I_1}}) \in \mathrm{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2),$$

proving that $\mathrm{Int}^{\uparrow\downarrow}(X_1, Y_2, \beta_f) \subseteq \mathrm{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2)$, and β_f is an a-bond from $\mathbb{K}_1$ to $\mathbb{K}_2$.

Similarly, let g be a c-morphism from $\mathrm{Ext}^{\uparrow\downarrow}(X_2, Y_2, I_2)$ to $\mathrm{Ext}^{\cap\cup}(X_1, Y_1, I_1)$. By Lemma 2 (b), there is an $\mathbf{L}$-relation $R \in \mathbf{L}^{X_2 \times X_1}$, such that $g(A) = A \rhd R$ for each $A \in \mathbf{L}^{X_2}$. That is equivalent to $g(A) = R^{\mathrm{T}} \lhd A$. Denoting $S_{\mathrm{e}} = R^{\mathrm{T}}$, we get $g(A) = S_{\mathrm{e}} \lhd A$ for each $A \in \mathbf{L}^{X_2}$. We consider $\beta_g = S_{\mathrm{e}} \lhd I_2$ and we need to show that it is an a-bond from $\mathbb{K}_1$ to $\mathbb{K}_2$.

By Theorem 2 (e), we directly have $\mathrm{Int}^{\uparrow\downarrow}(X_1, Y_2, \beta_g) \subseteq \mathrm{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2)$.

Now, all the elements in $\mathrm{Ext}^{\uparrow\downarrow}(X_1, Y_2, \beta_g)$ have the form $B^{\downarrow_{\beta_g}}$ for some $B \in \mathbf{L}^{Y_2}$. Thus, we can write

$$B^{\downarrow_{\beta_g}} = (S_{\mathrm{e}} \lhd I_2) \rhd B = S_{\mathrm{e}} \lhd (I_2 \rhd B)$$

and since $I_2 \rhd B = B^{\downarrow_{I_2}} \in \mathrm{Ext}^{\uparrow\downarrow}(X_2, Y_2, I_2)$,

$$S_{\mathrm{e}} \lhd (I_2 \rhd B) = S_{\mathrm{e}} \lhd B^{\downarrow_{I_2}} = g(B^{\downarrow_{I_2}}) \in \mathrm{Ext}^{\cap\cup}(X_1, Y_1, I_1),$$

proving that $\mathrm{Ext}^{\uparrow\downarrow}(X_1, Y_2, \beta_g) \subseteq \mathrm{Ext}^{\cap\cup}(X_1, Y_1, I_1)$, and β_g is an a-bond from $\mathbb{K}_1$ to $\mathbb{K}_2$.

The fact that the two pairs of mappings between bonds and a-morphisms (resp. c-morphisms) are mutually inverse can be checked as in the proof of Theorem 3.

The proof of (b) is similar. □

Theorem 10:

(a) The system of all a-bonds from $\mathbb{K}_1$ to $\mathbb{K}_2$ is an $\mathbf{L}$-closure system.

(b) The system of all c-bonds from $\mathbb{K}_1$ to $\mathbb{K}_2$ is an $\mathbf{L}$-closure system.

Proof: (a) Consider a family $\{\beta_j \in L^{X_1\times X_2} \mid j \in J\}$ of a-bonds from $\mathbb{K}_1$ to $\mathbb{K}_2$ and let us show that $\bigcap_j \beta_j$ is an a-bond. By Theorem 8, a-bonds β_j are of the form

$$\beta_j = I_1 \triangleleft S_{\mathrm{i}j} = S_{\mathrm{e}j} \triangleleft I_2 \quad \text{for each } j \in J.$$

We have the following two expressions for $\bigcap_{j\in J} \beta_j$

$$\bigcap_{j\in J} \beta_j = \bigcap_{j\in J} (I_1 \triangleleft S_{\mathrm{i}j}) = I_1 \triangleleft \left(\bigcap_{j\in J} S_{\mathrm{i}j}\right);$$
$$\bigcap_{j\in J} \beta_j = \bigcap_{j\in J} (S_{\mathrm{e}j} \triangleleft I_2) = \left(\bigcup_{j\in J} S_{\mathrm{e}j}\right) \triangleleft I_2.$$

Thus, by Theorem 8, $\bigcap_{j\in J} \beta_j$ is an a-bond.

Similarly, consider an a-bond β, hence $\beta = I_1 \triangleleft S_{\mathrm{i}} = S_{\mathrm{e}} \triangleleft I_2$. Let us show that $a \to \beta$ is an a-bond as well:

$$a \to \beta = \beta \triangleright \mathrm{Id}_a = (I_1 \triangleleft S_{\mathrm{i}}) \triangleright \mathrm{Id}_a = I_1 \triangleleft (S_{\mathrm{i}} \triangleright \mathrm{Id}_a);$$
$$a \to \beta = \mathrm{Id}_a \triangleleft \beta = \mathrm{Id}_a \triangleleft (S_{\mathrm{e}} \triangleleft I_2) = (\mathrm{Id}_a \circ S_{\mathrm{e}}) \triangleleft I_2.$$

Thus, $a \to \beta$ is an a-bond from $\mathbb{K}_1$ to $\mathbb{K}_2$ by Theorem 8. We showed that the system of all a-bonds is closed under intersections and shifts, whence it is an $\mathbf{L}$-closure system.

Proof of (b) is similar. □

4.1. Direct ◁-product and direct ▷-product

In this part, we focus on direct products of $\mathbf{L}$-contexts which are related to a-bonds and c-bonds.

Definition 6: Let $\mathbb{K}_1 = \langle X_1, Y_1, I_1\rangle, \mathbb{K}_2 = \langle X_2, Y_2, I_2\rangle$ be $\mathbf{L}$-contexts.

(a) A *direct ◁-product* of $\mathbb{K}_1$ and $\mathbb{K}_2$ is defined as the $\mathbf{L}$-context $\mathbb{K}_1 ⊟ \mathbb{K}_2 = \langle X_2 \times Y_1, X_1 \times Y_2, \Delta\rangle$ with $\Delta(\langle x_2, y_1\rangle, \langle x_1, y_2\rangle) = I_1(x_1, y_1) \to I_2(x_2, y_2)$ for all $x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1, y_2 \in Y_2$.

(b) A *direct ▷-product* of $\mathbb{K}_1$ and $\mathbb{K}_2$ is defined as the $\mathbf{L}$-context $\mathbb{K}_1 ⊟ \mathbb{K}_2 = \langle X_2 \times Y_1, X_1 \times Y_2, \Delta\rangle$ with $\Delta(\langle x_2, y_1\rangle, \langle x_1, y_2\rangle) = I_2(x_2, y_2) \to I_1(x_1, y_1)$ for all $x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1, y_2 \in Y_2$.

The following theorem shows that $\mathbb{K}_1 ⊟ \mathbb{K}_2$ (resp. $\mathbb{K}_1 ⊟ \mathbb{K}_2$) induces a-bonds (resp. c-bonds) as its intents.

Theorem 11:

(a) *The intents of* $\mathbb{K}_1 \boxed{\rightarrow} \mathbb{K}_2$ *wrt* $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ *are a-bonds from* $\mathbb{K}_1$ *to* $\mathbb{K}_2$*, i.e. for each* $\phi \in L^{X_2 \times Y_1}$*,* $\phi^{\uparrow}$ *is an a-bond from* $\mathbb{K}_1$ *to* $\mathbb{K}_2$*.*

(b) *The intents of* $\mathbb{K}_1 \boxed{\leftarrow} \mathbb{K}_2$ *wrt* $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ *are c-bonds from* $\mathbb{K}_1$ *to* $\mathbb{K}_2$*, i.e. for each* $\phi \in L^{X_2 \times Y_1}$*,* $\phi^{\uparrow}$ *is a c-bond from* $\mathbb{K}_1$ *to* $\mathbb{K}_2$*.*

Proof: (a) For $\phi \in L^{X_2 \times Y_1}$, we have

$$\begin{aligned}
\phi^{\uparrow}(x_1, y_2) &= \bigwedge_{\langle x_2, y_1 \rangle \in X_2 \times Y_1} \phi(x_2, y_1) \to \Delta(\langle x_2, y_1 \rangle, \langle x_1, y_2 \rangle) \\
&= \bigwedge_{x_2 \in X_2} \bigwedge_{y_1 \in Y_1} \phi(x_2, y_1) \to (I_1(x_1, y_1) \to I_2(x_2, y_2)) \\
&= \bigwedge_{x_2 \in X_2} \bigwedge_{y_1 \in Y_1} I_1(x_1, y_1) \to (\phi(x_2, y_1) \to I_2(x_2, y_2)) \\
&= \bigwedge_{y_1 \in Y_1} I_1(x_1, y_1) \to \bigwedge_{x_2 \in X_2} \phi(x_2, y_1) \to I_2(x_2, y_2) \\
&= \bigwedge_{y_1 \in Y_1} I_1(x_1, y_1) \to \bigwedge_{x_2 \in X_2} \phi^{\mathrm{T}}(y_1, x_2) \to I_2(x_2, y_2) \\
&= \bigwedge_{y_1 \in Y_1} I_1(x_1, y_1) \to (\phi^{\mathrm{T}} \triangleleft I_2)(y_1, y_2) \\
&= [I_1 \triangleleft (\phi^{\mathrm{T}} \triangleleft I_2)](x_1, y_2) \\
&= [(I_1 \circ \phi^{\mathrm{T}}) \triangleleft I_2](x_1, y_2).
\end{aligned}$$

Thus, $\phi^{\uparrow}$ is an a-bond by Theorem 8 (a). Proof of (b) is similar. □

A similar proposition can be stated also for extents of direct $\triangleleft$-products and direct $\triangleright$-products. More exactly, extents of $\mathbb{K}_1 \boxed{\leftarrow} \mathbb{K}_2$ are c-morphisms from $\mathbb{K}_2$ to $\mathbb{K}_1$, and extents of $\mathbb{K}_1 \boxed{\rightarrow} \mathbb{K}_2$ are a-morphisms from $\mathbb{K}_2$ to $\mathbb{K}_1$.

Remark 7: It is worth to note that not all a-bonds need to be intents of the direct product as the following examples show.

Example 3: Consider the **L**-context $\mathbb{K} = \langle \{x\}, \{y\}, \{{}^{0.5}/\langle x, y \rangle\} \rangle$ with **L** being the three-element **L**ukasiewicz chain. Consider β to be the **L**-relation $\{{}^{0.5}/\langle x, y \rangle\}$. We have

$$\mathrm{Ext}^{\cap\cup}(\{x\}, \{y\}, \beta) = \{{}^{0.5}/x, x\} = \mathrm{Ext}^{\uparrow\downarrow}(\{x\}, \{y\}, \{{}^{0.5}/\langle x, y \rangle\}),$$

and $\mathrm{Int}^{\uparrow\downarrow}(\{x\}, \{y\}, \beta) \subseteq \mathrm{Int}^{\uparrow\downarrow}(\{x\}, \{y\}, \{{}^{0.5}/\langle x, y \rangle\})$ is trivial. Thus, β is an a-bond from $\mathbb{K}$ to $\mathbb{K}$. We have $\mathbb{K} \boxed{\rightarrow} \mathbb{K} = \langle \{\langle x, y \rangle\}, \{\langle x, y \rangle\}, \{\langle x, y \rangle, \langle x, y \rangle\} \rangle$. The only intent of $\mathbb{K} \boxed{\rightarrow} \mathbb{K}$ is $\{\langle x, y \rangle\}$; thus, the a-bond $\beta = \{{}^{0.5}/\langle x, y \rangle\}$ is not among its intents.

Example 4: Consider the following **L**-context with **L** being three-element **L**ukasiewicz chain.

$$\mathbb{K}_1 = \begin{array}{|cccc} 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{array} \qquad \mathbb{K}_2 = \begin{array}{|ccc} 0 & 1 & 1 \\ 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{array}.$$

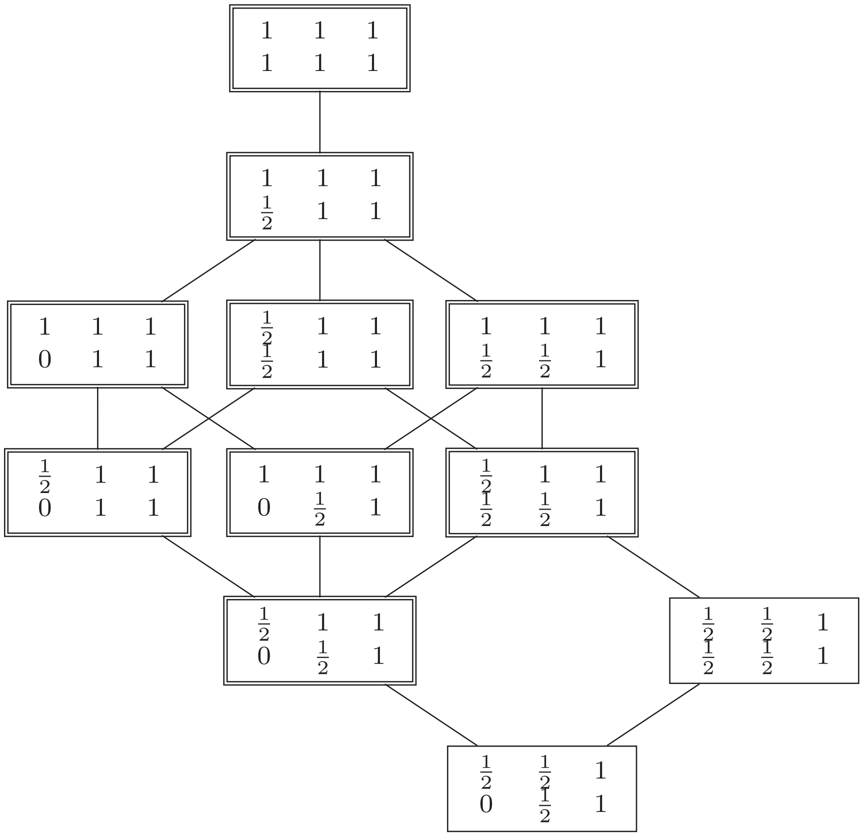

Figure 2. System of a-bonds between $\mathbb{K}_1$ and $\mathbb{K}_2$ from Example 4. The a-bonds with double border are those which are intents of $\mathbb{K}_1 \boxminus \mathbb{K}_2$.

There are 11 a-bonds from $\mathbb{K}_1$ to $\mathbb{K}_2$, but $\mathbb{K}_1 \boxminus \mathbb{K}_2$ has only 9 concepts (see Figure 2).

4.2. Relationship to homogeneous bonds with respect to $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$

In this section, we establish a relationship of a-bonds and c-bonds with homogeneous bonds with respect to $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$. Firstly, we will introduce the notion of strong heterogeneous bond and, then, will prove that they are a special case of homogeneous bond wrt $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$. Then, we study equality of homogeneous bonds wrt $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ with a-bonds and c-bonds under special conditions.

Definition 7: An **L**-relation β is called *strong heterogeneous bond* from $\langle X_1, Y_1, I_1 \rangle$ to $\langle X_2, Y_2, I_2 \rangle$ if it is both a-bond and c-bond from $\langle X_1, Y_1, I_1 \rangle$ to $\langle X_2, Y_2, I_2 \rangle$.

Let us start with the analogous version of Lemma 4 (with alternative characterizations) for homogeneous bonds wrt $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$.

Lemma 5: *The following statements are equivalent:*

(1) β *is a homogeneous bond wrt* $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ *from* $\mathbb{K}_1 = \langle X_1, Y_1, I_1 \rangle$ *to* $\mathbb{K}_2 = \langle X_2, Y_2, I_2 \rangle$.
(2) β *satisfies both* $\{y\}^{\downarrow_\beta} \in \mathrm{Ext}^{\uparrow\downarrow}(X_1, Y_1, I_1)$ *and* $\{x\}^{\uparrow_\beta} \in \mathrm{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2)$ *for each* $x \in X_1, y \in Y_2$.
(3) $\beta = S_{\mathrm{e}} \triangleleft I_2 = I_1 \triangleright S_{\mathrm{i}}$ *for some* $S_{\mathrm{e}} \in L^{X_1 \times X_2}$ *and* $S_{\mathrm{i}} \in L^{Y_1 \times Y_2}$.

Proof: (1) $\Rightarrow$ (2): trivially, we have that $\{y\}^{\downarrow_\beta} \in \mathrm{Ext}^{\uparrow\downarrow}(X_1, Y_2, \beta)$ and $\{x\}^{\uparrow_\beta} \in \mathrm{Int}^{\uparrow\downarrow}(X_1, Y_2, \beta)$.

(2) $\Rightarrow$ (3): each **L**-set A in $\mathbf{L}^{X_1}$ can be written in the form $\bigcup_{x\in X} A(x) \otimes \{x\}$. Thus, we have

$$\begin{aligned}
A^{\uparrow_\beta}(y) &= \bigwedge_{x'\in X} \left(\bigcup_{x\in X} A(x) \otimes \{x\}\right)(x') \to \beta(x', y)\\
&= \bigwedge_{x'\in X} \left(\bigvee_{x\in X} A(x) \otimes \{x\}(x')\right) \to \beta(x', y)\\
&= \bigwedge_{x'\in X} \bigwedge_{x\in X} A(x) \to (\{x\}(x') \to \beta(x', y))\\
&= \bigwedge_{x\in X} A(x) \to \bigwedge_{x'\in X} \{x\}(x') \to \beta(x', y)\\
&= \bigwedge_{x\in X} A(x) \to \{x\}^{\uparrow_\beta}(y)\\
&= \left(\bigcap_{x\in X} A(x) \to \{x\}^{\uparrow_\beta}\right)(y).
\end{aligned}$$

As a result, we have $A^{\uparrow_\beta} \in \mathrm{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2)$ since $\{x\}^{\uparrow_\beta} \in \mathrm{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2)$ for each $x \in X_1$ and $\mathrm{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2)$ is an **L**-closure system. Because each intent in $\mathrm{Int}^{\uparrow\downarrow}(X_1, Y_2, \beta)$ has the form $A^{\uparrow_\beta}$, we get $\mathrm{Int}^{\uparrow\downarrow}(X_1, Y_2, \beta) \subseteq \mathrm{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2)$. The existence of S_e now follows from Theorem 2. Similarly, the existence of S_i can be proved.

(c) $\Rightarrow$ (a): from Theorem 2 (c) and (d). □

One can easily observe that each strong heterogeneous bond is a homogeneous bond wrt $\langle^\uparrow,^\downarrow\rangle$. The following example shows that the converse is not true in general.

Example 5: Use $\mathbf{L} = \mathbf{2}$; obviously, the empty relation is a homogeneous bond wrt $\langle^\uparrow,^\downarrow\rangle$ between two formal contexts with empty incidence relation. On the other hand, it is not an a-bond because $|\mathrm{Ext}^{\cap\cup}(X_1, Y_1, \emptyset)| = 1 < 2 = |\mathrm{Ext}^{\uparrow\downarrow}(X_1, Y_2, \emptyset)|$. Specifically, the only concept in $\mathcal{B}^{\cap\cup}(X_1, Y_1, \emptyset)$ is $\langle X_1, \emptyset\rangle$, whereas the two concepts in $\mathcal{B}^{\uparrow\downarrow}(X_1, Y_2, \emptyset)$ are $\langle X_1, \emptyset\rangle$ and $\langle\emptyset, Y_2\rangle$.

4.2.1. Assuming the double negation law

If the double negation law holds true in **L**, each pair of the concept-forming operators we have been using so far (namely, $\langle^\uparrow,^\downarrow\rangle$, $\langle^\cap,^\cup\rangle$ and $\langle^\wedge,^\vee\rangle$) can define the other two.

As a consequence, for instance, we have that $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$ and $\mathcal{B}^{\cap\cup}(X, Y, \neg I)$ are isomorphic as lattices with $\langle A, B\rangle \mapsto \langle A, \neg B\rangle$ being the isomorphism.

In order to prove this, note that $A \in \mathrm{Ext}^{\uparrow\downarrow}(X, Y, I)$ iff $A = A^{\uparrow_I\downarrow_I}$ and that $A \in \mathrm{Ext}^{\cap\cup}(X, Y, \neg I)$ iff $A = A^{\cap_{\neg I}\cup_{\neg I}}$. We have

$$\begin{aligned}
A^{\cap_{\neg I}\cup_{\neg I}} &= \neg I \triangleleft (A \circ \neg I)\\
&= \neg I \triangleleft (\neg\mathrm{Id} \triangleright ((A \circ \neg I) \triangleleft \neg\mathrm{Id}))\\
&= \neg I \triangleleft (\neg\mathrm{Id} \triangleright (A \triangleleft (\neg I \triangleleft \neg\mathrm{Id})))\\
&= \neg I \triangleleft (\neg\mathrm{Id} \triangleright (A \triangleleft I))\\
&= (\neg I \triangleleft \neg\mathrm{Id}) \triangleright (A \triangleleft I))
\end{aligned}$$

$$= (I \triangleright (A \triangleleft I)) = A^{\uparrow_I \downarrow_I}.$$

That shows that

$$\mathrm{Ext}^{\uparrow\downarrow}(X, Y, I) = \mathrm{Ext}^{\cap\cup}(X, Y, \neg I) \tag{28}$$

As the ordering between the extents is defined to be the fuzzy subsethood ordering (which is independent from the concept-forming pair used to build the concept lattice), one can obtain that both lattices are isomorphic.

To justify the intent part of the isomorphism, note that for each $A \in \mathbf{L}^X, B \in \mathbf{L}^Y$, we have

$$\neg B = \neg(A^{\uparrow_I}) = \neg(A^{\uparrow_{\neg\neg I}}) = \neg(A \triangleleft \neg\neg I) = \neg(A \triangleleft (\neg I \triangleleft \neg \mathrm{Id})) =$$
$$= \neg((A \circ \neg I) \triangleleft \neg \mathrm{Id}) = \neg\neg(A \circ \neg I) = (A \circ \neg I) = A^{\cap_{\neg I}}.$$

Similarly, $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$ and $\mathcal{B}^{\wedge\vee}(X, Y, \neg I)$ are isomorphic as lattices with $\langle A, B\rangle \mapsto \langle \neg A, B\rangle$ being the isomorphism. The proof follows the line of the previous one, but showing

$$\mathrm{Int}^{\wedge\vee}(X, Y, \neg I) = \mathrm{Int}^{\uparrow\downarrow}(X, Y, I). \tag{29}$$

Using that we can state the following theorem.

Theorem 12: *Assume that the double negation holds true in* $\mathbf{L}$. *Then, homogeneous bonds wrt* $\langle \uparrow, \downarrow \rangle$ *from* $\langle X_1, Y_1, I_1\rangle$ *to* $\langle X_2, Y_2, I_2\rangle$ *are exactly a-bonds from* $\langle X_1, Y_1, \neg I_1\rangle$ *to* $\langle X_2, Y_2, I_2\rangle$; *and c-bonds from* $\langle X_1, Y_1, I_1\rangle$ *to* $\langle X_2, Y_2, \neg I_2\rangle$.

Proof: Directly from the definitions and Equations (28) and (29). □

Note that with the double negation law, the incidence relation Δ in $\triangleleft$-direct product $\langle X_1, Y_1, I_1\rangle \boxminus \langle X_2, Y_2, I_2\rangle$ becomes

$$\Delta(\langle x_2, y_1\rangle, \langle x_1, y_2\rangle) = \neg I_1(x_1, y_1) \to I_2(x_2, y_2)$$

and the incidence relation Δ in direct $\triangleright$-product $\mathbb{K}_1 \boxminus \mathbb{K}_2$ becomes

$$\Delta(\langle x_2, y_1\rangle, \langle x_1, y_2\rangle) = \neg I_2(x_2, y_2) \to I_1(x_1, y_1),$$

which coincides with the direct product (25) from Krídlo, Krajči, and Ojeda-Aciego (2012).

4.2.2. Using an alternative notion of complement

The mutual reducibility of concept-forming operators (17)–(19) does not hold generally. In Belohlavek and Konecny (2012a), a new notion of complement of $\mathbf{L}$-relation was proposed in order to overcome that. Using this notion we showed that each for each $I \in L^{X\times Y}$, and fixed an element $a \in L$, one can define $\urcorner I \in L^{X\times(Y\times L)}$ as

$$\urcorner I(x, \langle y, a\rangle) = I(x, y) \to a,$$

and obtain

$$\mathrm{Ext}^{\uparrow\downarrow}(X, Y \times L, \urcorner I) = \mathrm{Ext}^{\cap\cup}(X, Y, I), \tag{30}$$

and similarly,

$$\mathrm{Int}^{\uparrow\downarrow}(X \times L, Y, (\urcorner I^{\mathrm{T}})^{\mathrm{T}}) = \mathrm{Int}^{\wedge\vee}(X, Y, I). \tag{31}$$

That is, for any $I \in \mathbf{L}^{X\times Y}$, one can find a relation which induces the same structure of extents (resp. intents) wrt $\langle^{\downarrow},^{\uparrow}\rangle$ as I induces wrt $\langle\cup,\cap\rangle$ (resp. wrt $\langle\vee,\wedge\rangle$). Unfortunately, the converse does not hold true in general; i.e. there are relations $I \in \mathbf{L}^{X\times Y}$, such that no relation induces the same structure of extents wrt $\langle\cup,\cap\rangle$ (resp. intents wrt $\langle\vee,\wedge\rangle$), as I induces wrt $\langle^{\downarrow},^{\uparrow}\rangle$. Only for those $\mathbf{L}$-relations $I \in \mathbf{L}^{X\times Y}$ whose set of extents $\mathrm{Ext}^{\uparrow\downarrow}(X, Y, I)$ is a c-closure system (Belohlavek and Konecny 2011); i.e. an $\mathbf{L}$-closure system generated by a system of all a-complements of some $\mathcal{T} \subseteq \mathbf{L}^X$.

Theorem 13: *If* $\mathrm{Ext}^{\uparrow\downarrow}(X_1, Y_1, I_1)$ *is a c-closure system, the i-bonds wrt* $\langle^{\uparrow},^{\downarrow}\rangle$ *from* $\langle X_1, Y_1, I_1\rangle$ *to* $\langle X_2, Y_2, I_2\rangle$ *are exactly a-bonds from* $\langle X_1, Y_1 \times L, \neg I_1\rangle$ *to* $\langle X_2, Y_2, I_2\rangle$*. If* $\mathrm{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2)$ *is a c-closure system, the i-bonds wrt* $\langle^{\uparrow},^{\downarrow}\rangle$ *from* $\langle X_1, Y_1, I_1\rangle$ *to* $\langle X_2, Y_2, I_2\rangle$ *are exactly c-bonds from* $\langle X_1, Y_1, I_1\rangle$ *to* $\langle X_2 \times L, Y_2, (\neg I_2^{\mathrm{T}})^{\mathrm{T}}\rangle$.

Proof: Directly from Definitions and (28) and (29). □

5. Conclusions

Continuing with our study of generalized forms of FCA, we have focused on the different natural extensions of the notion of bond.

To the best of our knowledge, only Krídlo, Krajči, and Ojeda-Aciego (2012) had introduced a generalized definition of bond, but it turns out that, in a generalized framework, several alternatives can be considered, depending essentially on the pair(s) of concept-forming operators one relies on. In this paper, we have introduced the notion of homogeneous **L**-bond, namely, a generalized bond wrt a pair of isotone concept-forming operators, and presented a thorough study of them.

Specifically, homogeneous bonds with respect to $\langle\cap,\cup\rangle$ (resp. $\langle\wedge,\vee\rangle$) have been proved to be in one-to-one correspondence with c-morphisms from extents (resp. intents) of the corresponding concept lattices. Moreover, the set of all homogeneous bonds (of either case) is proved to form an **L**-interior system. The natural notion of homogeneous bond wrt the two pairs of isotone concept-forming operators simultaneously (strong homogeneous bond) is proved to be in one-to-one correspondence with i-morphisms between intents of $\langle\cap,\cup\rangle$ and also with i-morphisms between extents of $\langle\wedge,\vee\rangle$. Obviously, the set of all strong homogeneous bonds is an **L**-interior system. The study is concluded by presenting the existing relationship with the direct $\circ$-product of contexts.

A different notion of bond arises when one allows the interaction of isotone and antitone concept-forming operators, and this leads to the so-called heterogeneous bonds, which are proved to be closely related to the a-morphisms and c-morphisms. Specific types of product were needed in order to establish the connection between these new types of bonds with the direct product of contexts.

It is worth to remark that one can see applied papers in the area of information retrieval, see for instance (Valverde-Albacete 2006), which directly calls for heterogeneous bonds, specifically for some a-, c- and i-morphisms.

The obtained results shed new light on the structure and properties of generalized versions of bond between contexts: on the one hand, the results are abstract versions of those already known in the classical case and, on the other hand, generalize as well those published in Krídlo, Krajči, and Ojeda-Aciego (2012).

As future work, on the one hand, it seems worth to consider a further generalization in terms of complete idempotent semifields, which satisfy all the properties of residuated lattices, except that the multiplicative unit need not be the top element of the lattice. In this new framework, it makes sense to consider the fourth pair of concept-forming operators not considered in the present work, which can be viewed as a double dualization on the first pair, see Valverde-Albacete and Peláez-Moreno (2011) for these connections defined over completed idempotent semifields.

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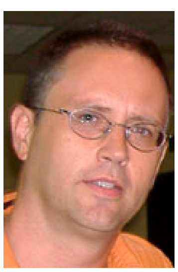

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